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THE AP-HENSTOCK EXTENSION OF THE DUNFORD AND PETTIS INTEGRALS

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Abstract. In this paper, we introduce the AP-Henstock Dunford, AP-Henstock Pettis and AP-Henstock Bochner integral Banach-valued functions and investigate some properties of the these integrals.

1. Introduction and preliminaries

The Denjoy-Dunford, Denjoy-Pettis, and Denjoy-Bochner integral are the extension of Dunford, Pettis, and Bocher integrals, respectively. These integrals were defined and studied by Gordon [4]. He Showed that Denjoy-Dunford(Denjoy-Bochner) integrable function on $[a, b]$ is Dunford(Bochner) integrable on some subinterval of $[a, b]$ and that for spaces that do not contains copy c_0, a Denjoy-Pettis integrable function on $[a, b]$ is Pettis integrable on some subinterval of $[a, b]$. In 2000, Park [5] introduced the Denjoy extension of the Riemann and McShane integral and proved some properties of these integral.

In this paper, we define and study the AP-Henstock extension of Dunford, Pettis, and Bochner integrals of functions mapping $[a, b]$ into Banach space X, respectively.

Troughout this paper, X will denote a real Banach space and X^* its dual.

Let E be measurable set and let c be a real number. The density of E at c is defined by

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$$d_cE = \lim_{h \to 0+} \frac{\mu(E \cap (c-h, c+h))}{2h},$$

provided the limit exists. The point c is called a point of density of E if $D_cE = 1$. The E^d represents the set of all point $x \in E$ such that x is s point of density of E.

A function $F : [a, b] \to R$ is said to be approximately differentiable at $c \in [a, b]$ if there exists a measurable set $E \subset [a, b]$ such that $c \in E^d$ and $\lim_{\substack{x \to c \\ x \in E}} \frac{f(x)-F(c)}{x-c}$ exists. The approximate derivative of F at c is denoted by $F'_{ap}(c)$.

An approximate neighborhood(or ap-nbd) of $x \in [a, b]$ is a measurable set $S_x \subset [a, b]$ containing x as a point of density. For every $x \in E \subset [a, b]$, choose an ap-nbd $S_x \subset [a, b]$ of x. then we say that $S = \{S_x : x \in E\}$ is a choice on E. A tagged interval $(x, [c, d])$ is said to be subordinate to the choice $S = \{S_x\}$ if $c, d \in S_x$. Let $P = \{(x_i, [c_i, d_i]) : 1 \le i \le n\}$ be a finite collection of non-overlapping tagged intervals. If $(x_i, [c_i, d_i])$ is subordinate to a choice S for each i , then we say that P is subordinate to S. If P is subordinate to S and $[a, b] = \cup_{i=1}^n [c_i, c_i]$, then we say that P is a tagged partition of $[a, b]$ that is subordinate to S

Definition 1.1. A function $f : [a, b] \to X$ is AP-Henstock integrable on $[a, b]$ if there exists a vector $A \in X$ with the following property : for each $\epsilon > 0$ there exists a choice S on $[a, b]$ such that $\|f(P) - A\| < \epsilon$ whenever P is a tagged partition of $[a, b]$ that is subordinate to S where $f(P) = (P) \sum f(x) \mid I \mid$. the vector A is called the AP-Henstock integral of F on $[a, b]$ and denoted by $(AP) \int_a^b f$.

Recall that $F : [a, b] \to R$ is AC_s on a measurable et $E \subset [a, b]$ if for each $\epsilon > 0$ there exists a position number η and a choice S on E such $\|(P) \sum F(I)\| < \epsilon$ for every finite collection P of non-overlapping tagged intervals that is subordinate to S and satisfies $(P) \sum |I| < \eta$, where $|I|$ is the Lebesgue measure of an interval I. The function F is ACG_s on E if E can be expressed as a countable union of measurable sets on each of which F is AC_s.

Definition 1.2. ([6]) A function $f : [a, b] \to X$ is AP-Denjoy integrable on $[a, b]$ if there exists an ACG_s function F on $[a, b]$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$.

Theorem 1.3. *([6]) A function $f : [a, b] \to X$ is AP-Denjoy integrable on $[a, b]$ if and only if f is AP-Henstock integrable on $[a, b]$*

DEFINITION 1.4. ([3])

(a) A function $f : [a,b] \to X$ is Denjoy-Dunford integrable on $[a,b]$ if for each x^* in X^* the function x^*f is Denjoy integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for x^* in X^*.
(b) A function $f : [a,b] \to X$ is Denjoy-Pettis integrable on $[a,b]$ if f is Denjoy-Dunford integrable on $[a,b]$ and if $x_I^{**} \in X$ for every interval I is $[a,b]$.
(c) A function $f : [a,b] \to X$ is Denjoy-Bochner integrable on $[a,b]$ if there exists an ACG function $F : [a,b] \to X$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$.

2. AP-Henstock-Dunford and AP-Henstock-Pettis integrability

we introduce the AP-Henstock-Dunford and AP-Henstock-Pettis integral of which is extension for Denjoy-Dunford and Denjoy-Pettis integral and investigate some properties of there integrals.

DEFINITION 2.1. (a) A function $f : [a,b] \to X$ is AP-Henstock-Dunford integrable on $[a,b]$ if for each x^* in X^* the function x^*f is AP-Henstock integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I$ “x^*f for all x^* in X^*.
(b) A function $f : [a,b] \to X$ is AP-Henstock-Pettis integrable on $[a,b]$ if f is AP-Henstock-Dunford integrable on $[a,b]$ and $x_I^{**} \in X$ for every interval I in $[a,b]$.
(c) A function $f : [a,b] \to X$ is AP-Henstock-Bochner integrable on $[a,b]$ if there exists an ACG_s function $F : [a,b] \to X$ such that F is approximately differentiable almost everywhere on $[a,b]$ and such that $F'_{ap} = f$ almost everywhere on $[a,b]$.

Throughout this paper, $(APD)\int_a^b$ and $(APP)\int_a^b f$ will denote the AP-Henstock-Dunford integral and the AP-Henstock-Pettis integral of F on $[a,b]$.

The following theorem was proved by J. I. Games and J. Mendoza [2] .

THEOREM 2.2. *A function $f : [a, b] \to X$ is Denjoy-Dunford integrable on $[a, b]$ if and only if x^*f is Denjoy integrable on $[a, b]$ for each x^* in X^*.*

The following theorems can easily obtained by definition 2.1 and theorem 1.3.

THEOREM 2.3. (a) *A function $f : [a, b] \to X$ is Denjoy-Dunford integrable on $[a, b]$, then f is AP-Henstock-Dunford integrable on $[a, b]$.*

(b) *A function $f : [a, b] \to X$ is Denjoy-Pettis integrable on $[a, b]$, the f is AP-Henstock-Pettis integrable on $[a, b]$.*

THEOREM 2.4. (a) *A function $f : [a, b] \to X$ is AP-Henstock-Dunford integrable on $[a, b]$, then f is is weakly measurable on $[a, b]$.*

(b) *A function $f : [a, b] \to X$ is a bounded and AP-Henstock-Dunford integrable on $[a, b]$, then f is Dunford integrable on $[a, b]$.*

Proof. Let $f : [a, b] \to X$ be AP-Henstock-Dunford integrable on $[a, b]$. then x^*f is AP-Henstock integrable on $[a, b]$ for all x^* in X^*. Hence x^*f is measurable[[4], theorem 16.14 (d)]. Hence f is weakly measurable. (b) Let $f : [a, b] \to X$ is a bounded and AP-Henstock-Dunford integrable on $[a, b]$ for x^*f is Lebesge integrable[[4], theorem 16.15 (a)]. Therefore f is Dunford integrable on $[a, b]$. □

The next corollary follows immediately from Pettis Measurability and Theorem 2.3.

COROLLARY 2.5. *If X is a separable Banach space and if $f : [a, b] \to X$ is AP-Henstock-Dunford integrable on $[a, b]$. Then f is measurable.*

THEOREM 2.6. *A function $f : [a, b] \to X$ is AP-Henstock-Dunford integrable on $[a, b]$ if and only if x^*f is AP-Henstock integrable on $[a, b]$ for each x^* in X^*.*

Proof. If a function $f : [a, b] \to X$ is AP-Henstock-Dunford integrable on $[a, b]$. By definition, x^*f is AP-Henstock integrable on $[a, b]$ for each x^* in X^*. Conversely, if x^*f is AP-Henstock integrable on $[a, b]$ for each X^*. Let $B = \{x^* :\| x^* \|\leq 1\}$ and for each positive integer n and let $V_n = \{x^* \in B : \int_a^b \mid x^*f \mid\leq n\}$. Then $B = \bigcup_n V_n$ and we show next that each V_n is closed. Let y^* be a limit point of V_n and let $\{x_k^*\}$ be a sequence in V_n that converges to y^*. Since $x_k * f$, x^*f are AP-Henstock integrable, $x_k * f$, x^*f are measurable functions[[4], theorem 16.14 (d)].

Also, the sequence $\{|x_k^* f|\}$ converges pointwise to $|y^* f|$. By the Fatou's Lemma, we have

$$\int_a^b |x * f| \leq \liminf_{k\to\infty} \int_a^b \mid x_k * f \mid \leq n$$

Hence , $y^* \in V_n$ and we conclude that V_n is closed. By the Baire Category theorem, there exist an integer N, a real number $r > 0$, and a vector $x_0^* \in B$ such that $\{x^* : \|x^* - x_0^*\| \leq r\} \subset V_N$. For $x^* \in B$,

$$\sup_B \mid \int_E x^f \mid \leq \sup_B \int_E \mid x^* f \mid \leq \sup_B \int_a^b \mid x^* f \mid \leq \frac{2N}{r},$$

the linear funtional T_B is bounded and hence $T_E \in X^{**}$. □

Theorem 2.7. *A function* $f : [a, b] \to X$ *os AP-Henstock-Dunford integrable on each interval* $[c, d] \subset (a, b)$. *If* $\lim_{\substack{c\to a+ \\ d\to b-}} (APD) \int_c^d f$ *exists in* X^{**}, *then* f *is AP-Henstock-Dunford integrable on* $[a, b]$ *and* $(APD) \int_a^b = \lim_{\substack{c\to a+ \\ d\to b-}} (APD) \int_c^d f$

Proof. Let $x_0^{**} = \lim_{\substack{c\to a+ \\ d\to b-}} (APD) \int_c^d f$. By hypothesis, for each $x^* \in X^*$, $x^* f : [a, b] \to R$ is AP-Henstock integrable on each interval $[c, d] \subset (a, b)$ and

$$< x^*, x_0^{**} >= \lim_{\substack{c\to a+ \\ d\to b-}} < x^*, (APD) \int_c^d f >= \lim_{\substack{c\to a+ \\ d\to b-}} \int_c^d x^* f.$$

Hence for each $x^* \in X^*$, $x^* f$ is AP-Henstock integrable on $[a, b]$. Thus f is AP-Henstock-Dundford integrable on $[a, b]$ by theorem 2.6 and

$$< x^*, x_0^{**} >= \lim_{\substack{c\to a+ \\ d\to b-}} < x^*, (APD) \int_c^d f >=< x^*, (APD) \int_a^b f >$$

for all $x^* \in X^*$. Hence $(APD) \int_a^b f = x_0^{**} = \lim_{\substack{c\to a+ \\ d\to b-}} (APD) \int_c^d f$ □

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