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PERIODIC POINTS WHOSE STABLE SETS HAVE NONEMPTY INTERIOR

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Abstract. In this paper, we show that if a homeomorphism has the pseudo-orbit-tracing-property and its nonwandering set is locally connected, then the points whose stable sets have nonempty interior are periodic points.

1. Introduction and preliminaries

Throughout this paper, let X be a compact metric space with a metric function d and f be a homeomorphism of X. Our purpose here is to study dynamical properties of homeomorphisms together with the related concepts of nonwanderingness, chain recurrence and the pseudo-orbit-tracing-property. In [7], Ruess and Summers studied the motions whose limit sets consist of a single periodic motion. In [6], Ombach gave necessary and sufficient conditions that a limit set of a point consists of a single periodic orbit under the condition that f is expansive homomorphism with the pseudo-orbit-tracing-property. Also, author studied stable points whose limit sets consist of single periodic orbit [4] and also study the dynamical properties of nonwandering points whose stable sets have nonempty interior [5].

Here, we show that if f has the pseudo-orbit-tracing-property and its nonwandering set is locally connected, then the points whose stable sets have nonempty interior are periodic points.

For x in X, $O_f(x)$, $O_f^+(x)$ and $O_f^-(x)$ denote the f-orbit, positive f-orbit and negative f-orbit of x, respectively. Let $C(f)$ and $\Omega(f)$ be the *recurrent set* and *nonwandering set* of f, respectively. Recall that $C(f)$ $=\{x \in X : x \in \omega_f(x) \cap \alpha_f(x)$, where $\omega_f(x)$ and $\alpha_f(x)$ denote the positive and negative limit set of x for f,respectively, and $\Omega(f) = \{x \in X :$ for every

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neighborhood U of x and integer $n_0 > 0$ there exists an integer $n \geq n_0$ such that $f^n(U) \cap U \neq \emptyset\}$.

A sequence of points $\{x_i\}_{i\in(a,b)}, (-\infty \leq a < b \leq \infty)$, is called a *$\delta$-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in (a, b-1)$. A finite pseudo-orbit $\{x_0, x_1, \ldots, x_n\}$ is called a pseudo-orbit from x_0 to x_n. Let $x, y \in X$. x is related to y (written $x \sim y$) if there are γ-pseudo-orbits of f from x to y and y to x for every $\gamma > 0$. $CR(f) = \{x \in X : x \sim x\}$ is called the *chain recurrent set* of f. The relation $\sim$ is an equivalence relation in $CR(f)$. A *chain component* is an equivalence class in $CR(f)$ under the relation $\sim$. A sequence of points $\{x_i\}_{i\in(a,b)}$ is called *ε-traced* by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ holds for $i \in (a, b)$. We say that f has the *pseudo-orbit-tracing-property* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that every δ-pseudo-orbit of f can be ε-traced by some point $x \in X$.

A subset M of X is called *f-minimal* if f-orbit of every point in M is dense in M.

Let $B(x, \varepsilon)$ denote $\{y \in X : d(x, y) < \varepsilon\}$ and $\overline{M}$ denote the closure of $M \subset X$.

2. Basic results

Here, we introduce several lemmas which are used in this paper.

LEMMA 1 [4]. *Each connected component of $CR(f)$ is contained in a chain component of $CR(f)$.*

LEMMA 2 [2,3]. *If f has the pseudo-orbit-tracing-property, then the following properties hold;*

(1) *f^k has the pseudo-orbit-tracing-property for every nonzero integer k;*
(2) *f restricted to its nonwandering set has the pseudo-orbit-tracing-property;*
(3) *if Y is an open and closed f-invariant subset of X, then f restricted to Y has the pseudo-orbit-tracing-property;*
(4) *$\overline{C(f)} = \Omega(f) = CR(f)$ holds.*

LEMMA 3 [1]. *If X is a nontrivial connected f-minimal set, then f does not satisfy the pseudo-orbit-tracing-property.*

From here to the end of this paper, we assume that f has the pseudo-orbit-tracing-property and its nonwandering set is locally connected. Note that if the nonwandering set is connected, then f must be a nonwandering homeomorphism.

THEOREM 4. *There is a decomposition of $\Omega(f)$ satisfying the followings;*

(1) *There is a decomposition of $\Omega(f)$ into disjoint closed sets ; $\Omega(f) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k$ such that each Ω_i is f-invariant and f restricted to each Ω_i is topologically transitive.*

(2) *Again, there is a decomposition of each Ω_i into disjoint closed sets ; $\Omega_i = \Omega_1^i \cup \Omega_2^i \cup \cdots \Omega_{n_i}^i$ and these sets are permuted by f.*

Here, each Ω_i is a chain component and each Ω_j^i is a connected component of $\Omega(f)$

Proof. By the local connectedness of $\Omega(f)$, we can find a finite number of pairwise disjoint connected components of $\Omega(f)$ which form a covering of $\Omega(f)$. According to Lemma 1, $\Omega(f)$ decomposes as a finite disjoint union of chain components of $\Omega(f)$:

$$\Omega(f) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k,$$

and each Ω_i decomposes as a finite disjoint union of connected components of $\Omega(f)$:

$$\Omega_i = \Omega_1^i \cup \Omega_2^i \cup \cdots \Omega_{n_i}^i.$$

It is well known that each chain component is f-invariant closed. It is not difficult to show that f restrictd to each Ω_i is topologically transitive.

Next, we prove the part (2) of this result. Fix $i \in \{1, 2, \cdots, k\}$. Observe that since continuous images of connected sets are also connected, we have

$$f(\Omega_{j_1}^i) \cap \Omega_{j_2}^i \neq \emptyset \quad \text{implies} \quad f(\Omega_{j_1}^i) = \Omega_{j_2}^i. \tag{1}$$

Using this fact, now, we prove that $\{\Omega^i_j\}$, $(1 \le j \le n_i)$, is permuted by f. Since $f|_{\Omega_i}$ is topologically transitive, there is a point in Ω^i_1 whose orbit is dense in Ω_i. In view of (1), this implies that, for every $1 \le j \le n_i$,

$$f^{n_i}(\Omega^i_1) = \Omega^i_j \quad \text{for some integer} \quad n_j. \tag{2}$$

Without loss of generality, let us assume that $\{\Omega^i_1, \Omega^i_2, \cdots, \Omega^i_l\}, (1 \le l \le n_i)$, is permuted by f. Then, for j with $l < j \le n_i$, we have

$$f^{n}(\Omega^i_1) \cap \Omega^i_j = \emptyset \quad \text{for every integer} \quad n. \tag{3}$$

The contradiction between (2) and (3) shows that $\{\Omega^i_j\}$ is permuted by f. By renumbering suitably, we get a decomposition of Ω_i, as given in the part (2) of this theorem. This completes the proof. □

For $x \in X$ the *stable set* and *unstable set* of a homeomorphism f are defined by

$$W^s(x,f) = \{y \in X : \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\},$$
$$W^u(x,f) = \{y \in X : \lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}.$$

Let $\mathrm{int}W^\sigma(x,f)$, $(\sigma = s,\ u)$ denote the set of interior points of $W^\sigma(x,f)$ and let $\mathrm{int}W^\sigma(x,f)_{\Omega(f)}$ denote the set of interior points of $W^\sigma(x,f) \cap \Omega(f)$ in the subspace $\Omega(f)$ of X.

LEMMA 5. *If K is a nontrivial connected component of $\Omega(f)$ and $x \in K$, then $\mathrm{int}W^\sigma(x,f) = \emptyset$, $(\sigma = s,\ u)$.*

Proof. Let K be a nontrivial connected component of $\Omega(f)$ and $x \in K$. We start with $\sigma = s$. Suppose, on the contrary that $\mathrm{int}W^s(x,f) \ne \emptyset$. By the Theorem 4, there is a positive integer m such that $f^m(K) = K$. Let g denote f^m for convenience and $0 < c_0 < \min\{d(K_i, K_j)\}$, where $\{(K_i, K_j)\}$ is the set of pairs of disjoint connected component of $\Omega(f)$. Note that $K \subset \Omega(g)$. It is clear that also $\mathrm{int}W^s(x,g)$ is not an empty set.

First, we claim that $\omega_g(x) = K$. To see this, assume on the contrary, that $\omega_g(x)$ is a proper subset of K. Let $p \in K \setminus \omega_g(x)$ and $d(p, \omega_g(x)) = \alpha$, since $\mathrm{int}W^s(x,g)_{\Omega(g)} \neq \emptyset$, we can take a point y and a positive number ξ with $\xi <\min\{c_0, \alpha/4\}$ satisfying that

$$y \in B(y,\xi) \cap \Omega(g) \subset \ \mathrm{int}W^s(x,g)_{\Omega(g)}.$$

Let $\delta = \delta(\xi)$ $(< \xi)$ be a positive number with the property of the pseudo-orbit-tracing-property of g. Since K is connected, we can take a δ-pseudo-orbit $\{z_i\} = \{z_1, z_2, \cdots, z_N\}$ of g from p to y. By the pseudo-orbit-tracing-property of g, there exists a point p_1 ξ-tracing this pseudo-orbit $\{z_i\}$. In particular, we get

$$p_1 \in B(p,\xi) \ \text{ and } \ g^N(p_1) \in B(y,\xi).$$

Using the continuity of g, we can choose an open neighborhood W of p_1 satisfying that

$$p_1 \in W \subset B(p,\xi) \ \text{ and } \ g^N(W) \subset B(y,\xi).$$

On the other hand, since $\overline{C(g)} = \Omega(g)$, we can choose a point p_2 satisfying that

$$p_2 \in W \cap C(g) \ \text{ and } \ g^N(p_2) \subset B(y,\xi) \subset \mathrm{int}W^s(x,g)_{\Omega(g)}.$$

Since $d(g^n(g^N(p_2)), g^n(x))$ tends to zero as n goes to infinity, there exists a positive integer L_1 satisfying that

$$d(g^n(g^N(p_2), g^n(x)) < \frac{\alpha}{4}, \ \text{ for every } \quad n > L_1.$$

Also, there is a positive integer L_2 satisfying that

$$d(g^n(x), \omega_g(x)) < \frac{\alpha}{4} \ \text{ for every } \quad n > L_2.$$

Let L_3 be a positive integer with $L_3 > \max\{L_1, L_2\}$. Then we have

$$(1) \quad d(\omega_g(x), g^n(g^N(p_2))) < d(\omega_g(x), g^n(x)) + d(g^n(x), g^n(g^N(p_2))) < \frac{\alpha}{2}.$$

for every $n > L_3$. Since $p_2 \in C(g)$, we can select a positive integer L with $L > L_3$ satisfying that

$$(2) \qquad g^L(g^N(p_2)) \in W \subset B(p,\xi) \subset B(p, \frac{\alpha}{4}).$$

By (1) and (2), we get the following inclusions.

$$g^L(g^N(p_2)) \subset W \cap B(\omega_g(x), \frac{\alpha}{2}) \subset B(p, \frac{\alpha}{2}) \cap B(\omega_g(x), \frac{\alpha}{2}).$$

This shows that $d(p, \omega_g(x)) < \alpha$. This contradicts the fact that $d(p, \omega_g(x)) = \alpha$. Hence, we obtain that $\omega_g(x) = K$ as desired.

Since restriction $g|_K$ has the pseudo-orbit-tracing-property and K is connected, K is not g-minimal by Lemma 3. Let M be a g-minimal proper subset of K. Let $q \in K \setminus M$ and $d(q, M) = \beta$. Choose $\varepsilon > 0$ with $\varepsilon < \min\{c_0, \beta/3\}$. Choose a point y and a positive number $\zeta < \varepsilon$ such that

$$B(y, \zeta) \cap \Omega(g) \subset \text{int}W^s(x, g)_{\Omega(g)}.$$

Let $\delta = \delta(\zeta)$ be a number with the property of the pseudo-orbit-tracing -property of g. Let $z \in M$. Take a δ-pseudo-orbit $\{v_0, v_1, \cdots, v_L\}$ of g from y to z. Consider the following δ pseudo-orbit of g ;

$$\{b_i\}_{i=0}^{\infty} = \{v_0, v_1, \cdots v_L, g(z), g^2(z), g^3(z), \cdots\}.$$

Then there exists a point y_b ζ-tracing the pseudo-orbit $\{b_i\}$. In particular, we have

$$(3) \qquad d(y, y_b) < \zeta \quad \text{and} \quad d(g^i(g^L(y_b)), g^i(z)) < \zeta$$

for every $i \geq 0$. Note that

$$(4) \qquad M = \overline{O_g^+(g^j(z))} \quad \text{for every integer } j.$$

Therefore, by (3) and (4), the following inclusions hold ;

$$(5) \qquad \omega_g(y_b) = \omega_g(g^L(y_b)) \subset \overline{B(M, \zeta)} \subset \overline{B(M, 2\zeta)}.$$

Also, $y_b \in B(y, \zeta) \cap \Omega(g) \subset \text{int}W^s(x, g)_{\Omega(g)}$ implies that there is a positive integer J satisfying that

$$d(g^n(x), g^n(y_b)) < \zeta \quad \text{for every} \quad n \geq J.$$

This implies that

$$\omega_g(x) = \omega_g(g^J(x)) \subset \overline{B(\omega_g(g^J(y_b)), \zeta))} = \overline{B(\omega_g(y_b), \varepsilon)}. \tag{6}$$

By (5) and (6), we obtain the following inclusions ;

$$K = \omega_g(x) \subset \overline{B(\omega_g(y_b), \varepsilon)} \subset \overline{B(M, 2\zeta + \varepsilon)} \subset B(M, 3\varepsilon).$$

Therefore, we have $d(q, M) < 3\varepsilon < \beta$ and this contradicts the fact that $d(q, M) = \beta$. Hence, we conclude that $\text{int}W^s(x, f) = \emptyset$. One can use the similar method used in the above argument to obtain the fact that $\text{int}W^u(x, f) = \emptyset$. This completes the proof. □

THEOREM 2.6. *Let* $\text{int}W^\sigma(x, f) \neq \emptyset$, $(\sigma = s,\ u)$. *Then the followings hold :*

(1) *If x is an wandering point of f, then its limit set consists of single periodic orbit.*

(2) *If x is a nonwandering point of f, then x is a periodic point.*

Proof. (1). See [5].

(2). Let x be a nonwandering point and $\sigma = s$. If x is in a nontrivial connected component of $\Omega(f)$ Then, by the previous lemma, $\text{int}W^s(x, f) = \emptyset$. Thus x must be in a trivial connected component K of $\Omega(f)$. Therefore $\{x\} = K$ and by Theorem 4, $f^n(x) = x$ for some nonnegative integer n. Thus x is a periodic point. The conclusion in the case that $\sigma = u$ is also obtained similarly. □

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