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# THE INTEGRATION BY PARTS FOR THE $M_\alpha$-INTEGRAL

Jae Myung Park*, Deok Ho Lee**, Ju Han Yoon***, and Hoe Kyoung Lee****

Abstract. In this paper, we define the $M_\alpha$-integral and prove the integration by parts formula for the $M_\alpha$-integral.

## 1. Introduction and preliminaries

It is well-known [8] that the integration by parts formula is valid for the Lebesgue, Denjoy, Perron, and Henstock integrals. In this paper, we prove the integration by parts formula for the $M_\alpha$-integral.

Throughout this paper, $[a, b]$ is a compact interval in $R$. Let $D$ be a finte collection of interval-point pairs $\{(I_i, \xi_i)\}_{i=1}^n$, where $\{I_i\}_{i=1}^n$ are non-overlapping subintervals of $[a, b]$, and let $\delta$ be a positive function on $[a, b]$, i.e. $\delta : [a, b] \to R^+$.

(1) $D$ is a $\delta$-fine McShane partition of $[a, b]$ if $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)$ and $\xi_i \in [a, b]$ for all $i = 1, 2, ..., n$, and $\cup_{i=1}^n I_i = [a, b]$.

(2) $D$ is a $\delta$-fine $M_\alpha$-partition of $[a, b]$ for a constant $\alpha > 0$ if it is a $\delta$-fine McShane partition of $[a, b]$ satisfying

$$\sum_{i=1}^{n} dist(\xi_i, I_i) < \alpha,$$

where $dist(\xi_i, I_i) = inf\{|t - \xi_i| : t \in \xi_i\}$.

(3) $D$ is a $\delta$-fine Henstock partition of $[a, b]$ if $\xi_i \in I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)$ for all $i = 1, 2, ..., n$, and $\cup_{i=1}^n I_i = [a, b]$.

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Correspondence should be addressed to Deok Ho Lee, dhlee2@kongju.ac.kr.

Given a $\delta$-fine partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ we write

$$S(f, D) = \sum_{i=1}^{n} f(\xi_i)|I_i|$$

for integral sums over $D$, whenever $f : [a, b] \to R$.

## 2. Propertise of the $M_\alpha$-integral

DEFINITION 2.1. Let $\alpha > 0$ be a constant. A function $f : [a, b] \to R$ is $M_\alpha$-integrable if there exists a real number $A$ such that for each $\epsilon > 0$ there exists a positive function $\delta : [a, b] \to R^+$ such that

$$|S(f, D) - A| < \epsilon$$

for each $\delta-$fine $M_\alpha$-partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of $[a, b]$. $A$ is called the $M_\alpha$-integral of $f$ on $[a, b]$, and we write $A = \int_a^b f$ or $A = (M_\alpha) \int_a^b f$.

The function $f$ is $M_\alpha$-integrable on the set $E \subset [a, b]$ if the function $f\chi_E$ is $M_\alpha$-integrable on $[a, b]$, and we write $\int_E f = \int_a^b f\chi_E$.

THEOREM 2.2. *A function $f : [a, b] \to R$ is $M_\alpha$-integrable if and only if for each $\epsilon > 0$ there exists a positive function $\delta : [a, b] \to R^+$ such that*

$$|S(f, D_1) - S(f, D_2)| < \epsilon$$

*for any $\delta$-fine $M_\alpha$ -partitions $D_1$ and $D_2$ of $[a, b]$.*

*Proof.* Assume that $f : [a, b] \to R$ is $M_\alpha$-integrable on $[a, b]$. For each $\epsilon > 0$ there is a positive function $\delta : [a, b] \to R^+$ such that

$$|S(f, D) - \int_{I_0} f| < \frac{\epsilon}{2}$$

for each $\delta$-fine $M_\alpha$-partition $D$ of $[a, b]$. If $D_1$ and $D_2$ are $\delta$-fine $M_\alpha$ -partitions, then

$$\begin{aligned} |S(f, D_1) - S(f, D_2)| &\leq |S(f, D_1) - \int_a^b f| + |\int_a^b -S(f, D_2)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Converesly, assume that for each $\epsilon > 0$, there is a positive function $\delta : [a, b] \to R^+$ such that $|S(f, D_m) - S(f, D_k)| < \epsilon$ for any $\delta$-fine $M_\alpha$-partitions $D_m$ and $D_k$ of $[a, b]$. For each $n \in N$, choose $\delta_n : [a, b] \to R^+$ such that $|S(f, D_1) - S(f, D_2)| < \frac{1}{n}$ for any $\delta_n$-fine $M_\alpha$-partitions $D_1$ and $D_2$ of $[a, b]$. Assume that $\{\delta_n\}$ is decreasing. For each $n \in N$, let $D_n$ be a $\delta_n$-fine $M_\alpha$-partition of $[a,b]$. Then $\{S(f,D_n)\}$ is a Cauchy sequence. Let $L = \lim_{n\to\infty} S(f,D_n)$ and let $\epsilon > 0$. Choose $N$ such that $\frac{1}{N} < \frac{\epsilon}{2}$ and $|S(f,D_n) - L| < \frac{\epsilon}{2}$ for all $n \geq N$. Let $D$ be a $\delta_N$-fine $M_\alpha$-partition of $[a,b]$. Then

$$\begin{aligned} |S(f,D) - L| &\leq |S(f,D) - S(f,D_N)| + |S(f,D_N) - L| \\ &< \frac{1}{N} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, $f$ is $M_\alpha$-integrable on $[a,b]$, and $\int_a^b f = L$. □

We can easily get the following theorems.

THEOREM 2.3. *Let $f : [a,b] \to R$. Then*
*(1) If $f$ is $M_\alpha$-integrable on $[a,b]$, then $f$ is $M_\alpha$-integrable on every subinterval of $[a,b]$.*
*(2) If $f$ is $M_\alpha$-integrable on each of the intervals $[a,c]$ and $[c,b]$, then $f$ is $M_\alpha$-integrable on $[a,b]$ and $\int_a^c f + \int_c^b f = \int_a^b f$.*

THEOREM 2.4. *Let $f$ and $g$ be $M_\alpha$-integrable functions on $[a,b]$. Then*
*(1) $kf$ is $M_\alpha$-integrable on $[a,b]$ and $\int_a^b kf = k\int_a^b f$ for each $k \in R$,*
*(2) $f+g$ is $M_\alpha$-integrable on $[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.*

LEMMA 2.5. *(Saks-Henstock Lemma) Let $f : [a,b] \to R$ be $M_\alpha$-integrable on $[a,b]$. Let $\epsilon > 0$. Suppose that $\delta$ is a positive function on $[a,b]$ such that*

$$|S(f,D) - \int_a^b f| < \epsilon$$

*for each $\delta$-fine $M_\alpha$-partition $D = \{(I_i,\xi_i)\}$ of $[a,b]$. If $D' = \{(I_i,\xi_i)\}_{i=1}^m$ is a $\delta$-fine partial $M_\alpha$-partition of $[a,b]$, then*

$$|S(f,D') - \sum_{i=1}^m \int_{I_i} f| \leq \epsilon.$$

*Proof.* Assume that $D' = \{(I_i,\xi_i)\}_{i=1}^m$ is an arbitrary $\delta$-fine partial $M_\alpha$-partition of $[a,b]$. Let $\overline{[a,b] - \cup_{i=1}^m I_i} = \cup_{j=1}^k I'_j$.

Let $\eta > 0$. Since $f$ is $M_\alpha$-integrable on each $I'_j$, there is a positive function $\delta_j : I'_j \to R^+$ such that

$$|S(f,D_j) - \int_{I'_j} f| < \frac{\eta}{k}.$$

for each $\delta_j$-fine $M_\alpha$-partition of $I'_j$.

Assume that $\delta_j(\xi) \leq \delta(\xi)$ for all $\xi \in I'_j$. Let $D_0 = D' \cup D_1 \cup \cdots \cup D_k$. Then $D_0$ is a $\delta$-fine $M_\alpha$-partition of $[a, b]$ and we have

$$|S(f, D_0) - \int_a^b f| < \epsilon.$$

Consequently, we have

$$\begin{aligned}
&|S(f, D') - \sum_{i=1}^{m} \int_{I_i} f| \\
&= |S(f, D_0) - \sum_{j=1}^{k} S(f, D_j) - (\int_a^b f - \sum_{j=1}^{k} \int_{I'_j} f)| \\
&\leq |S(f, D_0) - \int_a^b f| + \sum_{j=1}^{k} |S(f, D_j) - \int_{I'_j} f| \\
&< \epsilon + k \cdot \frac{\eta}{k} = \epsilon + \eta.
\end{aligned}$$

Since $\eta > 0$ was arbitrary, we have $|S(f, D') - \sum_{i=1}^{m} \int_{I_i} f| \leq \epsilon$. □

Now we recall the definition of the derivative of a function.

Definition 2.6. *A function $F : [a, b] \to R$ is differentiable at $\xi \in [a, b]$ if*

$$\lim_{\mu \to 0} \frac{F(\xi + \mu) - F(\xi)}{\mu}$$

*exists. The limit in case it exists, is called the derivative of $F$ at $\xi$, and is denoted by $F'(\xi)$.*

Theorem 2.7. *If the function $F : [a, b] \to R$ is differentiable on $[a, b]$ with $f(\xi) = F'(\xi)$ for each $\xi \in [a, b]$, then $f : [a, b] \to R$ is $M_\alpha$-integrable.*

*Proof.* Let $\epsilon > 0$. By the definition of derivative, for each $\xi \in [a, b]$ there is a positive function $\delta : [a, b] \to R^+$ such that

$$\Big| \frac{F(\zeta) - F(\xi)}{\zeta - \xi} - f(\xi) \Big| < \frac{\epsilon}{2(\alpha + b - a)}$$

for all $\zeta \in [a,b]$ with $0 < |\zeta - \xi| < \delta(\xi)$. Assume that $D = \{(I_i, \xi_i)\}_{i=1}^n$ is a $\delta$-fine $M_\alpha$-partition of $[a,b]$. Then we have

$$\Big|\sum_{i=1}^n [f(\xi_i)|I_i| - F(I_i)]\Big| \leq \sum_{i=1}^n |f(\xi_i)|I_i| - F(I_i)|$$
$$< \frac{\epsilon}{(\alpha + b - a)} \sum_{i=1}^n (dist(\xi_i, I_i) + |I_i|)$$
$$< \frac{\epsilon}{(\alpha + b - a)}(\alpha + b - a) = \epsilon.$$

Hence, $f : [a,b] \to R$ is $M_\alpha$-integrable on $[a,b]$. □

DEFINITION 2.8. *Let $\alpha > 0$ be a constant. Let $F : [a,b] \to R$ and let $E$ be a subset of $[a,b]$.*

*(a) $F$ is said to be $AC_\alpha$ on $E$ if for each $\epsilon > 0$ there exist a constant $\eta > 0$ and a positive function $\delta : [a,b] \to R^+$ such that $|\sum_{i=1} F(I_i)| < \epsilon$ for each $\delta$-fine partial $M_\alpha$-partition $D = \{(I_i, \xi_i)\}$ of $[a,b]$ satisfying $\xi_i \in E$ and $\sum_i |I_i| < \eta$.*

*(b) $F$ is said to be $ACG_\alpha$ on $E$ if $E$ can be expressed as a countable union of sets on each of which $F$ is $AC_\alpha$.*

By considering positive and negative parts, it is clear that there is no change if the part $|\sum_i F(I_i)| < \epsilon$ of the above definition is written as $\sum_i |F(I_i)| < \epsilon$.

THEOREM 2.9. *If a function $f : [a,b] \to R$ is $M_\alpha$-integrable on $[a,b]$ with the primitive $F$, then $F$ is $ACG_\alpha$ on $[a,b]$.*

*Proof.* By the definition of the $M_\alpha$-integral and the Saks-Henstock Lemma, for each $\epsilon > 0$ there is a positive function $\delta : [a,b] \to R^+$ such that

$$\Big|\sum_{i=1}^n [f(\xi_i)|I_i| - F(I_i)]\Big| \leq \epsilon$$

for each $\delta$-fine partial $M_\alpha$-partition $D = \{(I_i, \xi_i)\}$ of $[a,b]$.

Assume that $E_n = \{\xi \in [a,b] : n - 1 \leq |f(\xi)| < n\}$ for each $n \in \mathbb{N}$. Then we have $[a,b] = \cup E_n$. To show that $F$ is $AC_\alpha$ on each $E_n$, fix $n$ and take a $\delta$-fine partial $M_\alpha$-partition $D_0 = \{(I_i, \xi_i)\}$ of $[a,b]$ satisfying

$\xi_i \in E_n$ for all $i$. If $\sum_i |I_i| < \frac{\epsilon}{n}$, then

$$\begin{aligned} |F(I_i)| &\le \Big|\sum_i [F(I_i) - f(\xi_i) \cdot |I_i|]\Big| + \Big|\sum_i f(\xi_i)|I_i|\Big| \\ &\le \Big|\sum_i [F(I_i) - f(\xi_i)|I_i|]\Big| + \sum_i |f(\xi_i)| \cdot |I_i| \\ &\le \epsilon + n \sum_i |I_i| < 2\epsilon. \end{aligned}$$

□

Now we recall the definitions of the McShane and Henstock integrals.

A function $f : [a, b] \to R$ is McShane integrable if there exists a real number A such that for each $\epsilon > 0$ there exists a positive function $\delta : [a, b] \to R^+$ such that

$$|S(f, D) - A| < \epsilon$$

for each $\delta$-fine McShane partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of $[a, b]$.

A function $f : [a, b] \to R$ is Henstock integrable if there exists a real number A such that for each $\epsilon > 0$ there exists a positive function $\delta : [a, b] \to R^+$ such that

$$|S(f, D) - A| < \epsilon$$

for each $\delta$-fine Henstock partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of $[a, b]$.

Since every Henstock partition is an $M_\alpha$-partition and every $M_\alpha$-partition is a McShane partition, we get the following theorem.

THEOREM 2.10. *Let $f : [a, b] \to R$ be a function.*

*(a) If $f$ is McShane integrable on $[a, b]$, then $f$ is $M_\alpha$-integrable on $[a, b]$.*

*(b) If $f$ is $M_\alpha$-integrable on $[a, b]$, then $f$ is Henstock integrable on $[a, b]$.*

The following Theorem is well-known [12].

THEOREM 2.11. *If a function $f : [a, b] \to R$ is $M_\alpha$-integrable on $[a, b]$ if and only if there exists an $ACG_\alpha$ function $F$ on $[a, b]$ such that $F' = f$ almost everywhere on $[a, b]$.*

## 3. Integration by parts

THEOREM 3.1. *If $F$ is $ACG_\alpha$ on $[a, b]$, then $F$ is continuous on $[a, b]$.*

*Proof.* Let $[a,b] = \cup_{n=1}^{\infty} E_n$ where $F$ is $AC_\alpha$ on each $E_n$. Let $c \in [a,b]$ and choose an index $n$ such that $c \in E_n$. Let $\epsilon > 0$. Since $F$ is $AC_\alpha$ on $E_n$, there exist a positive number $\eta > 0$ and a positive function $\delta : [a,b] \to R^+$ such that $\sum_i |F(I_i)| < \epsilon$ for each $\delta$-fine partial $M_\alpha$-partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of $[a,b]$ satisfying $\sum_i |I_i| < \eta$ and $\xi_i \in E_n$. Let $r = min\{\delta(c), \eta\}$. Suppose that $x \in (c-r, c+r) \cap I_0$. Then $([c,x], c)(or\ ([x,c], c))$ is a $\delta$-fine partial $M_\alpha$-partition with $|x - c| < \eta$. Hence, $|F(x) - F(c)| < \epsilon$. It follows that $F$ is continuous at $c$. □

THEOREM 3.2. *If $F$ and $G$ are $ACG_\alpha$ on $[a,b]$, then $FG$ is $ACG_\alpha$ on $[a,b]$.*

*Proof.* Since $F$ and $G$ are continuous on $[a,b]$ by Theorem 3.1, there exist real numbers $M_1$ and $M_2$ with $M_1,\ M_2 \geq 1$ such that $|F(t)| \leq M_1$ and $|G(t)| \leq M_2$ for each $t \in [a,b]$. Since $F$ is $ACG_\alpha$ on $[a,b]$, we have $[a,b] = \cup_{n=1}^{\infty} E_n$ and $F$ is $AC_\alpha$ on each $E_n$. Since $G$ is $ACG_\alpha$ on $[a,b]$, we have $[a,b] = \cup_{k=1}^{\infty} A_k$ and $G$ is $AC_\alpha$ on each $A_k$. Then $[a,b] = \cup_{n=1}^{\infty} \cup_{k=1}^{\infty} (E_n \cap A_k)$.

To show that $FG$ is $AC_\alpha$ on each $E_n \cap A_k$, fix $n$ and $k$. Let $\epsilon > 0$. Since $F$ is $AC_\alpha$ on $E_n$, there exist a constant $\eta_1 > 0$ and a positive function $\delta_1 : [a,b] \to R^+$ such that

$$\sum_{i=1}^{n} |F(I_i)| < \frac{\epsilon}{2M_2}$$

for each $\delta_1$-fine partial $M_\alpha$-partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ of $[a,b]$ satisfying $\sum_{i=1}$ $|I_i| < \eta_1$ and $\xi_i \in E_n$. Since $G$ is $AC_\alpha$ on $A_k$, there exist a constant $\eta_2 > 0$ and a positive function $\delta_2 : [a,b] \to R^+$ such that

$$\sum_{j=1}^{n} |G(J_j)| < \frac{\epsilon}{2M_1}$$

for each $\delta_2$-fine partial $M_\alpha$-partition $\{(J_j, \zeta_j)\}_{j=1}^{q}$ of $[a,b]$ satisfying $\sum_{j=1}$ $|J_j| < \eta_2$ and $\zeta_j \in A_k$.

Let $\delta = min\{\delta_1, \delta_2\}$ and $\eta = min\{\eta_1, \eta_2\}$. Let $D = \{([c_i, d_i], \xi_i)\}_{i=1}^{m}$ be a $\delta$-fine partial $M_\alpha$-partition satisfying $\sum_{i=1} |d_i - c_i| < \eta$ and $\xi_i \in E_n \cap A_k$. Then we have

$$\begin{aligned}
&\sum_{i=1}^{m} |F(d_i)G(d_i) - F(c_i)G(c_i)| \\
&\leq \sum_{i=1}^{m} |F(d_i)G(d_i) - F(c_i)G(d_i)| + \sum_{i=1}^{m} |F(c_i)G(d_i) - F(c_i)G(c_i)|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} |G(d_i)||F(d_i) - F(c_i)| + \sum_{i=1}^{m} |F(c_i)||G(d_i) - G(c_i)| \\
&\le M_2 \sum_{i=1}^{m} |F(d_i) - F(c_i)| + M_1 \sum_{i=1}^{m} |G(d_i) - G(c_i)| \\
&< M_2 \frac{\epsilon}{2M_2} + M_1 \frac{\epsilon}{2M_1} = \epsilon
\end{aligned}
$$

Hence, $FG$ is $AC_\alpha$ on $E_n \cap A_k$. □

THEOREM 3.3. *Let $f : [a, b] \to R$ be $M_\alpha$-integrable on $[a, b]$ and let $F(x) = (M_\alpha)\int_a^x f$ for each $x \in [a, b]$. If $G : [a, b] \to R$ is $AC$ on $[a, b]$, then $fG$ is $M_\alpha$-integrable on $[a, b]$ and*

$$(M_\alpha)\int_a^b fG = F(b)G(b) - (L)\int_a^b FG'.$$

*Proof.* Since $F$ is $ACG_\alpha$ on $[a, b]$ and the $AC$ function $G$ is $AC_\alpha$ on $[a, b]$, $FG$ is $ACG_\alpha$ on $[a, b]$ by Theorem 3.2. Hence, $(FG)'$ is $M_\alpha$-integrable on $[a, b]$. Since $F$ is bounded and measurable, $FG'$ is Lebesgue integrable on $[a, b]$. Since $fG = (FG)' - FG'$ almost everywhere on $[a, b]$, $fG$ is $M_\alpha$-integrable on $[a, b]$ and

$$
\begin{aligned}
(M_\alpha)\int_a^b fG &= (M_\alpha)\int_a^b (FG)' - (L)\int_a^b FG' \\
&= F(b)G(b) - (L)\int_a^b FG'
\end{aligned}
$$

□

COROLLARY 3.4. *Let $f : [a, b] \to R$ be $M_\alpha$-integrable on $[a, b]$ and let $F(x) = (M_\alpha)\int_a^x f$ for each $x \in [a, b]$. If $G : [a, b] \to R$ is $AC$ on $[a, b]$, then $fG$ is $M_\alpha$-integrable on $[a, b]$ and*

$$(M_\alpha)\int_a^b fG = F(b)G(b) - \int_a^b FdG,$$

*where the second integral is the Riemann-Stieltjes integral of $F$ with respect to $G$.*

*Proof.* By Theorem 3.3, the function $fG$ is $M_\alpha$-integrable on $[a, b]$. Since $F$ is continuous and $G$ is $AC$ on $[a, b]$,

$$(L)\int_a^b FG' = \int_a^b FdG.$$

Hence,

$$\int_a^b fG = F(b)G(b) - \int_a^b FdG.$$

□

THEOREM 3.5. *Let $f : [a,b] \to R$ be $M_\alpha$-integrable on $[a,b]$ and let $F(x) = (M_\alpha)\int_a^x f$ for each $x \in [a,b]$. If $G : [a,b] \to R$ is an $ACG_\alpha$ function of bounded variation on $[a,b]$, then $fG$ is $M_\alpha$-integrable on $[a,b]$ and*

$$(M_\alpha)\int_a^b fG = F(b)G(b) - \int_a^b FdG.$$

*Proof.* Since $F$ is $ACG_\alpha$ on $[a,b]$, $FG$ is $ACG_\alpha$ on $[a,b]$ by Theorem 3.2. Hence, $(FG)'$ is $M_\alpha$-integrable on $[a,b]$. Since $F$ is bounded and measurable, $FG'$ is Lebesgue integrable on $[a,b]$. Since $fG = (FG)' - FG'$ almost everywhere on $[a,b]$, $fG$ is $M_\alpha$-integrable on $[a,b]$ and hence, $fG$ is Henstock integrable on $[a,b]$. By [8, Theorem 12.21],

$$(M_\alpha)\int_a^b fG = (H)\int_a^b fG$$

$$= F(b)G(b) - \int_a^b FdG.$$

□

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*
Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
*E-mail*: parkjm@cnu.ac.kr

**
Department of Mathematics Education
Kongju University
Kongju 314-701, Republic of Korea
*E-mail*: dhlee2@kongju.ac.kr

***
Department of Mathematics Education
Chungbuk University
Chongju 361-7631, Republic of Korea
*E-mail*: yoonjh@chungbuk.ac.kr

****
Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
*E-mail*: ghlrud98@nate.com