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A THIRD-ORDER VARIANT OF NEWTON-SECANT METHOD FINDING A MULTIPLE ZERO

Young Ik Kim* and Sang Deok Lee**

Abstract. A nonlinear algebraic equation $f(x) = 0$ is considered to find a root with integer multiplicity $m \geq 1$. A variant of Newton-secant method for a multiple root is proposed below: for $n = 0, 1, 2 \cdots$

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f'(x_n)\{f(x_n) - \lambda f(x_n - \frac{f(x_n)}{f'(x_n)})\}},$$

$$\lambda = \begin{cases} (\frac{m}{m-1})^{m-1}, \text{if } m \geq 2 \\ 1, \text{ if } m = 1. \end{cases}$$

It is shown that the method has third-order convergence and its asymptotic error constant is expressed in terms of m. Numerical examples successfully verified the proposed scheme with high-precision Mathematica programming.

1. Introduction

To find a numerical solution of a nonlinear algebraic equation $f(x) = 0$, many researchers[2-5,8-9,11] have improved Newton's method as well as secant method and proposed its variants. In 1982, Traub[11] introduced Newton-secant method finding a simple root as shown below:

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f'(x_n)\{f(x_n) - f(x_n - \frac{f(x_n)}{f'(x_n)})\}}, \ n = 0, 1, 2 \cdots. \quad (1.1)$$

It is known that(1.1) has third-order convergence and asymptotic error constant as $\frac{1}{4}\frac{f''(\alpha)}{f'(\alpha)}$. Although it is generally difficult to know a priori the multiplicity of a given root, the convergence study of a multiple root is of considerable theoretical interest for wide applications. Assuming

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Correspondence should be addressed to Sang Deok Lee, sdlee@dankook.ac.kr.

that a function $f : \mathbb{C} \to \mathbb{C}$ has a root α of integer multiplicity $m \geq 1$ and is analytic [7] in a small neighborhood of α, an extended variant of Traub's Newton-secant method is proposed below to treat multiple roots as wells as a simple root:

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f'(x_n)\{f(x_n) - \lambda f(x_n - \frac{f(x_n)}{f'(x_n)})\}}, \; n = 0, 1, 2 \cdots, \quad (1.2)$$

with

$$\lambda = \begin{cases} (\frac{m}{m-1})^{m-1}, \text{if } m \geq 2. \\ 1, \text{ if } m = 1. \end{cases} \quad (1.3)$$

It is easily seen that $\lambda = \lim_{m->1}(\frac{m}{m-1})^{m-1} = 1$ and if $m = 1$, (1.2) reduces to (1.1). Observe that (1.2) is free of second derivatives, unlike modified Halley method[4] and Euler-Chebyshev method[11] requiring second derivatives shown below respectively:

$$x_{n+1} = x_n - \frac{2}{(1 + \frac{1}{m})\frac{f'(x_n)}{f(x_n)} - \frac{f''(x_n)}{f'(x_n)}}, \; n = 0, 1, 2 \cdots, \quad (1.4)$$

$$x_{n+1} = x_n - \frac{mf(x_n)}{2f'(x_n)}\left(3 - m + \frac{mf''(x_n)f(x_n)}{f'(x_n)^2}\right), \; n = 0, 1, 2 \cdots. \quad (1.5)$$

We rewrite the given equation $f(x) = 0$ in the form $x - g(x) = 0$, where $g : \mathbb{C} \to \mathbb{C}$ is analytic in a sufficiently small neighborhood of α. Thus (1.2) can be written in the form of the following scheme

$$x_{n+1} = g(x_n), \; n = 0, \; 1, \; 2, \cdots, \quad (1.6)$$

for a given $x_0 \in \mathbb{C}$. Let $p \in \mathbb{N}$ be given and $g(x)$ satisfy the relation below:

$$\begin{cases} \left|\frac{d^p}{dx^p}g(x)\right|_{x=\alpha} = |g^{(p)}(\alpha)| < 1, & \text{if } p = 1. \\ g^{(i)}(\alpha) = 0 \text{ for } 1 \leq i \leq p-1 \text{ and } g^{(p)}(\alpha) \neq 0, & \text{if } p \geq 2. \end{cases} \quad (1.7)$$

Then it can be shown, by extending the similar analysis[3] in $\mathbb{C}$, that the asymptotic error constant η with order of convergence p is found to be with $e_n = x_n - \alpha$:

$$\eta = \lim_{n\to\infty}\left|\frac{e_{n+1}}{e_n^p}\right| = \frac{|g^{(p)}(\alpha)|}{p!}. \quad (1.8)$$

The main aim of this paper is to show iteration scheme (1.2) has cubic convergence by directly identifying $g'(\alpha) = g''(\alpha) = 0, g'''(\alpha) \neq 0$ as well as to express the asymptotic error constant[1,10] in terms of m, f and

α. With the fact that $f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0,\ f^{(m)} \neq 0$ and L'Hospital's rule, let us define

$$h(x) = \begin{cases} f(x)/f'(x), & if\ x \neq \alpha \\ \lim_{x\to\alpha} f(x)/f'(x) = 0, & if\ x = \alpha. \end{cases} \tag{1.9}$$

and

$$g(x) = \begin{cases} x - G(x), if\ x \neq \alpha \\ x,\ if\ x = \alpha. \end{cases}, \tag{1.10}$$

where $G(x) = \frac{f(x)^2}{f'(x)\left\{f(x)-\lambda f(x-f(x)/f'(x)\right\}}$.

To compute $g'(\alpha)$, $g''(\alpha)$ and $g'''(\alpha)$ efficiently, some local properties of $h(x)$ are shown in the following lemmas that can be proved by repeated applications of L'hospital's rule[6] and Leibnitz' rule[6] for differentiation.

LEMMA 1.1. *Let $f : \mathbb{C} \to \mathbb{C}$ have a root α of integer multiplicity $m \geq 1$ and be analytic in a small neighborhood of α. Then the function $h(x)$ defined by (1.9) and its derivatives up to order 3 evaluated at α have the following properties with $\theta_j = \frac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}$ for $j \in \mathbb{N}$:*

(1) $h(\alpha) = 0$
(2) $h'(\alpha) = \frac{1}{m}$
(3) $h''(\alpha) = -\frac{2}{m^2(m+1)}\theta_1$
(4) $h^{(3)}(\alpha) = \frac{6}{m^3(m+1)}(\theta_1^2 - \frac{2m}{m+2}\theta_2)$

2. Convergence analysis

Convergence behavior for (1.2) is investigated here by deriving the asymptotic error constant η in terms of m, f and α. We first need to investigate some local properties of $g(x)$ in a small neighborhood of α. It follows from (1.10) that

$$(g - x) \cdot f' \cdot \left\{f - \lambda f(z)\right\} = -f^2, \tag{2.1}$$

where $f = f(x), f' = f'(x), z = z(x) = x - h(x)$ and $g = g(x)$ are used for brevity and the symbol $'$ denotes the derivative with respect to x.

Differentiating $2m$ times both sides of Eq(2.1) with respect to x and substituting $x = \alpha$, we obtain the following with the aid of Leibnitz rule for differentiation:

$$\sum_{r=0}^{2m}\binom{2m}{r}(g-x)^{(2m-r)}\cdot[f'\cdot\{f-\lambda f(z)\}]^{(r)}\Big|_{x=\alpha}$$
$$=-\sum_{r=0}^{2m}\binom{2m}{r}f^{(r)}\cdot f^{(2m-r)}\Big|_{x=\alpha}. \tag{2.2}$$

Using $f(\alpha)=f'(\alpha)=\cdots=f^{(m-1)}(\alpha)=0, f^{(m)}(\alpha)\neq 0$ and Lemma 1.2, we favorably find that $[f'\cdot\{f-\lambda f(z)\}]\Big|_{x=\alpha}^{(r)}\neq 0$ in the left side of (2.2) only when $r=2m-1$. By Lemma 1.2, when $r=2m-1$, we have with $t=1-1/m$

$$\big[f'\cdot\{f-\lambda f(z)\}\big]_{x=\alpha}^{(2m-1)}=\binom{2m-1}{m-1}f^{(m)}(\alpha)^2(1-\lambda t^m). \tag{2.3}$$

Using (2.3) in (2.2) yields

$$2m\cdot(g'-1)\cdot\binom{2m-1}{m-1}f^{(m)}(\alpha)^2(1-\lambda t^m)=-\binom{2m}{m}\cdot f^{(m)}(\alpha)^2.$$

By substituting λ in (1.3) into the above equation and simplifying, we find that $g'(\alpha)=0$.

We differentiate $2m+1$ times both sides of (2.1) with respect to x and substitute $x=\alpha$:

$$\sum_{r=0}^{2m+1}\binom{2m+1}{r}(g-x)\Big|_{x=\alpha}^{(2m+1-r)}\big[f'\cdot\{f-\lambda f(z)\}\big]_{x=\alpha}^{(r)}$$
$$=-\sum_{r=0}^{2m+1}\binom{2m+1}{r}f^{(r)}\cdot f^{(2m+1-r)}\Big|_{x=\alpha}. \tag{2.4}$$

The left side of (2.4) can have nonzero terms, when $r=2m-1$ and $r=2m$.

$$\big[f'\cdot\{f-\lambda f(z)\}\big]_{x=\alpha}^{(2m-1)}=\binom{2m-1}{m-1}f^{(m)}(\alpha)^2(1-\lambda t^m). \tag{2.5}$$

$$\begin{aligned}\big[f'\cdot\{f-\lambda f(z)\}\big]_{x=\alpha}^{(2m)}&=\theta_1 f^{(m)}(\alpha)^2\Big\{\binom{2m}{m}(1-\lambda t^m)\\&\quad+\binom{2m}{m-1}(1-\lambda t^{m-1}(t^2-t+1))\Big\}\\&=2\theta_1 f^{(m)}(\alpha)^2\frac{2m!}{(m+1)!m!}.\end{aligned} \tag{2.6}$$

Therefore, we let $g'(\alpha)=0$ in the left side of Eq.(2.4) to obtain the following:

$$-2\frac{(2m+1)!}{(m+1)!m!}\theta_1 f^{(m)}(\alpha)^2+\frac{1}{2}g''(\alpha)\frac{(2m+1)!}{(m-1)!m!}f^{(m)}(\alpha)^2(1-\lambda t^m)$$

$$= -\left\{\binom{2m+1}{m} + \binom{2m+1}{m+1}\right\} f^{(m)}(\alpha)^2\theta_1. \tag{2.7}$$

Hence in view of the fact $\lambda t^m = 1 - 1/m$ and with θ_1 in Lemma 1.1, we obtain $g''(\alpha) = 0$. To obtain $g'''(\alpha)$, we further differentiate $2m+2$ times both sides of Eq(2.1) with respect to x and substitute $x = \alpha$ as follows:

$$\sum_{r=0}^{2m+2} \binom{2m+2}{r} (g-x)\Big|_{x=\alpha}^{(2m+2-r)} \cdot \big[f'\{f - \lambda f(z)\}\big]_{x=\alpha}^{(r)}$$

$$= -\sum_{r=0}^{2m+2} \binom{2m+2}{r} f^{(r)} \cdot f^{(2m+2-r)}\Big|_{x=\alpha}. \tag{2.8}$$

With $t = 1 - 1/m, \lambda t^m = 1 - 1/m, g(\alpha) = \alpha, g'(\alpha) = 0$ and $g''(\alpha) = 0$, the left side of (2.8) possibly has nonzero values if $r = 2m+1$. Using Lemma 1.2, we have

$$\big[f' \cdot \{f - \lambda f(z)\}\big]_{x=\alpha}^{(2m+1)} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} f'^{(k)}\Big|_{x=\alpha} \cdot (f - \lambda f(z))\Big|_{x=\alpha}^{(2m+1-k)}$$

$$= \binom{2m+1}{m-1} f^{(m)}(\alpha)^2 \cdot (W_1\theta_1^2 + W_2\theta_2),$$

where

$$\begin{cases} W_1 = \frac{(m+2)(m-1)}{m^3} - \frac{m}{m-1}q_1(t),\ W_2 = 1 + \frac{m+2}{m^2} - \frac{m}{m-1}q_2(t), & \text{if } m \geq 2. \\ W_1 = 3(1-t),\ W_2 = -(t-1)(t^2+t+2), \text{ if } m = 1. \end{cases}$$

with $q_1(t)$ and $q_2(t)$ defined in Lemma 1.2. Hence(2.8) now reduces to with $\lambda t^m = 1 - 1/m$:

$$-2(m+1) \cdot (W_1\theta_1^2 + W_2\theta_2) + \frac{(m+1)^2(m+2)}{3} g'''(\alpha)(1 - \lambda t^m)$$

$$= -\frac{2}{m}\left\{(m+2){\theta_1}^2 + 2(m+1)\theta_2\right\}.$$

From this, it follows that

$$g'''(\alpha) = \frac{6}{(m+1)(m+2)}(\phi_1\theta_1^2 + \phi_2\theta_2), \tag{2.9}$$

where $\phi_1 = \begin{cases} \frac{(m+2)}{2(m+1)}, \text{ if } m \geq 2. \\ 3/2, \text{ if } m = 1. \end{cases}$ and $\phi_2 = \frac{1}{m} - 1$.

In view of (1.8), we summarize our analysis done so far in the following theorem:

Theorem 2.1. *Let f, h, m, α and $\theta_k(k \in \mathbb{N})$ be described in Lemma 1.1. Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Then iteration scheme (1.2) converges with order 3 and its asymptotic error constant η is given by*

$$\eta = \frac{1}{(m+1)(m+2)} |\phi_1 \theta_1^2 + \phi_2 \theta_2|,$$

where ϕ_1 and ϕ_2 are described in (2.9), provided that $\phi_1 \theta_1^2 + \phi_2 \theta_2 \neq 0$.

Remark 2.2. If $m = 1$, Theorem 2.1 immediately gives the result of Traub[11].

3. Algorithm, numerical results and discussions

The theory stated in Sections 1 and 2 allows us to develop a zero-finding algorithm below to be implemented with *Mathematica*[12]:

Algorithm 3.1 (Zero-Finding Algorithm)
Step 1. Construct iteration scheme (1.2) with the given function f having a multiple zero α for $n \in \mathbb{N} \cup \{0\}$ as mentioned in Section 1.
Step 2. Set the minimum number of precision digits. With exact zero α or most accurate zero, supply the asymptotic error constant η, order of convergence p as well as $\theta_1, \theta_2, \phi_1$ and ϕ_2 stated in Section 2. Set the error range ϵ, the maximum iteration number n_{max} and the initial guess x_0. Compute $f(x_0)$ and $|x_0 - \alpha\ |$.
Step 3. Tabulate the computed values of n, x_n, $e_n = |x_n - \alpha|$, $e_{n+1}/{e_n}^p$ and η.

A variety of numerical examples have been experimented with error bound $\epsilon = 0.5 \times 10^{-235}$ and minimum number of precision digits 250. Symbol i is used to denote $\sqrt{-1}$. The computed asymptotic error constant agrees up to 10 significant digits with the theoretical one. The computed zero is actually rounded to be accurate up to the 235 significant digits, although displayed only up to 15 significant digits.

Iteration scheme (1.2) applied to test functions $(7 - x + x^2)^2/(x^2 + \cos(x))$ and $(e^{-x} \sin x + \log[1 + x - \pi])^2 (x - \pi)^2 \sin^2 x$ clearly shows successful asymptotic error constants with cubic convergence for suitable initial values chosen near α. Tables 1-2 list iteration indexes n, approximate zeros x_n, errors $e_n = |x_n - \alpha|$ and computational asymptotic error constants $e_{n+1}/{e_n}^3$ as well as the theoretical asymptotic error constant η.

TABLE 1. Asymptotic error constant for $f(x) = \frac{(x^2-x+7)^2}{x^2+\cos x}$ with $m=2,\ \alpha = \frac{1+3\sqrt{3}i}{2}$

n	x_n	$e_n = \mid x_n - \alpha \mid$	$e_{n+1}/e_n{}^3$	η
0	$0.36 + 2.387i$	0.253285		0.9449259108
1	$0.499577476751567 + 2.57388126341839i$	0.0241986	1.489236779	
2	$0.500012395983446 + 2.59807060633620i$	0.0000136043	0.9600703765	
3	$0.499999999999998 + 2.59807621135332i$	2.37911×10^{-15}	0.9449013940	
4	$0.500000000000000 + 2.59807621135332i$	1.27245×10^{-44}	0.9449259108	
5	$0.500000000000000 + 2.59807621135332i$	1.94677×10^{-132}	0.9449259108	
6	$0.500000000000000 + 2.59807621135332i$	$0. \times 10^{-249}$		

TABLE 2. Convergence for $f(x) = (e^{-x}\sin x + \log[1+x-\pi])^2(x-\pi)^2\sin^2 x$ with $m=7, \alpha=\pi$

n	x_n	$e_n = \mid x_n - \alpha \mid$	$e_{n+1}/e_n{}^3$	η
0	2.90000000000000	0.241593		0.2826722582
1	3.13818475241933	0.00340790	0.2416772773	
2	3.14159264244215	1.11476×10^{-8}	0.2816579059	
3	3.14159265358979	3.91590×10^{-25}	0.2826722549	
4	3.14159265358979	1.69738×10^{-74}	0.2826722582	
5	3.14159265358979	1.38236×10^{-222}	0.2826722582	
6	3.14159265358979	$0. \times 10^{-249}$		

Convergence behavior was confirmed for further test functions that are listed below:

$f_1(x) = x^9 - x^4 + 73,\ \alpha = -1.24943225052 - 1.04103553493i,$
$\quad m=1,\ \ x_0 = -1.57 - 0.78i$
$f_2(x) = (x-2)\cos(\frac{\pi}{x}),\ \alpha = 2,\ m = 2, x_0 = 1.97$
$f_3(x) = (x^2+16)\log^2(x^2+17),\ \alpha = -4i,\ m = 3, x_0 = -3.92$
$f_4(x) = \frac{(3-x+x^2)^4}{x^4+\sin x},\ \alpha = \frac{1-\sqrt{11}i}{2},\ m = 4, x_0 = 0.37 - 1.89i$
$f_5(x) = (x^3-4x^2-16x-35)\log^3(x-6)\ \sin\frac{\pi x}{7},\ \alpha = 7,\ m = 5, x_0 = 6.5$
$f_6(x) = (x-\pi)^3\cos^3\frac{x}{2},\ \alpha = \pi,\ m = 6, x_0 = 3.75$
$f_7(x) = (e^{x^2+7x-30} - 1)(x-3)^6,\ \alpha = 3,\ m = 7, x_0 = 2.87$
$f_8(x) = (x-\pi)\log^2(x-\pi+1)\sin^5 x/e^x,\ \alpha = \pi,\ m = 8, x_0 = 2.79$

Let p denote the order of convergence and d the number of new evaluations of $f(x)$ or its derivatives per iteration. Taking into account the computational cost, an efficiency of the given iteration function is measured by efficiency index $^*EFF = p^{1/d}$ introduced in [11]. The bigger efficiency index indicates the more efficient and less expensive iteration scheme. For our proposed iteration scheme, we find $p = 3$ and $d = 3$ to get $^*EFF = 3^{\frac{1}{3}} \approx 1.44224957$ which is better than $\sqrt{2}$, the efficiency index of modified Newton's method.

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Department of Applied Mathematics
Dankook University
Cheonan 330-714, Republic of Korea
E-mail: yikbell@yahoo.co.kr

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Department of Applied Mathematics
Dankook University
Cheonan 330-714, Republic of Korea
E-mail: sdlee@dankook.ac.kr