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HÀJECK-RÈNYI TYPE INEQUALITY AND STRONG LAW OF LARGE NUMBERS FOR AQSI RANDOM VARIABLES

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Abstract. In this paper we study the Hàjeck-Rènyi type inequality and strong law of large numbers for asymptotically quadrant sub-independent(AQSI) sequences. We also prove the integrability of supremum for AQSI sequences.

1. Introduction

Hàjeck-Rènyi(1955) proved the following important inequality: Let $\{X_n,\ n \geq 1\}$ be a sequence of centered independent random variables with finite variances and $\{b_n,\ n \geq 1\}$ a sequence of nondecreasing positive numbers. Then, for any $\epsilon > 0$ and any positive integer $m < n$, we obtain

$$P(\max_{m \leq k \leq n} \frac{|\sum_{i=1}^{k} X_i|}{b_k} \geq \epsilon) \leq \epsilon^{-2}(\sum_{k=m+1}^{n} \frac{EX_k^2}{b_k^2} + \sum_{k=1}^{m} \frac{EX_k^2}{b_m^2}).$$

Since then, the extensions of this type inequality for the dependent sequences defined below have be been studied by many authors. For example, Liu, Gan and Chen(1999) proved the Hàjeck-Rènyi inequality and the strong law of large numbers for negatively associated random variables, Fazekas and Klesov(2000) considered a general method for obtaining a Hàjeck-Rènyi type inequality and a strong law of large numbers and gave applications for some dependent sequences and Ko et al.(2005) proved the Hàjeck-Rènyi inequality and strong law of large numbers for AANA random variables. In associated sequence case, Prakasa

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Rao(2002) proved the Hàjeck-Rènyi type inequality for associated random variables and Sung(2008) improved the Hàjeck-Rènyi inequality of Prakasa Rao(2002). Shuhe et al.(2009) investigated the Hàjeck-Rènyi type inequality by using different methods from Sung's and improved the results of Sung(2008) and proved a strong law of large numbers for associated sequences by this type inequality.

Next, we turn to our attention to the dependence for random variables. Lehmann(1966) introduced a simple and natural definition of bivariate dependence: A sequence $\{X_n, n \geq 1\}$ of random variables is said to be pairwise positively quadrant dependent(pairwise PQD)[resp. pairwise negatively quadrant dependent(pairwise NQD)] if for any r_i, r_j and $i \neq j$, $P(X_i > r_i, X_j > r_j) - P(X_i > r_i)P(X_j > r_j) \geq 0$[resp. ≤ 0]. This definition subsequently extended to the multivariate case. Esary, Proschan and Walkup(1967) extended: A finite family $\{X_1, \cdots, X_n\}$ of random variables is said to be associated if $Cov(f(X_1, \cdots, X_n)), g(X_1, \cdots, X_n)) \geq 0$, for any real coordinatewise nondecreasing functions f and g on $\mathbb{R}^n$ such that this covariance exists. An infinite family is associated if every subfamily is associated. A finite family $\{X_1, \cdots, X_n\}$ of random variables is said to be negatively associated(NA) if for any disjoint subsets $A, B \subset \{1, 2, \cdot, n\}$ and any nondecreasing functions f on R^A and g on R^B, $Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$ where this covariance exists. An infinite family is NA if every subfamily is NA(see Joag-Dev and Proschan(1983)).

Chandra and Gohsal(1996) also introduced the following dependence notion which allows both positive and negative correlations. A sequence $\{X_n, n \geq 1\}$ of random variables is called asymptotically almost negatively associated(AANA) if there exists a nonnegative sequence $q(m) \to 0$ such that

$$
\begin{aligned}
&Cov(f(X_m), g(X_{m+1}, \cdots, X_{m+k})) \\
&\leq q(m)(Var(f(X_m)), Var(g(X_{m+1} \cdots, X_{m+k})))^{\frac{1}{2}}
\end{aligned}
$$

for all $n, k \geq 1$ and all cordinatewise nondecreasing functions f and g whenever the right-hand side is finite.

It is interesting to unify as well as weaken the concept of dependence. Chandra and Ghosal(1996) introduced the dependence which unifies, to some extent, the notion of mixing-type sequences and that of negatively dependent sequences as follows: A sequence $\{X_n, n \geq 1\}$ of dependent random variables is said to be asymptotically quadrant subindependent(AQSI) if there exists a nonnegative sequence $\{q(m)\}$ such that for all $i \neq j$,

(1.1)
$$P(X_i > s, X_j > t) - P(X_i > s)P(X_j > t) \leq q(|i-j|)\alpha_{ij}(s,t),\ \ s,t > 0,$$

(1.2)
$$P(X_i < s, X_j < t) - P(X_i < s)P(X_j < t) \leq q(|i-j|)\beta_{ij}(s,t),\ \ s,t < 0,$$

where $q(m) \rightarrow 0$ and $\alpha_{ij}(s,t) \geq 0,\ \beta_{ij}(s,t) \geq 0$.

Note that pairwise negative quadrant dependent and pairwise m-dependent random variables are a special cases of AQSI random variables(see Birkel(1992)).

In this paper we consider some notions of AQSI random variables and investigate Hàjeck-Rènyi type inequality, strong law of large number and integrability of supremum for AQSI random variables which have not been established previously in the literature.

2. Preliminaries

The following lemma is an extension of the well-known Rademacher-Mensov inequality.

Lemma 2.1 (Chandra, Ghosal(1996)). *Let $\{X_n,\ n \geq 1\}$ be a sequence of square-integrable dependent random variables with $EX_n = 0, n \geq 1$. Assume that there exists a sequence $\{a_n^2, n \geq 1\}$ of real numbers such that*

$$E(\sum_{i=m+1}^{m+p} X_i)^2 \leq \sum_{i=m+1}^{m+p} a_i^2 \tag{2.1}$$

for all $m, p \geq 1$ and $m + p \leq n$. Then, we have

$$E(\max_{1 \leq k \leq n}(\sum_{i=1}^{k} X_i)^2) \leq ((\log n/\log 3) + 2)^2 \sum_{i=1}^{n} a_i^2. \tag{2.2}$$

Proof. The proof is found in Theorem 10 of Chandra and Ghosal(1993) □

From Lemma 2.1 we obtain the following Hàjeck-Rènyi type inequality for square-integrable dependent random variables with mean zeros.

Lemma 2.2. *Let $\{X_n,\ n \geq 1\}$ be a sequence of square-integrable dependent random variables with mean zeros and let $\{b_n,\ n \geq 1\}$ be a* *sequence of nondecreasing positive numbers. Assume that there exist* $a_1^2, \cdots, a_n^2$ *satisfying*

$$E(\sum_{i=m+1}^{m+p} \frac{X_i}{b_i})^2 \leq \sum_{i=m+1}^{m+p} \frac{a_i^2}{b_i^2} \tag{2.3}$$

for all $m, p \geq 1,\ m+p \leq n$. *Then, for any* $\epsilon > 0$

$$E(\max_{1\leq k\leq n}(\sum_{i=1}^{k} \frac{X_i}{b_i})^2) \leq ((\log n/\log 3)+2)^2 \sum_{i=1}^{n} \frac{a_i^2}{b_i^2}, \tag{2.4}$$

where $S_n = X_1 + \cdots + X_n$.

LEMMA 2.3. *Let* X *and* Y *be random variables with finite second moments. Then, for any real numbers* x *and* y

(2.5)

$$Cov(X,Y) = \int_{-\infty}^{\infty}\{P(X>x, Y>y) - P(X>x)P(Y>y)\}dxdy.$$

Proof. See the proof of Lemma 2 in Lehmann(1966). □

LEMMA 2.4. *Let* $\{b_n, n \geq 1\}$ *be a sequence of nondecreasing positive numbers. Let* $\{X_n, n \geq 1\}$ *be a sequence of centered square-integrable AQSI random variables with* $\sum_{m=1}^{\infty} q(m) < \infty$. *If for all* $i < j$

$$\int_0^{\infty}\int_0^{\infty} \alpha_{ij}(x,y)dxdy \leq D(\frac{1}{b_i^2} + \frac{EX_i^2}{b_i^2} + \frac{EX_j^2}{b_j^2}) \tag{2.6}$$

and

$$\int_0^{\infty}\int_0^{\infty} \beta_{ij}(x,y)dxdy \leq D(\frac{1}{b_i^2} + \frac{EX_i^2}{b_i^2} + \frac{EX_j^2}{b_j^2}), \tag{2.7}$$

then,

$$E(\sum_{i=1}^{n} \frac{X_i}{b_i})^2 \leq C\sum_{i=1}^{n}(\frac{1+EX_i^2}{b_i^2}). \tag{2.8}$$

Proof. Since $\{X_n/b_n\}$ is a sequence of square-integrable AQSI random variables $\{X_n^+/b_n\}$ and $\{X_n^-/b_n\}$ are also square-integrable AQSI sequences, where X_n^+ means $\max\{X_n, 0\}$ and X_n^- means $\min\{X_n, 0\}$. Now by (1.1), (1.2), (2.5), (2.6) and (2.7) we have for $i < j$

$$Cov(\frac{X_i^+}{b_i}, \frac{X_j^+}{b_j}) \leq Dq(|i-j|)(\frac{1}{b_i^2} + \frac{EX_i^2}{b_i^2} + \frac{EX_j^2}{b_j^2}).$$

Hence,

$$Var(\sum_{i=1}^{n} \frac{X_i^+}{b_i}) \leq C \sum_{i=1}^{n} (\frac{1 + EX_i^2}{b_i^2}) \text{ for all } n$$

since $\sum_{m=1}^{\infty} q(m) < \infty$ and $EX_i^2 < \infty$ for all $i \geq 1$.

Similarly, by (1.1), (1.2), (2.5), (2.6) and (2.7)

$$Var(\sum_{i=1}^{n} \frac{X_i^-}{b_i}) \leq C \sum_{i=1}^{n} (\frac{1 + EX_i^2}{b_i^2}) \text{ for all } n.$$

Thus

$$\begin{aligned} E(\sum_{i=1}^{n} \frac{X_i}{b_i})^2 &= Var(\sum_{i=1}^{n} \frac{X_i}{b_i}) \\ &\leq 2Var(\sum_{i=1}^{n} \frac{X_i^+}{b_i}) + 2Var(\sum_{i=1}^{n} \frac{X_i^-}{b_i}) \\ &\leq C \sum_{i=1}^{n} (\frac{1 + EX_i^2}{b_i^2}) \text{ for all } n. \end{aligned}$$

□

3. The Hàjeck-Rènyi inequality for AQSI sequence

From Lemmas 2.2 and 2.4 we get the Hàjeck-Rènyi type inequality for AQSI random variables.

THEOREM 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of centered square integrable AQSI random variables with and $\sum_{m=1}^{\infty} q(m) < \infty$ and $\{b_n, n \geq 1\}$ a sequence of nondecreasing positive numbers. Assume that for all $i < j$ (2.6) and (2.7) hold, then, for any $\epsilon > 0$*

$$P(\max_{1 \leq k \leq n} \frac{|S_k|}{b_k} \geq \epsilon) \leq C((\log n / \log 3) + 2)^2 \sum_{k=1}^{n} \frac{1 + EX_k^2}{b_k^2}. \tag{3.1}$$

Proof. In (2.8) of Lemma 2.4 put $a_i^2 = 1 + EX_i^2$. Then by Lemma 2.2 we have

$$E(\max_{1 \leq k \leq n} (\sum_{i=1}^{k} \frac{X_i}{b_i})^2) \leq C((\log n / \log 3) + 2)^2 \sum_{i=1}^{n} \frac{1 + EX_i^2}{b_i^2}. \tag{3.2}$$

Next, it is obvious that

$$\begin{aligned}\{\max_{1\le k\le n}|\frac{S_k}{b_k}|\ge\epsilon\} &\subseteq \{\max_{1\le k\le n}\max_{1\le i\le k}|\sum_{i\le j\le k}\frac{X_j}{b_j}|\ge\epsilon\}\\ &= \{\max_{1\le i\le k\le n}|\sum_{j\le k}\frac{X_j}{b_j}-\sum_{j<i}\frac{X_j}{b_j}|\ge\epsilon\}\\ &\subseteq \{\max_{1\le i\le n}|\sum_{j=1}^{i}\frac{X_j}{b_j}|\ge\frac{\epsilon}{2}\},\end{aligned}$$

which yields

$$\begin{aligned}(3.3)\qquad P(\max_{1\le k\le n}\frac{|S_k|}{b_k}\ge\epsilon) &\le P(\max_{1\le k\le n}|\sum_{j=1}^{k}\frac{X_j}{b_j}|\ge\frac{\epsilon}{2})\\ \le\ P\{\max_{1\le k\le n}(\sum_{j=1}^{k}\frac{X_j}{b_j})\ge\frac{\epsilon}{2}\} &+ P\{\max_{1\le k\le n}(-\sum_{j=1}^{k}\frac{X_j}{b_j})\ge\frac{\epsilon}{2}\}.\end{aligned}$$

(See the proof of Theorem 2.1 in Liu et al.(1999), for more details.) Hence (3.2), (3.3) and Markov's inequality the result (3.1) follows. □

From Theorem 3.1 we can get the following more generalized Hàjeck-Rènyi inequality.

THEOREM 3.2. *Under conditions of Theorem 3.1, for any $\epsilon > 0$ and any positive integer $m < n$ we have*

$$\begin{aligned}(3.4)\qquad P(\max_{m\le k\le n}\frac{|S_k|}{b_k}\ge\epsilon) &\le C\{((\log m/\log 3)+2)^2(\sum_{k=1}^{m}\frac{1+EX_k^2}{b_m^2})\\ &+((\log n/\log 3)+2)^2(\sum_{k=m+1}^{n}\frac{1+EX_k^2}{b_k^2})\}.\end{aligned}$$

EXAMPLE 3.3. *Let $\{b_n, n\ge 1\}$ be a sequence of nondecreasing positive numbers and let $\{X_n,\ n\ge 1\}$ be a sequence of square integrable pairwise NQD random variables with mean zeros. Then, for any $\epsilon > 0$ and for any positive integer $m < n$,*

$$\begin{aligned}P(\max_{m\le k\le n}|\frac{1}{b_k}S_k|\ge\epsilon) &\le C\{((\log m/\log 3)+2)^2\sum_{k=1}^{m}\frac{EX_k^2}{b_m^2}\\ &+((\log n/\log 3)+2)^2\sum_{k=m+1}^{n}\frac{EX_k^2}{b_k^2}\}.\end{aligned}$$

4. Strong law of large numbers and integrability of supremum for AQSI sequence

THEOREM 4.1. *Let $\{b_n, n \geq 1\}$ be a sequence of nondecreasing unbounded positive numbers. Let $\{X_n, n \geq 1\}$ be a sequence of centered square-integrable AQSI random variables with $\sum_{m=1}^{\infty} q(m) < \infty$. Suppose that (2.6), (2.7) and*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(1+EX_k^2)(\log n)^2}{b_k^2} < \infty \tag{4.1}$$

hold. Then

$$\frac{S_n}{b_n} \to 0 \text{ a.s. as } n \to \infty. \tag{4.2}$$

Proof. By Theorem 3.2 we have

$$\begin{aligned} P(\max_{m\leq k\leq n} \frac{|S_k|}{b_k} \geq \epsilon) &\leq C(((\log m/\log 3)+2)^2(\sum_{k=1}^{m} \frac{1+EX_k^2}{b_m^2}) \\ &+((\log n/\log 3)+2)^2(\sum_{k=m+1}^{n} \frac{1+EX_k^2}{b_k^2})). \end{aligned} \tag{4.3}$$

But

$$\begin{aligned} & P(\sup_{k\geq m} |\frac{1}{b_k}\sum_{i=1}^{k} X_i| \geq \epsilon) \\ = & \lim_{m\to\infty} P(\max_{m\leq k\leq n} |\frac{1}{b_k}\sum_{i=1}^{k} X_i| \geq \epsilon) \\ \leq & C \lim_{m\to\infty} \{((\log m/\log 3)+2)^2 \sum_{k=1}^{m} \frac{1+EX_k^2}{b_m^2} \\ & +((\log n/\log 3)+2)^2 \sum_{k=m+1}^{n} \frac{1+EX_k^2}{b_k^2}\}. \end{aligned} \tag{4.4}$$

By the Kronecker lemma and (4.1) we get

$$\sum_{k=1}^{m}((\log m/\log 3)+2)^2 \frac{1+EX_k^2}{b_m^2} \to 0 \text{ as } m \to \infty. \tag{4.5}$$

Hence, by combining (4.1), (4.4) and (4.5) we obtain

$$\lim_{n\to\infty} P(\sup_{k\ge n} \frac{1}{b_k}|\sum_{i=1}^{k} X_i| \ge \epsilon) = 0.$$

So the proof is complete. □

COROLLARY 4.2. *Let $\{b_n, n \ge 1\}$ be a sequence of nondecreasing unbounded numbers such that*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(\log n)^2}{b_k^2} < \infty$$

and $\{X_n, n \ge 1\}$ be a sequence of centered AQSI random variables with $\sup_{k\ge1} EX_k^2 < \infty$ and $\sum_{m=1}^{\infty} q(m) < \infty$. Then, (2.6) and (2.7) imply (4.2).

Next we consider the integrability of supremum for AQSI random variables.

THEOREM 4.3. *Let $\{b_n, n \ge 1\}$ be a sequence of nondecreasing positive numbers. Let $\{X_n, n \ge 1\}$ be a sequence of centered square-integrable AQSI random variables with $\sum_{m=1}^{\infty} q(m) < \infty$ and satisfying (2.6) and (2.7).*

Suppose that (4.1) holds. Then, for $0 < r < 2$ we have

$$E\sup_n (\frac{|S_n|}{b_n})^r < \infty. \tag{4.6}$$

Proof.

$$E\sup_n (\frac{|S_n|}{b_n})^r < \infty \Leftrightarrow \int_1^{\infty} P(\sup_n \frac{|S_n|}{b_n} > t^{\frac{1}{r}})dt < \infty.$$

By Theorem 3.1, we get

$$\begin{aligned} & \int_1^{\infty} P(\sup_n |\frac{|S_n|}{b_n} > t^{\frac{1}{r}})dt \\ \le\ & C\int_1^{\infty} t^{-2/r}dt \lim_{n\to\infty}((\log n/\log 3)+2)^2 \sum_{k=1}^{n} \frac{1+EX_k^2}{b_k^2} \\ =\ & C\lim_{n\to\infty}(\log n/\log 3+2)^2 \sum_{k=1}^{n} \frac{1+EX_k^2}{b_k^2} \int_1^{\infty} t^{-\frac{2}{r}}dt \\ <\ & \infty. \end{aligned}$$

□

REMARK 4.4. The results in Section 4 are applied to pairwise negative quadrant dependent and mixing type random variables by the similar method.

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