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# $h-$STABILITY OF THE NONLINEAR PERTURBED DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we investigate $h-$stability of the nonlinear perturbed differential systems.

## 1. Introduction

As is traditional in a pertubation theory of nonlinear differential equations, the behavior of solutions of a perturbed equation is determined in terms of the behavior of solutions of an unperturbed equation. There are three useful methods for studying the qualitative behavior of the solutions of perturbed nonlinear system of differential equations: the method of variation of constants formula, the second method of Lyapunov and the use of inequalities.

The notion of $h$-stability (hS) was introduced by Pinto [11, 12] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called $h$-systems.

Choi and Ryu [3] studied the important properties about hS for the various differential systems. Recently, Choi and Koo [4] ,and Goo [7] obtained results for hS of nonlinear differential systems via $t_\infty$-similarity.

We investigated hS for the nonlinear Volterra integro-differential system [9] and for the linear perturbed Volterra integro-differential systems [8]. In this paper, we investigate $h-$stability of the nonlinear perturbed differential systems.

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## 2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean $n$-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^n \times \mathbb{R}^n$ and $f(t, 0) = 0$. For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. For an $n \times n$ matrix $A$, define the norm $|A|$ of $A$ by $|A| = \sup_{|x| \leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $J = [t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \ v(t_0) = v_0 \tag{2.2}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \ z(t_0) = z_0. \tag{2.3}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.2).

We recall some notions of $h$-stability [12] and the notion of $t_\infty$-similarity [5].

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called
(hS) *$h-$stable* if there exist $c \geq 1$, $\delta > 0$, and a positive bounded continuous function $h$ on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\, |x_0|\, h(t)\, h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \delta$,
(hSV) *$h-$stable in variation* if (2.3) (or $z = 0$ of (2.3)) is $h$-stable.

Let $M$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+ = [0, \infty)$ and $N$ be the subset of $M$ consisting of those nonsingular matrices $S(t)$ that are of class $C^1$ with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of $t_\infty$-similarity in $M$ was introduced by Conti [5]. Consider two differential systems $x' = A(t)x(t)$ and $y' = B(t)y(t), \ t \in \mathbb{R}^+$.

DEFINITION 2.2. A matrix $A(t) \in M$ is $t_\infty$*-similar* to a matrix $B(t) \in M$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{2.4}$$

for some $S(t) \in N$.

The notion of $t_\infty$-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [5, 9].

In this paper, we investigate $h$-stability of the nonlinear perturbed differential systems.

We give some related properties that we need in the sequal.

LEMMA 2.3. [13] *The linear system*

$$x' = A(t)x, \ x(t_0) = x_0, \tag{2.5}$$

*where $A(t)$ is an $n \times n$ continuous matrix, is hS if and only if there exist $c \geq 1$ and a positive bounded continuous function $h$ defined on $\mathbb{R}^+$ such that*

$$|\phi(t, t_0, x_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{2.6}$$

*for $t \geq t_0 \geq 0$, where $\phi(t, t_0, x_0)$ is a fundamental matrix of (2.5).*

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \ y(t_0) = y_0, \tag{2.7}$$

where $g \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.7) passing through the point $(t_0, y_0)$ in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 2.4. *If $y_0 \in \mathbb{R}^n$, for all $t$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

THEOREM 2.5. [3] *If the zero solution of (2.1) is hS, then the zero solution of (2.2) is hS.*

THEOREM 2.6. [4] *Suppose that $f_x(t,0)$ is $t_\infty$-similar to $f_x(t,x(t,t_0,x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (2.2) is hS, then the solution $z = 0$ of (2.3)is hS.*

THEOREM 2.7. [11] *Let $f \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, and $f_x = \partial f/\partial x$ exist and be continuous on $\mathbb{R}^+ \times \mathbb{R}^n$. Assume that $x(t,t_0,x_0)$ and $x(t,t_0,y_0)$ are the solutions of (2.1) through $(t_0,x_0)$ and $(t_0,y_0)$, respectively, existing for $t \geq t_0$, such that $x_0$, $y_0$ belong to a convex subset of $\mathbb{R}^n$. Then, for $t \geq t_0$,*

$$x(t,t_0,x_0) - x(t,t_0,y_0) = \left[\int_0^1 \Phi(t,t_0,sx_0 + (1-s)y_0)ds)\right](x_0 - y_0).$$

The following theorem is a modification of Theorem 3.6 in [3].

THEOREM 2.8. [3] *Suppose that the solution $x = 0$ of (2.1) is hS with a nondecreasing function $h$ and the perturbed term $g$ in (2.7) satisfies*

$$|\Phi(t,s,z)\,k(t,z)| \leq \gamma(s)|z|, \quad t \geq t_0 \geq 0,$$

*where $\gamma \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\int_{t_0}^\infty \gamma(s)ds < \infty$. Then $y = 0$ of (2.7) is hS.*

The following comparison result is well-known.

LEMMA 2.9. [11] *Let the following condition hold for functions $u(t), v(t) \in C[[t_0,\infty), \mathbb{R}^+)$ and $k(t,u) \in C[[t_0,\infty) \times \mathbb{R}^+, \mathbb{R}^+)$:*

$$u(t) - \int_{t_0}^t k(s,u(s))ds \leq v(t) - \int_{t_0}^t k(s,v(s))ds,$$

*$t \geq t_0$ and $k(s,u)$ is strictly in $u$ for each fixed $s \geq 0$. If $u(t_0) < v(t_0)$, then $u(t) < v(t)$, $t \geq t_0 \geq 0$.*

## 3. Main results

In this section, we investigate hS for the nonlinear perturbed differential systems.

THEOREM 3.1. *If the solution $z = 0$ of (2.3) is hS, then the solution $v = 0$ of (2.2) is hS.*

*Proof.* Suppose the solution $z = 0$ of (2.3) is hS. Let $x(t) = x(t,t_0,x_0)$ be any solution of (2.1). Then by Theorem 2.7, we have

$$x(t,t_0,x_0) = \left[\int_0^1 \Phi(t,t_0,sx_0)ds\right]x_0.$$

By Lemma 2.3, since the solution $z = 0$ of (2.3) is hS, there exist $c \geq 1$ and a positive bounded continuous function $h$ on $\mathbb{R}^+$ such that

$$|\Phi(t, t_0, x_0)| \leq c\, h(t)\, h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$, where $\Phi(t, t_0, x_0)$ is a fundamental matrix of (2.3). From (2.6), we have

$$|x(t, t_0, x_0)| \leq \int_0^1 |\Phi(t, t_0, sx_0)|\, ds\, |x_0| \leq c\, |x_0|\, h(t)\, h(t_0)^{-1}.$$

This implies that the zero solution of (2.1) is hS. Therefore, by Theorem 2.5, the solution $v = 0$ of (2.2) is hS and so the proof is complete. □

Corollary 3.2. *Suppose that the solution $z = 0$ of (2.3) is hS with a nondecreasing function $h$, and for all $t \geq t_0 \geq 0$,*

$$|\Phi(t, s, z)\, g(t, z)| \leq \gamma(s)|z|,$$

*where $\gamma \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\int_{t_0}^{\infty} \gamma(s)ds < \infty$. Then, $y = 0$ of (2.7) is hS.*

*Proof.* It follows from Theorem 3.1 that the solution $v = 0$ of (2.2) is hS. In the proof of Theorem 3.1, the solution $x = 0$ of (2.1) is hS. Hence, by Theorem 2.8, the solution $y = 0$ of (2.7) is hS. This completes the proof. □

We also examine the properties of hS for the perturbed system of (2.7)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds,\ y(t_0) = y_0, \tag{3.1}$$

where $g \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$ and $g(t, 0) = 0$.

Theorem 3.3. *Suppose that $f_x(t, 0)$ is $t_\infty$-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (2.1) is hS, and $g$ in (3.1) satisfies*

$$\left| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \right| \leq \gamma(s)|y(s)|, \quad t \geq t_0 \geq 0,$$

*where $\gamma \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\int_{t_0}^{\infty} \gamma(s)ds < \infty$. Then, the solution $y = 0$ of (3.1) is hS.*

*Proof.* Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$. By Theorem 2.5, since the solution $x = 0$ of (2.1) is hS, the solution $v = 0$ of (2.2) is hS.

Therefore, by Theorem 2.6, the solution $z = 0$ of (2.3) is hS. By Lemma 2.4, we have

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \right| ds \\ &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\,h(s)^{-1}\gamma(s)|y(s)|ds \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Gronwall's inequality, we obtain

$$\begin{aligned} |y(t)| &\le c_1|y_0|h(t)\,h(t_0)^{-1}e^{c_2\int_{t_0}^{t}\gamma(s)ds} \\ &\le c|y_0|h(t)\,h(t_0)^{-1}, \quad c = c_1 e^{c_2\int_{t_0}^{\infty}\gamma(s)ds} \end{aligned}$$

It follows that $y = 0$ of (3.1) is hS. Hence, the proof is complete. □

Theorem 3.4. *For the system (3.1), suppose that*

$$\left| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \right| \le r(s, |y|),$$

*where $r \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+]$ is strictly increasing in $u$ for each fixed $t \ge t_0 \ge 0$ with $r(t,0) = 0$. Assume also that $x = 0$ of (2.1) is hSV with the nonincreasing function $h$. Consider the scalar differential equation*

$$u' = cr(t,u), u(t_0) = u_0 = c|y_0|. \tag{3.2}$$

*If $u = 0$ of (3.2) is hS, then $y = 0$ of (3.1) is also hS whenever $u_0 = c|y_0|$.*

*Proof.* Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$. By Lemma 2.4, we have

$$|y(t)| \le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \right| ds,$$

where $\Phi(t, s, y(s))$ is the fundamental matrix of (2.3). Then, by assumptions, we obtain

$$\begin{aligned} |y(t)| &\le c|y_0|h(t)\,h(t_0)^{-1} + c\int_{t_0}^{t} h(t)\,h(s)^{-1} \left| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \right| ds \\ &\le c|y_0| + c\int_{t_0}^{t} r(s, |y(s)|)ds \end{aligned}$$

since $h(t)$ is nonincreasing. Thus we have

$$|y(t)| - c\int_{t_0}^{t} r(s, |y(s)|)ds \le c|y_0| = u_0 = u(t) - c\int_{t_0}^{t} r(s, u(s))ds.$$

By Lemma 2.9, we get $|y(t)| < u(t)$ for all $t \geq t_0 \geq 0$. In view of assumption, since $u = 0$ of (3.2) is hS,

$$|y(t)| < u(t) \leq c_1|u_0|h(t)\,h(t_0)^{-1} \\ = c_1c|y_0|h(t)\,h(t_0)^{-1} = M|y_0|h(t)h(t_0)^{-1}, \quad M = c_1c > 1.$$

This completes the proof. □

REMARK 3.5. In the linear case, we can obtain that if the zero solution $x = 0$ of (2.5) is hS, then the perturbed system

$$y' = A(t)y + \int_{t_0}^{t} g(s, y(s))ds, \quad y(t_0) = y_0,$$

is also hS under the same hypotheses in Theorem 3.3 except the condition of $t_\infty$-similarity.

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