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A GENERALIZATION OF STURM'S BOUND TO A GROUP $\Gamma_0^+(p)$

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Abstract. Let p be a prime. We generalized Sturm's bound to a group $\Gamma_0^+(p)$ of which the genus is zero

1. Introduction

For any positive integer N, let

$$\Gamma_0(N) := \{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) | c \equiv 0 \pmod N\}$$

and

$$\Gamma(N) := \{\gamma \in SL_2(\mathbb{Z}) | \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod N\}.$$

We call a subgroup of $SL_2(\mathbb{Z})$ containing $\Gamma(N)$ for some N a congruence subgroup of $SL_2(\mathbb{Z})$. Let $\Gamma_0^+(p)$ be the group generated by the congruence group $\Gamma_0(p)$ and a Fricke involution $W_p := \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$. Let F be a fixed number field, R_F be the ring of integers in F and λ be a prime ideal of R_F. Sturm [3] gave a criterion for deciding when two modular forms with algebraic integer coefficients for the congruence subgroups of $SL_2(\mathbb{Z})$ are congruent modulo a prime λ. In this paper we generalize Sturm's result to the group $\Gamma_0^+(p)$ when p is a prime and the genus of $\Gamma_0^+(p)$ is zero.

Definition 1.1. Let $f := \sum_{n=1}^{\infty} c(n)q^n$ be a formal sum with $c(n) \in R_F$. Then

$$\mathrm{ord}_\lambda f := \inf\{n \in \mathbb{N} \cup \{0\} | \lambda \nmid c(n)\},$$

with the convention $\mathrm{ord}_\lambda f = \infty$ if $\lambda | c(n)$ for all n.

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Let $M_k(\Gamma)$ be the space of modular forms of integral weight k for Γ. We start with stating Sturm's result.

THEOREM 1.2. ([3] Sturm) *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. Let $f, g \in M_k(\Gamma)$ with Fourier coefficients in R_F. Then we have that if*

$$ord_\lambda(f-g) > \frac{k[SL_2(\mathbb{Z}):\Gamma]}{12},$$

then

$$ord_\lambda(f-g) = \infty.$$

we now generalize Sturm's result to the group $\Gamma_0^+(p)$ when p is a prime and the genus of $\Gamma_0^+(p)$ is zero. Our main result is as follows:

THEOREM 1.3. *Let p be a prime such that the genus of $\Gamma_0^+(p)$ is equal to zero. Let $f \in M_k(\Gamma_0^+(p))$ have the Fourier expansion $f(z) = \sum_{n=0}^{\infty} c(n)q^n$ at the cusp ∞ such that $c(n) \in R_F$ for all $n \geq 0$. Then we have that if*

$$ord_\lambda f > \frac{p+1}{24}k,$$

then

$$ord_\lambda f = \infty.$$

2. Proof of Theorem 1.3

Let

$$\delta = \begin{cases} 8, & \text{if } p = 2 \\ 12, & \text{if } p = 3 \\ 12, & \text{if } p \equiv 1 \pmod{12} \\ 4, & \text{if } p \equiv 5 \pmod{12} \\ 12, & \text{if } p \equiv 7 \pmod{12} \\ 4, & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

By using [1, Theorem 3.6] we can check that $\Delta_p^+(z) := (\eta(z)\eta(pz))^\delta$ is a cusp form for the group $\Gamma_0^+(p)$. Here we note that Δ_p^+ has the following a Fourier expansion at ∞ of the following form:

$$\Delta_p^+(z) = q^{\frac{p+1}{24}\delta} + O(q^{\frac{p+1}{24}\delta+1}).$$

Let j_p^+ be a Hauptmodul for $\Gamma_0^+(p)$ with integral coefficients in its Fourier expansion at the cusp ∞. We are ready to prove Theorem 1.3.

Suppose that $\mathrm{ord}_\lambda f > \frac{p+1}{24}k$. Then $\mathrm{ord}_\lambda f^\delta > \frac{p+1}{24}k\delta$. Since

$$(\Delta_p^+(z))^k = q^{\frac{p+1}{24}k\delta} + O(q^{\frac{p+1}{24}k\delta+1}),$$

we have that

$$f(z)^\delta(\Delta_p^+(z))^{-k} = \sum_{n=-(p+1)k\delta/24}^{\infty} d(n)q^n$$

for some $d(n) \in R_F$ and $\lambda | d(n)$ if $n \leq 0$.

On the other hand, $f^\delta(\Delta_p^+)^{-k}$ is a weakly holomorphic modular function on $\Gamma_0^+(p)$ and hence it is a polynomial in j_p^+ with coefficients in R_F. For each positive integer m, let $j_{p,m}^+$ be a unique weakly holomorphic modular function for $\Gamma_0^+(p)$ with the following Fourier expansion:

$$j_{p,m}^+(z) = \frac{1}{q^m} + O(q).$$

Then one can show that $j_{p,m}^+$ is a polynomial in j_p^+ with integer coefficients. Let $j_{p,0}^+ = 1$. We obtain that

$$f(z)^\delta(\Delta_p^+(z))^{-k} = \sum_{n=0}^{(p+1)k\delta/24} d(-n)j_{p,n}^+.$$

and hence

$$f^\delta(\Delta_p^+)^{-k} \in \lambda R_F[j_p^+]$$

which implies

$$f^\delta \in \lambda R_F[j_p^+](\Delta_p^+)^k$$

Consequently we come up with $\mathrm{ord}_\lambda f^\delta = \infty$ and we see that $\mathrm{ord}_\lambda f = \infty$.

REMARK 2.1. We note that if $f \in M_k(\Gamma_0^+(p))$ then $f \in M_k(\Gamma_0(p))$. By Theorem 1.2, we see that if

$$\mathrm{ord}_\lambda f > \frac{p+1}{12}k,$$

then

$$\mathrm{ord}_\lambda f = \infty.$$

But our theorem says that if

$$\mathrm{ord}_\lambda f > \frac{p+1}{24}k,$$

then

$$\mathrm{ord}_\lambda f = \infty.$$

This means that our bound $(p+1)k/24$ is sharper than Sturm's bound.

References

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