JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **23**, No. 4, December 2010

ALMOST AUTOMORPHIC MILD SOLUTIONS OF SEMILINEAR ABSTRACT DIFFERENTIAL EQUATIONS

Sung Kyu Choi* and Namjip Koo**

Abstract. We investigate the existence and uniqueness of almost automorphic mild solutions of the semilinear abstract differential equations.

1. Introduction

Bochner [1] introduced the concept of almost automorphy as an important generalization of almost periodicity. In the last decade, the almost automorphy and almost periodicity of the solutions to various evolution equations have been widely investigated (see, e.g., [5, 6, 8–10]).

Stepanov [11] introduced a generalization of almost periodic functions - Stepanov almost periodic functions, which are not necessarily continuous. Recently, N'Guérékata and Pankov [10] studied a generalization of almost automorphic functions - Stepanov almost automorphic functions, which are also not necessarily continuous.

Diagana [4] started the research on Stepanov-like pseudo-almost automorphy. Now, the study of pseudo-almost automorphy, which is a meaningful generalization of almost automorphy as well as pseudo-almost periodicity is very active (see, e.g., [3–6, 9, 10]).

Received September 16, 2010; Revised November 09, 2010; Accepted November 18, 2010.

2010 Mathematics Subject Classification: Primary 34G20, 47D06.

Key words and phrases: almost automorphy, composition theorem, mild solution, semilinear abstract differential equation.

Correspondence should be addressed to Namjip Koo, njkoo@cnu.ac.kr.

This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(NRF-2010-0008835).

In this paper, we investigate the existence and uniqueness of almost automorphic mild solutions to the semilinear abstract differential equation in a Banach space X:

$$x'(t) = Ax(t) + f(t, x(t)),\ t \in \mathbb{R}, \tag{1.1}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$; that is, there exist $M > 0$ and $\omega < 0$ such that

$$||T(t)|| \leq Me^{\omega t},\ t \geq 0 \tag{1.2}$$

and the forcing term $f : \mathbb{R} \times X \to X$ is an almost automorphic function.

Definition 1.1. [8] A continuous function $f : \mathbb{R} \to X$ is called *almost automorphic* if for every sequence (s'_n) in $\mathbb{R}$, there exists a subsequence $(s_n) \subset (s'_n)$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well-defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. Denote by $AA(\mathbb{R}, X)$ the set of all such functions.

If the convergence in Definition 1.1 is uniform on $\mathbb{R}$, the function f is said to be *almost periodic* in Bochner's sense.

A classical example of an almost automorphic function (not almost periodic) is

$$f(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t},\ t \in \mathbb{R}$$

[6].

The following hold true [6]:

(i) For $f \in AA(\mathbb{R}, X)$, the range of f is precompact in X, and so f is bounded.

(ii) If $f, g \in AA(\mathbb{R}, X)$, then $f + g \in AA(\mathbb{R}, X)$.

(iii) If $f_n \in AA(\mathbb{R}, X)$ and $\lim_{n\to\infty} f_n = f$ uniformly on $\mathbb{R}$, then $f \in AA(\mathbb{R}, X)$.

(iv) $AA(\mathbb{R}, X)$ with the sup norm

$$||f|| = \sup_{t\in\mathbb{R}} ||f(t)||$$

becomes a Banach space.

Definition 1.2. [8] A continuous function $f : \mathbb{R} \times X \to X$ is called *almost automorphic in t for each $x \in X$* if for every real sequence (s'_n), there exists a subsequence $(s_n) \subset (s'_n)$ such that

$$g(t, x) := \lim_{n\to\infty} f(t + s_n, x)$$

is well-defined for each $t \in \mathbb{R}, x \in X$ and

$$\lim_{n\to\infty} g(t - s_n, x) = f(t, x)$$

for each $t \in \mathbb{R}, x \in X$. Denote by $AA(\mathbb{R} \times X, X)$ the set of all such functions.

For more details on the basic results for almost automorphic functions, see [8].

2. Main results

We need the following composition theorem for almost automorphic functions, which is an extension of [8, Theorem 2.2.6].

Theorem 2.1. *Assume that $f \in AA(\mathbb{R} \times X, X)$ satisfies*

$$||f(t, x) - f(t, y)|| \leq L(t)||x - y||,\ x, y \in X, t \in \mathbb{R},$$

where $L \in L^1(\mathbb{R}, \mathbb{R})$, and $h \in AA(\mathbb{R}, X)$. Then $F \in AA(\mathbb{R}, X)$, where $F(t) = f(t, h(t)),\ t \in \mathbb{R}$.

Proof. Let (s'_n) be any sequence in $\mathbb{R}$. Then there exists a subsequence $(s_n) \subset (s'_n)$ such that

$$\begin{aligned}
&\lim_{n\to\infty} f(t + s_n, x) = g(t, x),\ t \in \mathbb{R}, x \in X,\\
&\lim_{n\to\infty} h(t + s_n) = \phi(t),\ t \in \mathbb{R},\\
&\lim_{n\to\infty} g(t - s_n, x) = f(t, x),\ t \in \mathbb{R}, x \in X,\\
&\lim_{n\to\infty} \phi(t - s_n) = h(t),\ t \in \mathbb{R}.
\end{aligned}$$

We show that $G(t) = \lim_{n\to\infty} F(t + s_n),\ t \in \mathbb{R}$, is well defined and $\lim_{n\to\infty} G(t - s_n) = F(t), t \in \mathbb{R}$. Consider a function $G : \mathbb{R} \to X$ defined by

$$G(t) = g(t, \phi(t)),\ t \in \mathbb{R}.$$

Then

$$\begin{aligned} ||F(t+s_n)-G(t)|| &= ||f(t+s_n,h(t+s_n))-g(t,\phi(t))|| \\ &\le ||f(t+s_n,h(t+s_n))-f(t+s_n,\phi(t))|| \\ &+ ||f(t+s_n,\phi(t))-g(t,\phi(t))|| \\ &\le L(t+s_n)||h(t+s_n)-\phi(t)|| \\ &+ ||f(t+s_n,\phi(t))-g(t,\phi(t))|. \end{aligned}$$

Thus we obtain that $\lim_{n\to\infty} F(t+s_n)=G(t),\ t\in\mathbb{R}$. Also, $\lim_{n\to\infty} G(t-s_n)=F(t), t\in\mathbb{R}$, follows from the inequality

$$\begin{aligned} ||G(t-s_n)-F(t)|| &= ||g(t-s_n,\phi(t-s_n))-f(t,h(t))|| \\ &\le ||g(t-s_n,\phi(t-s_n))-f(t,\phi(t-s_n))|| \\ &+ ||f(t,\phi(t-s_n))-f(t,h(t))|| \\ &\le ||g(t-s_n,\phi(t-s_n))-f(t,\phi(t-s_n))|| \\ &+ L(t)||\phi(t-s_n)-h(t)||. \end{aligned}$$

Hence $F\in AA(\mathbb{R},X)$. □

We consider the semilinear abstract differential equation in a Banach space $(X,||\cdot||)$:

$$x'(t)=Ax(t)+f(t),\ t\in\mathbb{R}, \tag{2.1}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\ge 0}$; that is, there exist $M>0$ and $\omega<0$ such that

$$||T(t)||\le Me^{\omega t},\ t\ge 0 \tag{2.2}$$

and the forcing term f belongs to $AA(\mathbb{R},X)$. A mild solution of (2.1) is given by

$$x(t)=T(t-a)x(a)+\int_a^t T(t-s)f(s)ds,\ a\in\mathbb{R}, t\ge a, \tag{2.3}$$

[7].

Theorem 2.2. *The mild solution x given by (2.3) belongs to $AA(\mathbb{R},X)$.*

Proof. Let

$$\begin{aligned} u(t) &= \int_{-\infty}^t T(t-s)f(s)ds \\ &= \lim_{r\to-\infty}\int_r^t T(t-s)f(s)ds. \end{aligned}$$

Then $u(t)$ is absolutely convergent since

$$||\int_r^t T(t-s)f(s)ds|| \leq \frac{M}{|\omega|}||f||, \ r < t.$$

We show that $u \in AA(\mathbb{R}, X)$. Let (s'_n) be any sequence in $\mathbb{R}$. Since $f \in AA(\mathbb{R}, X)$, there exists a subsequence $(s_n) \subset (s'_n)$ such that

$$g(t) = \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n) = f(t), \ t \in \mathbb{R}.$$

Note that

$$\begin{aligned} u(t+s_n) &= \int_{-\infty}^{t+s_n} T(t+s_n-s)f(s)ds = \int_{-\infty}^{t} T(t-\sigma)f(\sigma+s_n)d\sigma \\ &= \int_{-\infty}^{t} T(t-\sigma)f_n(\sigma)d\sigma, \end{aligned}$$

where $f_n(\sigma) = f(\sigma+s_n), \ n = 1, 2, \cdots$. Moreover, we have

$$||u(t+s_n)|| \leq \frac{M}{|\omega|}||f||, \ n = 1, 2, \cdots .$$

From the continuity of the C_0-semigroup, we have

$$T(t-\sigma)f_n(\sigma) \to T(t-\sigma)g(\sigma) \text{ as } n \to \infty, \ \sigma \in \mathbb{R}, t \geq \sigma.$$

Put $v(t) = \int_{-\infty}^t T(t-s)g(s)ds$. Then $v(t)$ is also absolutely convergent for all $t \in \mathbb{R}$. Thus

$$\lim_{n\to\infty} u(t+s_n) = v(t), \ t \in \mathbb{R}$$

by the Lebesgue's dominated convergence theorem. Similarly, we get

$$\lim_{n\to\infty} v(t-s_n) = u(t), \ t \in \mathbb{R}.$$

It follows that $u \in AA(\mathbb{R}, X)$.

Now, for all $t \geq a$,

$$\begin{aligned} \int_a^t T(t-s)f(s)ds &= \int_{-\infty}^t T(t-s)f(s)ds - \int_{-\infty}^a T(t-s)f(s)ds \\ &= u(t) - T(t-a)u(a). \end{aligned}$$

Hence

$$u(t) = T(t-a)u(a) + \int_a^t T(t-s)f(s)ds.$$

If we fix $x(a) = u(a)$, then $x(t) = u(t)$. Therefore $x \in AA(\mathbb{R}, X)$. □

Consider the semilinear abstract differential equation

$$x'(t) = Ax(t) + f(t, x(t)),\ t \in \mathbb{R}, \tag{2.4}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$, and $f \in AA(\mathbb{R} \times X, X)$. We obtain the following result as an extension of [9, Theorem 3.2].

Theorem 2.3. *If $f \in AA(\mathbb{R} \times X, X)$ satisfies*

$$||f(t,x) - f(t,y)|| \leq L(t)||x - y||,\ x, y \in X, t \in \mathbb{R},$$

where $L \in L^1(\mathbb{R},\mathbb{R})$ with $||L||_1 < \frac{|\omega|}{M}$, then (2.4) has a unique almost automorphic mild solution.

Proof. Note that a mild solution x of (2.4) is given by

$$x(t) = T(t-a)x(a) + \int_a^t T(t-s)f(s, x(s))ds,\ a \in \mathbb{R}, t \geq a$$

and is continuous [7]. Define the nonlinear operator $F : AA(\mathbb{R}, X) \to AA(\mathbb{R}, X)$ defined by

$$(F\phi)(t) := \int_{-\infty}^t T(t-s)f(s, \phi(s)))ds.$$

Then F is well-defined by Theorem 2.1. It is clear that F is continuous. We show that F is a contraction.

Let $\phi_1, \phi_2 \in AA(\mathbb{R}, X)$. Then

$$\begin{aligned} ||F\phi_1 - F\phi_2|| &= \sup_{t\in\mathbb{R}} || \int_{-\infty}^t T(t-s)[f(s,\phi_1(s)) - f(s,\phi_2(s))]ds|| \\ &\leq \sup_{t\in\mathbb{R}} \int_{-\infty}^t ||T(t-s)||\ ||L||_1||\phi_1(s) - \phi_2(s)||ds \\ &\leq ||L||_1||\phi_1 - \phi_2|| \sup_{t\in\mathbb{R}} \int_{-\infty}^t Me^{\omega(t-s)}ds \\ &= \frac{||L||_1 M}{|\omega|}||\phi_1 - \phi_2||. \end{aligned}$$

Thus F is a contraction and so there exists a unique fixed point $u \in AA(\mathbb{R}, X)$.

Note that

$$\begin{aligned}\int_a^t T(t-s)f(s,u(s))ds &= \int_{-\infty}^t T(t-s)f(s,u(s))ds \\ &- \int_{-\infty}^a T(t-s)f(s,u(s))ds \\ &= u(t) - T(t-a)u(a).\end{aligned}$$

Hence

$$u(t) = T(t-a)u(a) + \int_a^t T(t-s)f(s,u(s))ds, \ a \in \mathbb{R}, t \geq a$$

is a mild solution of (2.4).

Moreover, it follows that $u \in AA(\mathbb{R}, X)$ along the same lines as in Theorem 2.2.

We assume that $x : \mathbb{R} \to X$ is bounded and satisfies the equation $x'(t) = Ax(t), t \in \mathbb{R}$. Then

$$x(t) = T(t-a)x(a) + \int_a^t T(t-s)f(s,x(s))ds.$$

Note that

$$\begin{aligned}||T(t-a)x(a)|| &\leq ||T(t-a)|| \, ||x(a)|| \\ &\leq Me^{\omega(t-a)}||x||.\end{aligned}$$

Thus $\lim_{a\to-\infty} T(t-a)x(a) = 0$. Therefore

$$x(t) = \int_{-\infty}^t T(t-s)f(s,x(s))ds = F(x)(t)$$

as $a \to -\infty$ and this implies that x is the unique fixed point u of F. This completes the proof. □

References

[1] S. Bochner, *A new approach to almost-periodicity*, Proc. Nat. Acad. Sci. USA **48** (1962), 2039-2043.

[2] S. K. Choi and N. Koo, *Almost periodic solutions of nonlinear discrete Volterra equations with unbounded delay*, Adv. Difference Equ. **2008**, Art. ID 692713, 15pp.

[3] C. Cuevas and M. Pinto, *Existence and uniqueness of pseudo almost periodic solutions of semilinear Cauchy problems with nondense domain*, Nonlinear Anal. **45** (2001), 73-83.

[4] T. Diagana, *Existence of pseudo-almost automorphic solutions to some abstract differential equations with S^p-pseudo-almost automorphic coefficients*, Nonlinear Anal. **70** (2009), 3781-3790.

[5] H. Ding, J. Liang and T. Xiao, *Some properties of Stepanov-like almost automorphic functions and applications to abstract evolution equations*, Appl. Anal. **88** (2009), 1079-1091.

[6] H. Ding, J. Liang and T. Xiao, *Almost automorphic solutions to nonautomous semilinear evolution equations in Banach spaces*, Nonlinear Anal. **73** (2010), 1426-1438.

[7] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford University Press, Oxford, 1985.

[8] G. M. N'Guérékata, *Almost Automorphic Functions and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic / Plenum Publishers, New York, 2001.

[9] G. M. N'Guérékata, *Existence and uniqueness of almost automorphic mild solutions to some semilinear abstract differential equations*, Semigroup Forum, **69** (2004), 80-86.

[10] G. M. N'Guérékata and A. Pankov, *Stepanov-like almost automorphic functions and monotone evolution equations*, Nonlinear Anal. **68** (2008), 2658-2667.

[11] W. Stepanov, *Über einige Verallgemeinerungen der fastperiodischen funktionen*, Math. Ann. **95** (1926), 473-498.

*

Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: sgchoi@cnu.ac.kr

**

Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: njkoo@cnu.ac.kr