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NONLINEAR HEAT EQUATIONS WITH TRANSCENDENTAL NONLINEARITY IN BESOV SPACES

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Abstract. The existence of solutions in Besov spaces for nonlinear heat equations having transcendental nonlinearity:

$$\frac{\partial}{\partial t}u - \Delta u = F(u)$$

is investigated. In particular, it is proved the local existence and blow-up phenomena of the solutions in Besov spaces for nonlinear heat equations corresponding to two transcendental nonlinear functions $F(u) \equiv |u|e^{u^2}$ and $F(u) \equiv e^u$ of rapid growth.

1. Introduction

We are concerned with the initial value problem of nonstationary nonlinear heat equations:

$$\frac{\partial}{\partial t}\,u(x,t) - \Delta u(x,t) = F(u(x,t)),$$
$$u(x,0) = u_0(x),$$

where $x \in \mathbb{R}^n$, F is a given nonlinear function and u is unknown. Existence and uniqueness theories of solutions of nonlinear heat equations have been extensively studied by many mathematicians and physicists. Unlike other studies, we focus on nonlinear heat equations with *transcendental* nonlinearities such as:

$$\frac{\partial}{\partial t}\,u - \Delta u = |u|e^{u^2} \tag{1.1}$$

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and

$$\frac{\partial}{\partial t}u - \Delta u = e^u. \tag{1.2}$$

Both nonlinearities in the above problems grow so fast that the solutions may blow up very fast. Even though we present problems with specific nonlinear functions, the nonlinearity $F(u) \equiv |u|e^{u^2}$ in the equation (1.1) speaks for the nonlinear heat equations with transcendental nonlinearities that grow very fast with $F(0) = 0$. On the other hand, the equation (1.2) with the nonlinearity $F(u) \equiv e^u$ represents for the nonlinear heat equations with transcendental nonlinearities with $F(0) \neq 0$. We recognized that nonlinear heat equations with other transcendental nonlinearities could be handled similarly as either (1.1) or (1.2). So, we mainly focus on the two kinds of problems like (1.1) and (1.2) which exemplify the nonlinearities with rapid growth.

The existence theory has been developed in many function spaces. For example, in [5], (nonlinear) heat equations have been studied in the space $C_0(\Omega)$ of continuous functions with compact support. Those equations were investigated on Hölder spaces $C^{m,\alpha}(\Omega)$ in [8] and [11]. F.B. Weissler [19] handled the problems in Lebesgue spaces $L^p(\Omega)$. Moreover, in Lebesgue spaces $L^p(\Omega)$, certain singular phenomena, such as non-uniqueness, have been observed (see [1], [4], and [10]). There are some cases in which the regularizing effect allows one to solve the Cauchy problem of nonlinear heat equations with singular initial data, such as measures (see [3]). In this paper, we treat the problems on Besov spaces $B^s_{p,q}$.

We prove a local in time existence and persistence of Besov space regularity of solutions for the nonlinear heat equations of the type (1.1) and (1.2). The Besov spaces we deal with cover almost all the Besov spaces $B^s_{p,q}$ which have more regularity than the Sobolev borderline spaces - we even include the critical cases when p, $q = 1$ or ∞.

To verify local existence of the solution, we choose to use a *fixed point theorem* rather than a compactness argument([13]) or a standard iterative algorithm ([6]). The essential tools for a priori estimates are Bony's para-product formula and Littlewood-Paley decomposition.

Notation Throughout this paper, C denotes various real positive constants.

2. Preliminaries and a priori estimates

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions. We consider a nonnegative radial function $\chi \in \mathcal{S}(\mathbb{R}^n)$ satisfying supp $\chi \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \frac{5}{6}\}$, and $\chi = 1$ for $|\xi| \leq \frac{3}{5}$. We set $h_j(\xi) := \chi(2^{-j-1}\xi) - \chi(2^{-j}\xi)$, and we notice that $\chi(\xi) + \sum_{j=0}^{\infty} h_j(\xi) = 1$, for $\xi \in \mathbb{R}^n$. Let φ_j and Φ be defined by $\varphi_j := \mathcal{F}^{-1}(h_j)$, $j \geq 0$ and $\Phi := \mathcal{F}^{-1}(\chi)$, where $\mathcal{F}(f)$ denotes the Fourier transform of f on $\mathbb{R}^n$. Note that φ_j is a mollifier of φ_0, that is, $\varphi_j(x) := 2^{jn}\varphi_0(2^j x)$ (or $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$). For $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote $\Delta_j f \equiv h_j(D)f = \varphi_j * f$ if $j \in \mathbb{Z}$, and the partial sums: $S_k f := f - \displaystyle\sum_{j=k+1}^{\infty} \Delta_j f$ for $k \in \mathbb{Z}$. Assume $s \in \mathbb{R}$, and $1 \leq p, q \leq \infty$. The Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are defined by

$$f \in B^s_{p,q}(\mathbb{R}^n) \Leftrightarrow \{\|2^{js}\Delta_j f\|_{L^p}\}_{j\in\mathbb{Z}} \in l^q.$$

We define *Bony's para-product formula* which decomposes the product $f \cdot g$ of two functions f and g into three parts:

$$f \cdot g = T_f g + T_g f + R(f,g),$$

where $T_f g$ represents Bony's *para-product* of f and g defined by $T_f g \equiv \sum_j S_{j-2}f \;\; \Delta_j g = \sum_{i\leq j-2} \Delta_i f \Delta_j g$ and $R(f,g)$ denotes the remainder of the para-product $R(f,g) \equiv \sum_{|i-j|\leq 1} \Delta_i f \Delta_j g$. The following remark exhibits the relationship between the role of the scaling factor 2^{js} and the role of the differentiation index s in $B^s_{p,q}$. The proof of the remark can be found in [6, p. 16].

Remark 2.1 (Bernstein's Lemma). *1. For $f \in \mathcal{S}'(\mathbb{R}^n)$ with*

$$\text{supp}\ \widehat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq r\},$$

there exists a constant $C = C(s)$ such that

$$\|f\|_{L^{p_1}} \leq C r^{n\left(\frac{1}{p}-\frac{1}{p_1}\right)} \|f\|_{L^p}, \quad 1 \leq p \leq p_1 \leq \infty,$$
$$\|D^s f\|_{L^p} \leq C\, r^s \|f\|_{L^p}, \quad 1 \leq p \leq \infty.$$

2. For $f \in L^p(\mathbb{R}^n)$ with $p \in [1,\infty]$ and

$$\text{supp}\ \widehat{f} \subset \left\{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\right\},$$

there exists a constant C such that

$$C^{-1} 2^j \|f\|_{L^p} \leq \|\nabla f\|_{L^p} \leq C\, 2^j \|f\|_{L^p}.$$

Now, we provide with a-priori estimates that are essential for proving the main theorems. The main techniques are Littlewood-Paley decomposition and Bony's para-product formula. First, we begin with Sobolev-type imbedding lemma in Besov spaces.

REMARK 2.2. *Let $1 \le p \le \infty$ and $s > \frac{n}{p}$. We have*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \le C\|u\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Proof. By applying Bernstein's lemma and Hölder's inequality on l^p, we get the estimate

$$\begin{aligned}\|u\|_{L^\infty} &\le C\sum_{j\in\mathbb{Z}} 2^{js}\|\Delta_j u\|_{L^p}\ 2^{\frac{jn}{p}-js} \\ &\le C\left(\sum_{j\in\mathbb{Z}}(2^{js}\|\Delta_j u\|_{L^p})^q\right)^{\frac{1}{q}}\left(\sum_{j\in\mathbb{Z}}(2^{\frac{n}{p}-s})^{\frac{jq}{q-1}}\right)^{\frac{q-1}{q}} = C\|u\|_{B^s_{p,q}(\mathbb{R}^n)}\end{aligned}$$

with the usual modification in the case of $q = \infty$. $\square$

The following product rule claims that most of Besov spaces are Banach algebra if the differential index s is bigger than the dimension of the space.

PROPOSITION 2.3. *Let $s > 0$ and $1 \le p,\ q \le \infty$. For $f, g \in B^s_{p,q}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, we have*

$$\|fg\|_{B^s_{p,q}} \le C\|g\|_{L^\infty}\|f\|_{B^s_{p,q}} + C\|f\|_{L^\infty}\|g\|_{B^s_{p,q}}.$$

Therefore the spaces $B^s_{p,q}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ are Banach algebras, and in particular, for $s > \frac{n}{p}$ and $1 \le p,\ q \le \infty$, the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are Banach algebras.

Proof. We omit the details since the result can be obtained by modifying Proposition 4.1 in [14](page 10). $\square$

The following proposition asserts that the transcendental function of the type ue^{u^2} is locally Lipschitz continuous in Besov spaces.

PROPOSITION 2.4. *Let $s > \frac{n}{p}$, $1 \le p,\ q \le \infty$. For $u, v \in B^s_{p,q}(\mathbb{R}^n)$, there exist constants $L > 0$, $L' > 0$ satisfying*

$$\| |u|e^{u^2} - |v|e^{v^2} \|_{B^s_{p,q}(\mathbb{R}^n)} \le L\|u - v\|_{B^s_{p,q}(\mathbb{R}^n)},$$
$$\|e^u - e^v\|_{B^s_{p,q}(\mathbb{R}^n)} \le L'\|u - v\|_{B^s_{p,q}(\mathbb{R}^n)},$$

respectively. The constants $L(\|u\|_{B^s_{p,q}}, \|v\|_{B^s_{p,q}})$, $L'(\|u\|_{B^s_{p,q}}, \|v\|_{B^s_{p,q}})$ depend only on the Besov norms $\|\cdot\|_{B^s_{p,q}}$ of the given functions u and v.

Proof. We deliver the proof of the first assertion, and then the second one can be proved similarly. We start with the estimate

$$\left\|\sum_{i=0}^{N} u^{N-i}v^{i}\right\|_{B^s_{p,q}(\mathbb{R}^n)} \leq (2C_0)^{N-1}\sum_{i=0}^{N}\|u\|^{N-i}_{B^s_{p,q}(\mathbb{R}^n)}\|v\|^{i}_{B^s_{p,q}(\mathbb{R}^n)},$$

for some positive constant C_0. In fact, by Proposition 2.3, we can easily get that for $i = 1, 2, \cdots, 2N$,

$$\|u^{N-i}v^{i}\|_{B^s_{p,q}} \leq (2C_0)^{N-1}\|u\|^{N-i}_{B^s_{p,q}}\|v\|^{i}_{B^s_{p,q}},$$

for some positive constant C_0. Hence we compute

$$\left\|\sum_{i=0}^{N} u^{N-i}v^{i}\right\|_{B^s_{p,q}} \leq (2C_0)^{N-1}\sum_{i=0}^{N}\|u\|^{N-i}_{B^s_{p,q}}\|v\|^{i}_{B^s_{p,q}}.$$

By the continuity of the Besov norm, we have

$$\||u|e^{u^2} - |v|e^{v^2}\|_{B^s_{p,q}(\mathbb{R}^n)} \leq \sum_{k=0}^{\infty}\frac{1}{k!}\left\||u-v|\left(\sum_{i=0}^{2k}|u|^{2k-i}|v|^{i}\right)\right\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Choose any number $M \geq \max\left\{\|u\|_{B^s_{p,q}}(\mathbb{R}^n), \|v\|_{B^s_{p,q}}(\mathbb{R}^n)\right\}$ and we obtain

$$\begin{aligned}
\||u|e^{u^2} - |v|e^{v^2}\|_{B^s_{p,q}} &\leq \sum_{k=0}^{\infty}\frac{1}{k!}(2C_0)\|u-v\|_{B^s_{p,q}}\left\|\sum_{i=0}^{2k}|u|^{2k-i}|v|^{i}\right\|_{B^s_{p,q}} \\
&\leq \sum_{k=0}^{\infty}\frac{(2C_0)^{2k}}{k!}\sum_{i=0}^{2k}\|u\|^{2k-i}_{B^s_{p,q}}\|v\|^{i}_{B^s_{p,q}}\|u-v\|_{B^s_{p,q}} \\
&\leq \sum_{k=0}^{\infty}\frac{1}{k!}(2C_0M)^{2k}(2k+1)\|u-v\|_{B^s_{p,q}} \\
&\leq (8C_0^2M^2+1)e^{(2C_0M)^2}\|u-v\|_{B^s_{p,q}}.
\end{aligned}$$

Take $L \equiv (8C_0^2M^2+1)e^{(2C_0M)^2}$ and we get the desired estimate:

$$\||u|e^{u^2} - |v|e^{v^2}\|_{B^s_{p,q}(\mathbb{R}^n)} \leq L\|u-v\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

□

The following proposition is *essential* to prove the existence of solutions for nonlinear heat equations on Besov spaces.

PROPOSITION 2.5. *Let $1 \le p,\ q \le \infty, s \ge 0$. Then for any $u \in B^{s+2}_{p,q}(\mathbb{R}^n)$, we have the following estimate:*

$$\|u\|_{B^{s+2}_{p,q}(\mathbb{R}^n)} \le C\|u - \Delta u\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

Proof. We refer [9]. □

REMARK 2.6. *Let $1 \le p, q \le \infty$. The Laplacian $\Delta : B^{s+2}_{p,q} \to B^s_{p,q}$ is a dissipative operator. In fact, in the proof of Proposition 2.5, one has for $\lambda > 0$,*

$$\|\Delta_j u\|_{L^p} \le \|\Delta_j(u - \lambda\Delta u)\|_{L^p}, \qquad j = -1, 0, 1, \cdots .$$

Therefore we have for $\lambda > 0$,

$$\|u\|_{B^s_{p,q}} \le \left(\sum_{j=-1}^{\infty} \|2^{js}\Delta_j(u - \lambda\Delta u)\|^q_{L^p}\right)^{1/q} = \|u - \lambda\Delta u\|_{B^s_{p,q}}.$$

3. Main theorems

In this section we present existence theorems for the two kinds of nonlinear heat equations (1.1) and (1.2) in Besov spaces.

The following proposition presents the linear stationary problem which explains the m-dissipative property of the Laplacian on Besov spaces.

PROPOSITION 3.1. *Let $s \ge 0$, $1 \le p,\ q \le \infty$, and $\lambda > 0$. For any $f \in B^s_{p,q}(\mathbb{R}^n)$, there exists $u \in B^{s+2}_{p,q}(\mathbb{R}^n)$ satisfying the following stationary problem:*

$$\lambda u - \Delta u = f.$$

Proof. It suffices to verify the case when $\lambda = 1$([5], page 19): that is, for $f \in B^s_{p,q}(\mathbb{R}^n)$, we will find $u \in B^{s+2}_{p,q}(\mathbb{R}^n)$ satisfying $u - \Delta u = f$. To accomplish it, we will first choose a possible candidate for u. We start with splitting f into pieces by Littlewood-Paley decomposition

$$f = \sum_{j=-1}^{\infty} \Delta_j f.$$

For each $j = -1, 0, 1, \cdots$, we shall find u_j satisfying

$$u_j - \Delta u_j = \Delta_j f. \tag{3.1}$$

To provide an explicit expression for u_j, we take the Fourier transform on both sides of (3.1) to get

$$\widehat{u_j} = \frac{1}{1+|\xi|^2}\widehat{\Delta_j f}.$$

Then from the well-known fact that

$$\mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^2}\right) = \frac{1}{2^{\frac{n}{2}}}\int_0^\infty \frac{e^{-t-\frac{|x|^2}{4t}}}{t^{\frac{n}{2}}}dt$$

(by the principle of subordination and the Fourier inversion theorem), we have

$$u_j(x) = \frac{1}{(4\pi)^{\frac{n}{2}}}\ \Delta_j f * \int_0^\infty \frac{e^{-t-\frac{|x|^2}{4t}}}{t^{\frac{n}{2}}}dt.$$

Next, we will show that the series $\sum\limits_{j=-1}^{\infty} \Delta_j u$ converges in $B^{s+2}_{p,q}(\mathbb{R}^n)$, and so $u \equiv \sum\limits_{j=-1}^{\infty} \Delta_j u$ is an element of $B^{s+2}_{p,q}(\mathbb{R}^n)$.

Let $F_m \equiv \sum\limits_{j=-1}^{m} \Delta_j f$. Then F_m converges to f in $B^s_{p,q}(\mathbb{R}^n)$. In fact, we note that

$$\begin{aligned}
\|F_m - f\|_{B^s_{p,q}(\mathbb{R}^n)} &= \left(\sum_{k=-1}^{\infty} 2^{ksq}\left\|\Delta_k \sum_{j=m+1}^{\infty}\Delta_j f\right\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} \\
&\le \left(\sum_{k=-1}^{\infty}\sum_{j=m+1}^{\infty} 2^{ksq}\|\Delta_k\Delta_j f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}.
\end{aligned}$$

For a fixed k, consider the supports of the functions, and we see that $\Delta_k\Delta_j f$ survives only when $|k-j|\le 1$. Hence we have the estimate

$$\begin{aligned}
\|F_m - f\|_{B^s_{p,q}(\mathbb{R}^n)} &\le \left(\sum_{j=m+1}^{\infty}\sum_{|k-j|\le 1} 2^{ksq}\|\Delta_k\Delta_j f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} \\
&\le C\left(\sum_{j=m+1}^{\infty} 2^{jsq}\|\Delta_j f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}. \qquad (3.2)
\end{aligned}$$

Since $\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum\limits_{j=-1}^{\infty}\|2^{js}\Delta_j f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} < \infty$, we can make (3.2) as small as we want by choosing m sufficiently large.

Now, let $S_m \equiv \sum_{j=-1}^{m} u_j$. Then we claim that $\{S_m\}$ is a Cauchy sequence in $B^{s+2}_{p,q}(\mathbb{R}^n)$. For any $m, n \in \mathbb{N}$ with $n > m$, Proposition 2.5 leads to

$$\begin{aligned}\|S_m - S_n\|_{B^{s+2}_{p,q}(\mathbb{R}^n)} \le \sum_{j=m+1}^{n} \|u_j\|_{B^{s+2}_{p,q}(\mathbb{R}^n)} &\le C \sum_{j=m+1}^{n} \|u_j - \Delta_j u_j\|_{B^{s}_{p,q}(\mathbb{R}^n)} \\ &\le C \sum_{j=m+1}^{\infty} \|\Delta_j f\|_{B^{s}_{p,q}(\mathbb{R}^n)}. \qquad (3.3)\end{aligned}$$

We can make (3.3) as small as possible with m being sufficiently large. Therefore, we see that $\{S_m\}$ is a Cauchy sequence in $B^{s+2}_{p,q}(\mathbb{R}^n)$. Let u be the limit of S_m in $B^{s+2}_{p,q}(\mathbb{R}^n)$, i.e., $u \equiv \sum_{j=-1}^{\infty} u_j$.

Finally, since $(I–\Delta)u_j = \Delta_j f$, we have $(I–\Delta)\sum_{j=-1}^{m} u_j = \sum_{j=-1}^{m} (I–\Delta)u_j = \sum_{j=-1}^{m} \Delta_j f$. Now we claim that $I - \Delta : B^{s+2}_{p,q}(\mathbb{R}^n) \to B^{s}_{p,q}$ is continuous, so then we have

$$f = \sum_{j=-1}^{\infty} \Delta_j f = \lim_{m\to\infty} (I - \Delta) \sum_{j=-1}^{m} u_j = (I - \Delta) \lim_{m\to\infty} \sum_{j=-1}^{m} u_j = (I - \Delta)u.$$

To verify it, we look at the estimate

$$\begin{aligned}\left\|(I - \Delta) \sum_{j=-1}^{m} u_j - (I - \Delta) \sum_{j=-1}^{\infty} u_j\right\|_{B^{s}_{p,q}(\mathbb{R}^n)} &= \left\| \sum_{j=m+1}^{\infty} \Delta_j f\right\|_{B^{s}_{p,q}(\mathbb{R}^n)} \\ &\le \sum_{j=m+1}^{\infty} \|\Delta_j f\|_{B^{s}_{p,q}(\mathbb{R}^n)}. \qquad (3.4)\end{aligned}$$

The right hand side of (3.4) goes to zero as $m \to \infty$. In all, for all $f \in B^{s}_{p,q}$, there exists $u \in B^{s+2}_{p,q}$ such that $u - \Delta u = f$. This completes the proof. □

The proposition above and Remark 2.6 imply the m-dissipative property of the Laplacian Δ.

Corollary 3.2. *The Laplacian $\Delta : D \to B^{s}_{p,q}$ with $D \equiv B^{s+2}_{p,q}$ is m-dissipative.*

Remark 3.3. *Besides the case of Sobolev spaces, the domain of the unbounded operator Δ is the Besov space $B^{s+2}_{p,q}$ itself, without any constraint on the domain.*

Now, we present our main theorems: the existence theorems of the solutions for the two kinds of nonlinear heat equations (1.1) and (1.2) in Besov spaces.

Theorem 3.4. *Let $1 \leq p,\ q \leq \infty$, $s > \frac{n}{p}$. For any $u_0 \in B^s_{p,q}(\mathbb{R}^n)$, there exists a $T^* \in (0, \infty]$ such that the initial value problem for the nonlinear heat equation (1.1) with initial datum $u(x,0) = u_0(x)$ has a unique solution $u \in C([0,T^*), B^s_{p,q})$.*

Proof. Let $u_0 \in B^s_{p,q}(\mathbb{R}^n)$ and $K = 2\|u_0\|_{B^s_{p,q}(\mathbb{R}^n)}$. Then by Proposition 2.4, there exists a constant $L > 0$ such that

$$\||u|e^{u^2} - |v|e^{v^2}\|_{B^s_{p,q}(\mathbb{R}^n)} \leq L\|u - v\|_{B^s_{p,q}(\mathbb{R}^n)}$$

for any $u, v \in B^s_{p,q}(\mathbb{R}^n)$ satisfying $\|u\|_{B^s_{p,q}} \leq K$ and $\|v\|_{B^s_{p,q}} \leq K$. Set $T \equiv \frac{1}{2L+2}$ and let

$$E \equiv \left\{u \in C\left([0,T], B^s_{p,q}\right) :\ \|u(t)\|_{B^s_{p,q}(\mathbb{R}^n)} \leq K,\ \text{for all } t \in [0,T]\right\},$$

and we equip E with the distance d generated by the norm of the space $C([0,T], B^s_{p,q})$, that is,

$$d(u,v) = \max_{t\in[0,T]} \|u(t) - v(t)\|_{B^s_{p,q}(\mathbb{R}^n)}, \quad u, v \in E.$$

Since $C([0,T], B^s_{p,q})$ is a Banach space, (E,d) is a complete metric space. On the other hand, since $\Delta : D \to B^s_{p,q}$ with $D = B^{s+2}_{p,q}$ is m-dissipative, it produces a semi-group on $B^s_{p,q}(\mathbb{R}^n)$, say $\mathcal{T}$. For all $u \in E$, we define $\Lambda_u \in C([0,T], B^s_{p,q})$ by

$$\Lambda_u(t) \equiv \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)|u(s)|e^{u(s)^2} ds,$$

for all $t \in [0,T]$. We claim that $\Lambda_u \in E$. Indeed, note that for $s \in [0,T]$, we have, by Proposition 2.4,

$$\left\||u(s)|e^{u(s)^2}\right\|_{B^s_{p,q}(\mathbb{R}^n)} \leq L\|u(s)\|_{B^s_{p,q}} \leq L\left(2\|u_0\|_{B^s_{p,q}}\right) = \frac{\|u_0\|_{B^s_{p,q}}}{T}.$$

It follows that

$$\|\Lambda_u(t)\|_{B^s_{p,q}} \leq \left(1 + t\frac{1}{T}\right)\|u_0\|_{B^s_{p,q}} \leq 2\|u_0\|_{B^s_{p,q}} = K.$$

Consequently, we have $\Lambda : E \longrightarrow E$, $\Lambda(u) = \Lambda_u$. Furthermore, for any $u, v \in E$, we get

$$\begin{aligned}\|\Lambda_u(t) - \Lambda_v(t)\|_{B^s_{p,q}} &\le L \int_0^t \|u(s) - v(s)\|_{B^s_{p,q}} ds \\ &\le \frac{L}{2L+2}\, d(u,v) < \frac{1}{2}\, d(u,v).\end{aligned}$$

Therefore, Λ_u is a contraction mapping in E with the Lipschitz constant $\frac{1}{2}$, and so by Banach fixed point theorem, Λ_u has a fixed point $\Lambda_u = u \in E$. This leads to

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)|u(s)|e^{u(s)^2} ds$$

for $0 \le t \le \frac{1}{2L+2} = T$. In all, we obtain a unique solution u in $C([0,T], B^s_{p,q})$.

We may continue to use this argument until the value $\|u(T^*)\|_{B^s_{p,q}}$ blows up, i.e., $\lim_{t\uparrow T^*} \|u(t)\|_{B^s_{p,q}} = \infty$. □

Theorem 3.5. *Let $1 \le q \le \infty$, $s \ge 0$. For any $u_0 \in B^s_{\infty,q}(\mathbb{R}^n)$, there exists a positive constant $T^* \in (0, \infty]$ such that the initial value problem for the nonlinear heat equation (1.2) with initial datum $u(x,0) = u_0(x)$ has a unique solution $u \in C([0,T^*), B^s_{\infty,q})$.*

Proof. The proof is very much the same as that of Theorem 3.4. So we only point out the differences.

Let $u_0 \in B^s_{\infty,q}(\mathbb{R}^n)$ and $K = 2\|u_0\|_{B^s_{\infty,q}}$. Proposition 2.4 provides the existence of a constant $L > 0$ with

$$\|e^u - e^v\|_{B^s_{\infty,q}(\mathbb{R}^n)} \le L\|u - v\|_{B^s_{\infty,q}(\mathbb{R}^n)}$$

for any $u, v \in B^s_{\infty,q}(\mathbb{R}^n)$ satisfying $\|u\|_{B^s_{\infty,q}} \le K$ and $\|v\|_{B^s_{\infty,q}} \le K$. We set

$$T \equiv \frac{1}{2L + c_0} \text{ with } c_0 = \frac{2^{-s}}{\|u_0\|_{B^s_{\infty,q}}}$$

and define the complete metric space E and the distance d by the exactly same way as in the proof of Theorem 3.4. For the same $\mathcal{T}$ and Λ_u with $u \in E$, we need to check that $\Lambda_u(t) \in E$. The reason is as follows: for $s \in [0,T]$, by Proposition 2.4, we have

$$\left\|e^{u(s)}\right\|_{B^s_{\infty,q}(\mathbb{R}^n)} \le L\|u(s)\|_{B^s_{\infty,q}(\mathbb{R}^n)} + \|1\|_{B^s_{\infty,q}(\mathbb{R}^n)}.$$

We observe that for $j = 0, 1, 2, \cdots$,

$$\|\Delta_j 1\|_{L^\infty} = \sqrt[n]{2\pi}\,\|\hat{\phi}_j(0)\|_{L^\infty} = 0.$$

Hence we have that

$$\|1\|_{B^s_{\infty,q}(\mathbb{R}^n)} = (2^{-sq}\|\hat{\Phi}(0)\|^q_{L^\infty(\mathbb{R}^n)})^{\frac{1}{q}} = 2^{-s}.$$

Therefore we obtain

$$\|e^u\|_{B^s_{\infty,q}} \le LK + 2^{-s} \le L(2\|u_0\|_{B^s_{p,q}}) + c_0\|u_0\|_{B^s_{\infty,q}} = \frac{\|u_0\|_{B^s_{p,q}}}{T}.$$

It follows that

$$\|\Lambda_u(t)\|_{B^s_{\infty,q}(\mathbb{R}^n)} \le 2\|u_0\|_{B^s_{\infty,q}(\mathbb{R}^n)} = K.$$

Consequently, we have $\Lambda : E \longrightarrow E$, $\Lambda(u) = \Lambda_u$, and for all $u, v \in E$, we get

$$\|\Lambda_u(t) - \Lambda_v(t)\|_{B^s_{\infty,q}(\mathbb{R}^n)} < \frac{1}{2}\, d(u, v).$$

Therefore, Λ_u is a contraction mapping in E, and so by Banach fixed point theorem, Λ_u has a fixed point $\Lambda_u = u \in E$. This leads to

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)e^{u(s)}ds$$

for $0 \le t \le \frac{1}{2L+c_0} = T$. In all, we obtain a unique solution u in $C([0,T], B^s_{\infty,q})$. This completes the proof of the local existence. $\Box$

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