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BIPROJECTIVITY OF $C_r^*(G)$ AS A $L^1(G)$-BIMODULE

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Abstract. We investigate biprojectivity of $C_r^*(G)$ as a $L^1(G)$-bimodule for a locally compact group G. The main results are the following. As a $L^1(G)$-bimodule $C_r^*(G)$ is biprojective if G is compact and is not biprojective if G is an infinite discrete group or G is a non-compact abelian group.

1. Introduction

Let G be a locally compact group. The space $L^1(G)$ is a Banach algebra under the convolution product, and clearly it is commutative if and only if G is commutative. Recall that the convolution $*$ is defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy, \ \ f, g \in L^1(G),$$

where dy denotes the left Haar measure on G. Moreover, the convolution algebra $L^1(G)$ contains every information about the group G itself, which is why $L^1(G)$ is regarded as a central object in abstract harmonic analysis.

Since the fundamental work of B. E. Johnson there have been many investigations about relating (co-)homological properties of $L^1(G)$ with the properties of G. For example, Johnson himself ([7]) proved that $L^1(G)$ is amenable as a Banach algebra if and only if G is amenable, and Helemskii ([5]) proved that $L^1(G)$ is a biprojective Banach algebra if and only if G is compact. Recently, H. G. Dales and M. E. Polyakov ([2]) turned their attention to the modules over $L^1(G)$. They showed,

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among many others, that as a left-module of $L^1(G)$, $C_0(G)$, the algebra of continuous function on G vanishing at infinity, is projective if and only if G is compact.

In this paper we will continue the same line of research of Dales/ Polyakov with $C^*_r(G)$, the reduced group C^*-algebra in the category of $L^1(G)$-bimodule. Recall that $C^*_r(G)$ is the C^*-algebra generated by $\{L_f : f \in L^1(G)\} \subseteq B(L^2(G))$, where L_f is the left convolution operator with respect to f given by $L_f(g) = f * g$, $g \in L^2(G)$. It is well-known that the *Fourier transform*

$$\mathcal{F} : L^1(G) \to C^*_r(G),\ f \mapsto L_f$$

is an injective homomorphism with norm ≤ 1, and clearly the image of $\mathcal{F}$ is norm-dense in $C^*_r(G)$. Since $C^*_r(G)$ appears more frequently in the dual side of convolution algebra theory, namely, the Fourier algebra theory of G, we might say that this paper deals with a mixture of two theories.

Although we could not get the whole characterization of biprojectivity of $C^*_r(G)$, we were able to show the following positive and negative results. When viewd as a $L^1(G)$-bimodule, $C^*_r(G)$ is biprojective if G is compact and is not biprojective if G is an infinite discrete group or G is a non-compact abelian group.

This paper is organized as follows. In section 2 we will present a general theory of Banach algebras focusing on biprojectivity, and our main results will be presented in the last section. We will assume that the reader is familiar with standard functional analysis concepts including projective and injective tensor products of Banach spaces.

2. A general theory related to biprojectivity

A standard reference for the homological treatment of Banach algebras is [6].

Let $\mathcal{A}$ be a Banach algebra with the multiplication map $m : \mathcal{A} \otimes_\gamma \mathcal{A} \to \mathcal{A}$, where $\otimes_\gamma$ implies the projective tensor product of Banach spaces. We say that a Banach space X is a $\mathcal{A}$-*bimodule* if we have a contractive multiplication map $\pi_X : \mathcal{A} \otimes_\gamma X \otimes_\gamma \mathcal{A} \to X$. We usually denote by $a \cdot x \cdot b$ for $a, b \in \mathcal{A}$, $x \in X$ instead of $\pi_X(a \otimes x \otimes b)$. Let $\mathcal{A}_+ = \mathcal{A} \oplus \mathbb{C}$ be the *unitization* of $\mathcal{A}$, then π_X can be naturally extended to $\pi^+_X : \mathcal{A}_+ \otimes_\gamma X \otimes_\gamma \mathcal{A}_+ \to X$.

When we have two $\mathcal{A}$-bimodules X and Y, $X \otimes_\gamma Y$ has a $\mathcal{A}$-bimodule structure given by $a \cdot (x \otimes y) \cdot b := (a \cdot x) \otimes (y \cdot b)$ for any $a, b \in \mathcal{A}$, $x \in X$, and $y \in Y$. If we have an additional Banach space Z, then the space $X \otimes_\gamma Z \otimes_\gamma Y$ has a $\mathcal{A}$-bimodule structure given by $a \cdot (x \otimes z \otimes y) \cdot b := (a \cdot x) \otimes z \otimes (y \cdot b)$ for any $a, b \in \mathcal{A}$, $x \in X$, $y \in Y$, and $z \in Z$. In this case, a bounded linear map $T : X \to Y$ is called a $\mathcal{A}$-*bimodule map* if $T(a \cdot x \cdot b) = a \cdot T(x) \cdot b$ for any $a, b \in \mathcal{A}$, $x \in X$.

A $\mathcal{A}$-bimodule X is called *biprojective* if there is a bounded $\mathcal{A}$-bimodule map $\rho : \mathcal{A}_+ \to \mathcal{A}_+ \otimes_\gamma X \otimes_\gamma \mathcal{A}_+$ such that $\pi^+_X \circ \rho = id_{\mathcal{A}_+}$.

We will present a useful consequence of biprojectivity when we have an object satisfying AP, the Grothendieck's classical *approximation property.* Recall that a Banach space X is said to have AP if the natural map

$$X \otimes_\gamma Y \hookrightarrow X \otimes_\epsilon Y$$

is one-to-one for any Banach space Y, where $\otimes_\epsilon$ implies the injective tensor product of Banach spaces. Note that $L^1(G)$ has AP all the time.

LEMMA 2.1. *Let $\mathcal{A}$ be a Banach algebra, and let X be a A-bimodule. Assume that X is biprojective and X or A have AP. Then for any non-zero element $x \in X$, there is a bounded $\mathcal{A}$-bimodule map $T : X \to \mathcal{A}_+ \otimes_\gamma \mathcal{A}_+$ such that $T(x) \neq 0$.*

Proof. The proof is essentially the same as [6, Corollary 4.5]. We include a detailed argument for the reader's convenience. Since X is biprojective, we have a bounded $\mathcal{A}$-bimodule map $\rho : X \to \mathcal{A}_+ \otimes_\gamma X \otimes_\gamma \mathcal{A}_+$ which is a right inverse of π^+_X. Since $\pi^+_X \circ \rho = id_X$ for any non-zero x, we have $\rho(x) \neq 0$. We can assume that $\mathcal{A}$ has AP since the proof for the case X has AP is the same. It is clear that $\mathcal{A}_+$ also has AP. Then, the following canonical map is injective.

$$\mathcal{A}_+ \otimes_\gamma X \otimes_\gamma \mathcal{A}_+ \to \mathcal{A}_+ \otimes_\epsilon X \otimes_\epsilon \mathcal{A}_+ \hookrightarrow (\mathcal{A}^*_+ \otimes_\gamma X^* \otimes_\gamma \mathcal{A}^*_+)^*.$$

Thus, we can find $g_1, g_2 \in \mathcal{A}^*_+$ and $f \in X^*$ such that $(g_1 \otimes f \otimes g_2)(\rho(x)) \neq 0$ and consequently $(I_{\mathcal{A}_+} \otimes f \otimes I_{\mathcal{A}_+})(\rho(x)) \neq 0$. If we set $T = (I_{\mathcal{A}_+} \otimes f \otimes I_{\mathcal{A}_+})\rho$, then T is the map we desired. $\square$

3. Main results

First of all, we need to understand the $L^1(G)$-bimodule structure of $C^*_r(G)$. It is given by the following multiplication map.

$$\pi : L^1(G) \otimes_\gamma C^*_r(G) \otimes_\gamma L^1(G) \to C^*_r(G), \quad f \otimes L_g \otimes h \mapsto L_{f*g*h}.$$

Since $L_{f*g*h} = L_f L_g L_h$, it is clear that π is a contraction. We denote the natural extension of π to the setting of unitization by π^+ as before.

We begin with a positive result.

THEOREM 3.1. *Let G is a compact group. Then $C_r^*(G)$ is a biprojective $L^1(G)$-bimodule.*

Proof. Let

$$\Gamma : L^1(G) \to L^1(G \times G \times G) \cong L^1(G) \otimes_\gamma L^1(G) \otimes_\gamma L^1(G)$$

given by $\Gamma(f)(s,t,u) = f(stu)$, $s,t,u \in G$. Note that $L^1(G)$ is naturally embedded in $L^1(G)_+$ by the map $f \mapsto (f, 0)$. Then, the map

$$\rho : C_r^*(G) \to L^1(G)_+ \otimes_\gamma C_r^*(G) \otimes_\gamma L^1(G)_+$$

given by $\rho(L_f) = (I_{L^1(G)_+} \otimes \mathcal{F} \otimes I_{L^1(G)_+}) \circ \Gamma(f)$ is a well defined isometry. Indeed, since $L^1(G) \otimes_\gamma X \cong L^1(G;X)$ the vector valued L^1-space for any Banach space X, we have

$$\|\rho(L_f)\|_{L^1(G)_+ \otimes_\gamma C_r^*(G) \otimes_\gamma L^1(G)_+} = \int_G \int_G \|\mathcal{F}({}_{s^{-1}}f_u)\|_{C_r^*(G)}\, dsdu,$$

where ${}_{s^{-1}}f_u$ is the translation of f on both sides given by

$${}_{s^{-1}}f_u(t) = f(stu).$$

By the homomorphic property of Fourier transform we get

$$\mathcal{F}({}_{s^{-1}}f_u) = \lambda(s^{-1}) L_f \lambda(u^{-1}),$$

where $\lambda(x)$, $x \in G$ is the left translation operator in $B(L^2(G))$ given by $\lambda(x)f(y) = f(x^{-1}y)$ for $f \in L^2(G)$ and $x, y \in G$. Note that $\lambda(x)$ is always a unitary map so that we have

$$\|\mathcal{F}({}_{s^{-1}}f_u)\|_{C_r^*(G)} = \left\|\lambda(s^{-1}) L_f \lambda(u^{-1})\right\|_{C_r^*(G)} = \|L_f\|_{C_r^*(G)}.$$

Thus, we get

$$\|\rho(L_f)\|_{L^1(G)_+ \otimes_\gamma C_r^*(G) \otimes_\gamma L^1(G)_+} = \|L_f\|_{C_r^*(G)}.$$

Since it is straightforward to check that ρ is a $L^1(G)$-bimodule map and it is a right inverse of π^+, we got the map we wanted. □

REMARK 3.2. For a compact group G it is well-known that $L^1(G)$ is biprojective as a $L^1(G)$-bimodule ([5]). Then a general theory of Banach algebras tells us that every essential left $L^1(G)$-module is left projective, which was the main tool Dales/Polyakov used in their work. Recall that a left $\mathcal{A}$-module X for a Banach algebra $\mathcal{A}$ is called *essential* if $\mathcal{A} \cdot X$ is dense in X. $C_r^*(G)$ is clearly an essential module of $L^1(G)$, but the above general theory only gives us the information of left projectivity.

We also have negative results when the group is an infinite discrete group or non-compact abelian group. The approach in [3, section 5] and [4, section 3] will be used in a modified form. We usually denote $L^1(G)$ by $\ell^1(G)$ in case of discret group G to emphasize discreteness.

Lemma 3.3. *Let G be an infinite discrete group. Then, the space of all bounded $\ell^1(G)$-bimodule maps from $C_r^*(G)$ into $\ell^1(G) \otimes_\gamma \ell^1(G)$ is trivial.*

Proof. Let $T : C_r^*(G) \to \ell^1(G) \otimes_\gamma \ell^1(G) \cong \ell^1(G \times G)$ be a bounded $\ell^1(G)$-bimodule map. Then we will show that $T(L_f) = 0$ for any $f \in \ell^1(G)$, so that $T = 0$ by continuity. Indeed, by the bimodule property of T we have $T(L_f) = T(L_{f*\delta_e}) = T(f \cdot L_{\delta_e}) = f \cdot T(L_{\delta_e})$, where δ_x, $x \in G$ is the point mass function on x. Similarly we have $T(L_f) = T(L_{\delta_e}) \cdot f$, so that we have $f \cdot F = F \cdot f$ for any $f \in \ell^1(G)$, where $F = T(L_{\delta_e}) \in \ell^1(G \times G)$. If we put $f = \delta_x$, $x \in G$, then we get

$$F(x^{-1}s, t) = F(s, tx), \ \ \forall s, t \in G,$$

which is equivalent to

$$F(s, t) = F(xs, tx), \ \ \forall s, t \in G.$$

Then we have $F(s, t) = 0$ for any $s, t \in G$. Indeed, we have

$$\sum_{x \in G} |F(s,t)| = \sum_{x \in G} |F(xs, tx)| \leq \|F\|_1 < \infty,$$

which means $F(s, t) = 0$ since G is an infinite group. □

Theorem 3.4. *Let G be an infinite discrete group. Then, $C_r^*(G)$ is not a biprojective $\ell^1(G)$-bimodule.*

Proof. Suppose that $C_r^*(G)$ is biprojective, then, by Lemma 2.1 and the fact $\ell^1(G)$ has AP we have the following: for any non-zero element $x \in C_r^*(G)$ we can find a $L^1(G)$-bimodule map $T : C_r^*(G) \to \ell^1(G)_+ \otimes_\gamma \ell^1(G)_+$ such that $T(x) \neq 0$. If we convolve appropriate non-zero functions in $\ell^1(G)$ to $T(x)$ on the right for the first variable and on the left for the second variable, we can actually find a $\ell^1(G)$-bimodule map $T' : C_r^*(G) \to \ell^1(G) \otimes_\gamma \ell^1(G)$ such that $T'(x) \neq 0$. However, this is impossible by Lemma 3.3. □

Theorem 3.5. *Let G be a non-compact abelian group. Then, $C_r^*(G)$ is not a biprojective $L^1(G)$-bimodule.*

Proof. We will follow the approach in the proof of Theorem 3.4. Thus, it is enough to show that the space of all bounded $L^1(G)$-bimodule maps from $C_r^*(G)$ into $L^1(G) \otimes_\gamma L^1(G)$ is trivial. Let $T : C_r^*(G) \to L^1(G) \otimes_\gamma L^1(G) \cong L^1(G \times G)$ be a bounded $L^1(G)$-bimodule map. Since G is abelian, $L^1(G)$ is a commutative algebra, so that we have

$$f \cdot T(L_g) = T(f \cdot L_g) = T(L_{f*g}) = T(L_{g*f}) = T(L_g \cdot f) = T(L_g) \cdot f$$

for any $f, g \in L^1(G)$. Let $F = T(L_g) \in L^1(G \times G)$. By w^*-continuity we can conclude that

$$\delta_x \cdot F = F \cdot \delta_x$$

for any $x \in G$, so that we have

$$F(s,t) = F(xs, tx), \ \ \forall s, t, x \in G.$$

Then since G is non-compact, for any nonempty compact subset $K \subset G$ there is a sequence $(x_n)_{n\geq 1} \subset G$ such that $x_n K$'s are disjoint. Moreover, we have

$$\begin{aligned}\sum_{n\geq 1} \int_{K\times K} |F(s,t)|\, dsdt &= \sum_{n\geq 1} \int_{K\times K} |F(x_n s, t x_n)|\, dsdt \\ &= \sum_{n\geq 1} \int_{x_n K \times x_n K} |F(s,t)|\, dsdt \\ &\leq \|F\|_1 < \infty.\end{aligned}$$

Thus, we have $\int_{K\times K} |F(s,t)|\, dsdt = 0$ for any nonempty compact subset $K \subset G$, which implies that $T(L_g) = F = 0$ by inner regularity of the Haar measure. Since $g \in L^1(G)$ is arbitrary we get $T = 0$ by continuity. □

We end this paper with the following open question.

PROBLEM 3.1. Is the biprojectivity of $C_r^*(G)$ as a $L^1(G)$-bimodule equivalent to the compactness of G?

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