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JORDAN HIGHER LEFT DERIVATIONS AND COMMUTATIVITY IN PRIME RINGS

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Abstract. Let R be a 2-torsionfree prime ring. Our goal in this note is to show that the existence of a nonzero Jordan higher left derivation on R implies R is commutative. This result is used to prove a noncommutative extension of the classical Singer-Wermer theorem in the sense of higher derivations.

1. Preliminaries

Throughout this note, R will represent an associative ring with center $Z(R)$ and we will write $[a, b]$ for the commutator $ab - ba$. Recall that R is *prime* if $aRb = 0$ implies $a = 0$ or $b = 0$. A *derivation* (resp. *Jordan derivation*) is an additive mapping $\delta : R \to R$ satisfying $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in R$ (resp. $\delta(a^2) = a\delta(a) + \delta(a)a$ for all $a \in R$). An additive mapping $\delta : R \to R$ is called a *left derivation* (resp. *Jordan left derivation*) if $\delta(ab) = a\delta(b) + b\delta(a)$ (resp. $\delta(a^2) = 2a\delta(a)$ for all $a \in R$) holds for all $a, b \in R$.

Higher derivations as a generalization of derivations have been studied in rings (mainly in commutative rings), but also in noncommutative rings (see [2], [3], [5]).

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of all nonnegative integers and

$$c_{ij} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j \end{cases} .$$

Definition 1.1. Let $\Delta = (\delta_i)_{i\in\mathbb{N}}$ (resp. $\Delta = \{\delta_1, \delta_2, \cdots, \delta_m\}$) be a sequence of additive mappings on a ring R. Δ is said to be:

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(i) a *higher derivation* (resp. *higher derivation of rank* m) if for each $n \in \mathbb{N}$ and $i, j \in \mathbb{N}_0$,

$$\delta_n(ab) = \sum_{i+j=n} \delta_i(a)\delta_j(b) \quad \text{for all } a, b \in R$$

$$(\text{or } \delta_n(ab) = \sum_{\substack{i+j=n \\ i \le j}} [\delta_i(a)\delta_j(b) + c_{ij}\delta_j(a)\delta_i(b)] \quad \text{for all } a, b \in R),$$

where $\delta_0 = id_R$.

(ii) a *Jordan higher derivation* (resp. *Jordan higher derivation of rank* m) if for each $n \in \mathbb{N}$ and $i, j \in \mathbb{N}_0$,

$$\delta_n(a^2) = \sum_{i+j=n} \delta_i(a)\delta_j(a) \quad \text{for all } a \in R$$

$$(\text{or } \delta_n(a^2) = \sum_{\substack{i+j=n \\ i \le j}} [\delta_i(a)\delta_j(a) + c_{ij}\delta_j(a)\delta_i(a)] \quad \text{for all } a, b \in R),$$

where $\delta_0 = id_R$

(iii) a *higher left derivation* (resp. *higher left derivation of rank* m) if for each $n \in \mathbb{N}$ and $i, j \in \mathbb{N}_0$,

$$\delta_n(ab) = \sum_{\substack{i+j=n \\ i \le j}} [\delta_i(a)\delta_j(b) + c_{ij}\delta_i(b)\delta_j(a)] \quad \text{for all } a, b \in R,$$

where $\delta_0 = id_R$.

(iv) a *Jordan higher left derivation* (resp. *Jordan higher left derivation of rank* m) if for each $n \in \mathbb{N}$ and $i, j \in \mathbb{N}_0$,

$$\delta_n(a^2) = \sum_{\substack{i+j=n \\ i \le j}} (c_{ij} + 1)\delta_i(a)\delta_j(a) \quad \text{for all } a, b \in R,$$

where $\delta_0 = id_R$.

It is easy to see that every higher left derivation is a Jordan higher left derivation. But the converse, in general, is not true.

M. Brešar and J. Vukman [1, Theorem 1.2] showed that the existence of a nonzero Jordan left derivation of R into a 2-torsionfree and 3-torsionfree left R-module X under the assumption that $aRx = 0$ ($a \in R$, $x \in X$) implies $a = 0$ or $x = 0$, guarantees the commutativity of R.

The main purpose in this note is to introduce Jordan higher left derivations to improve the above Brešar and Vukman's result [1, Theorem 1.2] when $R = X$, that is, R is a 2-torsionfree and 3-torsionfree prime ring. Using this result, we also prove a noncommutative extension of the classical Singer-Wermer theorem [6] in the sense of higher derivations. The classical Singer-Wermer theorem is as follows: *every continuous derivation on a commutative Banach algebra A maps into its Jacobson radical $rad(A)$ which is the intersection of all the primitive ideals of A.*

2. Main results

The following lemma is due to M. Brešar and J. Vukman [1, Proposition 1.1].

LEMMA 2.1. *Let R be a ring and X be a 2-torsionfree left R-module. If $\delta : R \to X$ is a Jordan left derivation, then for all $a, b, c \in R$:*

(i) $\delta(ab + ba) = 2a\delta(b) + 2b\delta(a)$
(ii) $\delta(aba) = a^2\delta(b) + 3ab\delta(a) - ba\delta(a)$
(iii) $[a, b]a\delta(a) = a[a, b]\delta(a)$
(iv) $[a, b](\delta(ab) - a\delta(b) - b\delta(a)) = 0$

Our main theorem is

THEOREM 2.2. *Let R be a 2-torsionfree prime ring and $\Delta = (\delta_n)_{n\in\mathbb{N}}$ be a Jordan higher left derivation on R. If $\Delta \neq 0$, i.e., there exists $n \in \mathbb{N}$ such that $\delta_n \neq 0$, then R is commutative.*

Proof. We use the induction. Let $n = 1$, i.e., δ_1 is a Jordan left derivation on R. Assume that R is noncommutative. By Lemma 2(iii), we have

$$(x^2y - 2xyx + y^2)\delta_1(x) = 0$$

for all $x, y \in R$. Replacing x by $[a, b]$ in this relation, we get

$$[a, b]^2y\delta_1[a, b] - 2[a, b]y\delta_1([a, b]) + y[a, b]^2\delta_1([a, b]) = 0 \tag{2.1}$$

for all $a, b, y \in R$. Since it follows from Lemma 2(iv) that

$$[a, b](\delta_1(ab) - a\delta_1(b) - b\delta_1(a)) = 0 \tag{2.2}$$

and

$$[a, b](\delta_1(ba) - a\delta_1(b) - b\delta_1(a)) = 0 \tag{2.3}$$

for all $a, b \in R$, comparing (2) with (3) yields

$$[a, b]\delta_1([a, b]) = 0 \tag{2.4}$$

for all $a, b \in R$. Then (4) makes (1) to

$$[a, b]^2 y \delta_1([a, b]) = 0 \tag{2.5}$$

for all $a, b, y \in R$. By the primeness of R, equation (5) yields either $[a, b]^2 = 0$ or $\delta_1([a, b]) = 0$ for all $a, b \in R$.

Suppose that $[a, b]^2 = 0$ for all $a, b \in R$. Using Lemma 2(i-ii) and (4), we see that

$$\begin{aligned} W &= \delta_1([a, b]x[a, b]y[a, b] + [a, b]y[a, b]([a, b]x)) \\ &= 2\{[a, b]x\delta_1([a, b]y[a, b]) + [a, b]y[a, b]\delta_1([a, b]x)\} \\ &= 6[a, b]x[a, b]y\delta_1[a, b] + [a, b]y\{2[a, b]\delta_1([a, b]x)\} \end{aligned} \tag{2.6}$$

for all $a, b, x, y \in R$.

On the other hand, we have

$$\begin{aligned} W &= \delta_1([a, b](x[a, b]y)[a, b]) \\ &= 3[a, b]x[a, b]y\delta_1([a, b]) \end{aligned} \tag{2.7}$$

for all $a, b, x, y \in R$. In comparison of (6) and (7), we obtain

$$3[a, b]x[a, b]y\delta_1([a, b]) + [a, b]y\{2[a, b]\delta_1([a, b])x\} = 0 \tag{2.8}$$

for all $a, b, x, y \in R$. Also, we get

$$\begin{aligned} V &= \delta_1([a, b]x[a, b] + x[a, b]^2) \\ &= \delta_1([a, b]x[a, b]) \\ &= 3[a, b]x\delta_1([a, b]) \end{aligned} \tag{2.9}$$

for all $a, b, x \in R$.

On the other hand, we have

$$\begin{aligned} V &= 2\{[a, b]\delta_1(x[a, b]) + x[a, b]\delta_1([a, b])\} \\ &= 2[a, b]\delta_1(x[a, b]) \end{aligned} \tag{2.10}$$

for all $a, b, x \in R$. Combining (9) and (10), we obtain

$$3[a, b]x\delta_1([a, b]) = 2[a, b]\delta_1(x[a, b]) \tag{2.11}$$

for all $a, b, x \in R$. From Lemma 2(i) and the hypothesis $[a, b]^2 = 0$, we get

$$\begin{aligned} &[a, b]\delta_1(x[a, b] + [a, b]x) \\ &= 2[a, b]^2\delta_1(x) + 2[a, b]x\delta_1([a, b]) \\ &= 2[a, b]x\delta_1([a, b]) \end{aligned} \tag{2.12}$$

for all $a, b, x \in R$. Now, (11) and (12) give

$$3[a, b]\{\delta_1(x[a, b]) + \delta_1([a, b]x)\} = 4[a, b]\delta_1(x[a, b])$$

which reduces to

$$[a,b]\delta_1(x[a,b]) = 3[a,b]\delta_1([a,b]x) \tag{2.13}$$

for all $a, b, x \in R$. By (13), we have

$$\begin{aligned} &[a,b]\delta_1(x[a,b]) + [a,b]x) \\ &= 3[a,b]\delta_1([a,b]x) + [a,b]\delta_1([a,b]x) \\ &= 4[a,b]\delta_1([a,b]x) \end{aligned} \tag{2.14}$$

for all $a, b, x \in R$. We also obtain that

$$[a,b]\delta_1(x[a,b]) + [a,b]x) = 2[a,b]\{x\delta_1[a,b] + [a,b]\delta_1(x)\} \tag{2.15}$$

for all $a, b, x \in R$.

From (14) and (15), it follows that

$$4[a,b]\delta_1([a,b]x) - 2[a,b]x\delta_1([a,b]) = 0$$

for all $a, b, x \in R$. Since R is 2-torsion free, this equation comes to

$$2[a,b]\delta_1([a,b]x) = [a,b]x\delta_1([a,b]) \tag{2.16}$$

for all $a, b, x \in R$. Hence (8) and (16) yield

$$3[a,b]x[a,b]y\delta_1([a,b]) + [a,b]y[a,b]y\delta_1([a,b]) = 0 \tag{2.17}$$

for all $a, b, x \in R$. Substituting $y[a,b]y$ for x in (16), we have

$$2[a,b]\delta_1(([a,b]y)^2) = [a,b]y[a,b]y\delta_1([a,b]) \tag{2.18}$$

for all $a, b, y \in R$. Therefore, equation (18) gives

$$4[a,b]^2y\delta_1([a,b]y) = [a,b]y[a,b]y\delta_1([a,b])$$

which, from hypothesis $[a,b]^2 = 0$, implies

$$[a,b]y[a,b]y\delta_1([a,b]) = 0 \tag{2.19}$$

for all $a, b, y \in R$. Replacing y by $x + y$ in (19) and utilizing (19), we get

$$[a,b]y[a,b]y\delta_1([a,b]) + [a,b]y[a,b]x\delta_1([a,b]) = 0 \tag{2.20}$$

for all $a, b, x, y \in R$. Thus the relations (17) and (20) give

$$[a,b]x[a,b]y\delta_1([a,b]) = 0$$

for all $a, b, x, y \in R$. Since R is noncommutative and prime, we see that $\delta_1([a,b]) = 0$ holds for all $a, b \in R$, i.e., $\delta_1(ab) = \delta_1(ba)$ for all $a, b \in R$. This means that

$$2\delta_1((ba)a) = \delta_1((ba)a + a(ba)) = 2\{a^2\delta_1(b) + ab\delta_1(a) + ba\delta_1(a)\}$$

for all $a, b \in R$ by using Lemma 2(i) which yields

$$\delta_1((ba)a) = a^2\delta_1(b) + ab\delta_1(a) + ba\delta_1(a) \tag{2.21}$$

for all $a, b \in R$ since R is 2-torsion free.

On the other hand, it follows from Lemma 2(i) that

$$\delta_1(aba + ba^2) = 2\{a\delta_1(ba) + ba\delta_1(a)\} \tag{2.22}$$

and

$$\delta_1(a^2b + aba) = 2\{a\delta_1(ab) + ab\delta_1(a)\} \tag{2.23}$$

for all $a, b \in R$. Comparing (22) with (23), we have

$$\delta_1(a^2b + ba^2) = 2\{a\delta_1([a, b]) + [a, b]\delta_1(a)\} \tag{2.24}$$

for all $a, b \in R$. Setting $a = a^2$ in Lemma 2(i) gives

$$\delta_1(a^2b + ba^2) = 2\{a^2\delta_1(b) + 2ba\delta_1(a)\} \tag{2.25}$$

for all $a, b \in R$ and so we combine (24) and (25) to obtain

$$\begin{aligned} \delta_1(ba^2) &= a^2\delta_1(b) + (3ba - ab)\delta_1(a) - a\delta_1([a, b]) \\ &= a^2\delta_1(b) + (3ba - ab)\delta_1(a) \end{aligned} \tag{2.26}$$

for all $a, b \in R$. According to (21) and (26), we get

$$[a, b]\delta_1(a) = 0 \tag{2.27}$$

for all $a, b \in R$ since R is 2-torsion free. Finally, the substitution bx for b in (27) yields

$$\begin{aligned} 0 &= (abx - bxa)\delta_1(a) \\ &= (ab - ba)x\delta_1(a) + b(ax - xa)\delta_1(a) \\ &= [a, b]x\delta_1(a), \end{aligned}$$

that is,

$$[a, b]x\delta_1(a) = 0$$

for all $a, b, x \in R$. Since R is noncommutative and prime, we arrive at $\delta_1 = 0$.

Assume that $n \geq 2$ and $\delta_m = 0$ for all $m < n$. Then δ_n is a Jordan left derivation on R and, from the above argument, it follows that $\delta_n = 0$. Hence we conclude that $\Delta = 0$. This completes the proof. □

Let A be a Banach algebra. As a noncommutative version of the Singer-Wermer theorem [6], B. Yood [7] proved the following: *every continuous linear derivation δ on A which satisfies $[\delta(a), b] \in rad(A)$ for all $a, b \in A$, maps A into $rad(A)$.* We now improve Yood's result in the sense of higher derivations by applying Theorem 3.

THEOREM 2.3. *Let A be a Banach algebra and $\Delta = \{\delta_1, \delta_2, \cdots, \delta_m\}$ be a continuous linear higher derivation of rank m on A, i.e., δ_n is linear and continuous for each $n = 1, 2, \cdots, m$. If for each $i, j = 0, 1, 2, \cdots, m$ with $i \neq j$, we have $[\delta_i(a), \delta_j(a)] \in rad(A)$ for all $a \in A$, where $\delta_0 = id_A$. Then Δ maps A into $rad(A)$, that is, $\delta_n(A) \subseteq rad(A)$ for each $n = 1, 2, \cdots, m$.*

Proof. From [1, Theorem 2], it follows that $\delta_1(A) \subseteq rad(A)$. In case when R is noncommutative, the proof of [4, Theorem 1] also is true and so we see that for each $n = 1, 2, \cdots, m$, $\delta_n(P) \subseteq P$ for any primitive ideal P. Hence, for each $n = 1, 2, \cdots, m$, δ_n can be dropped to a linear mapping d_n on A/P defined by

$$d_n(a + P) = \delta_n(a) + P$$

for all $a \in A$. Thus $D = \{d_1, d_2, \cdots, d_m\}$ is a higher derivation of rank m on A/P. Since every higher derivation is a Jordan higher derivation and for each $i, j = 1, 2, \cdots, m$ with $i \neq j$,

$$d_i(a + P)d_j(a + P) = d_j(a + P)d_i(a + P)$$

holds for all $a \in A$, we see that $D = \{d_1, d_2, \cdots, d_m\}$ is a Jordan higher left derivation of rank m on A/P. If A/P is noncommutative, $D = 0$ by Theorem 3 since A/P is prime. If A/P is commutative, then [4, Theorem 1] guarantees $D = 0$ since A/P is semisimple. Therefore, we deduce that for each $n = 1, 2, \cdots, m$, $\delta_n(A) \subseteq P$ for any primitive ideal P which gives the conclusion of the theorem. □

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