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STABILITY OF THE RECIPROCAL DIFFERENCE AND ADJOINT FUNCTIONAL EQUATIONS IN $m-$VARIABLES

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Abstract. In this paper, we prove stability of the reciprocal difference functional equation

$$r\left(\frac{\sum_{i=1}^m x_i}{m}\right) - r\left(\sum_{i=1}^m x_i\right) = \frac{(m-1)\prod_{i=1}^m r(x_i)}{\sum_{i=1}^m \prod_{k\neq i, 1\leq k\leq m} r(x_k)}$$

and the reciprocal adjoint functional equation

$$r\left(\frac{\sum_{i=1}^m x_i}{m}\right) + r\left(\sum_{i=1}^m x_i\right) = \frac{(m+1)\prod_{i=1}^m r(x_i)}{\sum_{i=1}^m \prod_{k\neq i, 1\leq k\leq m} r(x_k)}$$

in m-variables. Stability of the reciprocal difference functional equation and the reciprocal adjoint functional equation in two variables were proved by K. Ravi, J. M. Rassias and B. V. Senthil Kumar [13]. We extend their result to m-variables in similar types.

1. Introduction

In 1940, Ulam [14] proposed the stability problem of the functional equation;

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all x, $y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In 1941, Hyers [5] answered the Ulam's question for the case of the additive mapping on the Banach spaces.

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Let G_1 and G_2 be Banach spaces. Assume that a mapping $f : G_1 \to G_2$ satisfies the inequality

$$||f(x+y) - f(x) - f(y)|| \le \varepsilon$$

for all $x, y \in G_1$. Then the limit $g(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in G_1$ and g is the unique additive mapping such that

$$||f(x) - g(x)|| \le \varepsilon, \quad \forall x \in G_1.$$

In 1978, Th. M. Rassias [12] generalized the Hyers's result bounded by constant to an approximation involving a sum of norms;

Let G_1 be a vector space and G_2 a Banach space. Assume that a mapping $f : G_1 \to G_2$ satisfies

$$||f(x+y) - f(x) - f(y)|| \le \varepsilon(||x||^p + ||y||^p)$$

for all $x, y \in G_1$, $\varepsilon > 0$ and $p < 1$. Then the limit $g(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in G_1$ and g is the unique additive mapping such that

$$||f(x) - g(x)|| \le \frac{2\varepsilon}{2 - 2^p}||x||^p, \quad \forall x \in G_1.$$

During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [1]-[14]). P. Gâvruta [4] provided a further generalization of Th. M. Rassias' Theorem.

K. Ravi, J. M. Rassias and B. V. Senthil Kumar [13] proved some interesting results on the stability of a reciprocal difference functional equation

$$r\left(\frac{x+y}{2}\right) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)}$$

and the reciprocal adjoint functional equation

$$r\left(\frac{x+y}{2}\right) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)}.$$

In this paper, we prove the stability of a reciprocal difference functional equation

$$r\left(\frac{\sum_{i=1}^m x_i}{m}\right) - r\left(\sum_{i=1}^m x_i\right) = \frac{(m-1)\prod_{i=1}^m r(x_i)}{\sum_{i=1}^m \prod_{k\neq i, 1\le k\le m} r(x_k)} \tag{1.1}$$

and also prove stability of a reciprocal adjoint functional equation

$$r\left(\frac{\sum_{i=1}^{m} x_i}{m}\right) + r\left(\sum_{i=1}^{m} x_i\right) = \frac{(m+1)\prod_{i=1}^{m} r(x_i)}{\sum_{i=1}^{m}\prod_{k\neq i,1\leq k\leq m} r(x_k)} \tag{1.2}$$

with m-variables in the sense of Gâvruta, which is bounded by an arbitrary function.

2. Stability

THEOREM 2.1. *Let X and Y be spaces of non-zero real numbers and $2 \leq m \in N$. Assume that $f : X \to Y$ satisfies the functional inequality*

$$\left| f\left(\frac{\sum_{i=1}^{m} x_i}{m}\right) - f\left(\sum_{i=1}^{m} x_i\right) - \frac{(m-1)\prod_{i=1}^{m} f(x_i)}{\sum_{i=1}^{m}\prod_{k\neq i,1\leq k\leq m} f(x_k)} \right| \leq \phi(x_1, x_2, \cdots, x_m) \tag{2.1}$$

for all $x_1, x_2, \cdots, x_m \in X$, where $\phi : X^m \to Y$ is a function such that

$$\Phi(x_1, x_2, \cdots, x_m) := \sum_{i=0}^{\infty} \frac{1}{m^i}\phi\left(\frac{x_1}{m^{i+1}}, \frac{x_2}{m^{i+1}}, \cdots, \frac{x_m}{m^{i+1}}\right) < \infty.$$

Then there exists a unique reciprocal difference mapping $r : X \to Y$ which satisfies (1.1) and

$$|f(x) - r(x)| \leq \Phi(x, x, \cdots, x)$$

for all $x \in X$.

Proof. For each $i = 1, 2, \cdots, m$, replacing x_i by $\frac{x}{m}$ in (2.1), we obtain

$$\left|\frac{1}{m} f\left(\frac{x}{m}\right) - f(x)\right| \leq \phi\left(\frac{x}{m}, \frac{x}{m}, \cdots, \frac{x}{m}\right). \tag{2.2}$$

Again replacing x by $\frac{x}{m}$ in (2.2) and dividing by m we get

$$\left|\frac{1}{m^2} f\left(\frac{x}{m^2}\right) - f(x)\right| \leq \phi\left(\frac{x}{m}, \cdots, \frac{x}{m}\right) + \frac{1}{m}\phi\left(\frac{x}{m^2}, \cdots, \frac{x}{m^2}\right).$$

Again replacing x by $\frac{x}{m}$ in the above inequality and dividing by m we get

$$\left|\frac{1}{m^3} f\left(\frac{x}{m^3}\right) - f(x)\right| \leq \sum_{i=0}^{2} \frac{1}{m^i}\phi\left(\frac{x}{m^{i+1}}, \cdots, \frac{x}{m^{i+1}}\right).$$

Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned}\left|\frac{1}{m^n}f\left(\frac{x}{m^n}\right)-f(x)\right| &\le \sum_{i=0}^{n-1}\frac{1}{m^i}\phi\left(\frac{x}{m^{i+1}},\cdots,\frac{x}{m^{i+1}}\right)\\ &\le \sum_{i=0}^{\infty}\frac{1}{m^i}\phi\left(\frac{x}{m^{i+1}},\cdots,\frac{x}{m^{i+1}}\right)\end{aligned} \tag{2.3}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{\frac{1}{m^n}f\left(\frac{x}{m^n}\right)\right\}$, replace x by $\frac{x}{m^p}$ in (2.3) and divide by m^p , we find that for $n > p > 0$

$$\begin{aligned}\left|\frac{1}{m^p}f(\frac{x}{m^p})-\frac{1}{m^{n+p}}f(\frac{x}{m^{n+p}})\right| &= \frac{1}{m^p}\left|f(\frac{x}{m^p})-\frac{1}{m^n}f(\frac{x}{m^{n+p}})\right|\\ &\le \sum_{i=0}^{n-1}\frac{1}{m^{p+i}}\phi\left(\frac{x}{m^{p+i+1}},\cdots,\frac{x}{m^{p+i+1}}\right)\\ &\le \sum_{i=p}^{\infty}\frac{1}{m^i}\phi\left(\frac{x}{m^{i+1}},\cdots,\frac{x}{m^{i+1}}\right).\end{aligned}$$

Allowing $p \to \infty$, the right-hand side of the above inequality tends to 0. Thus the sequence $\left\{\frac{1}{m^n}f\left(\frac{x}{m^n}\right)\right\}$ is a Cauchy sequence. Thus we may define a mapping $r : X \to Y$ by

$$r(x) := \lim_{n\to\infty}\frac{1}{m^n}f\left(\frac{x}{m^n}\right), \quad \forall x \in X.$$

By (2.3) with $n \to \infty$, we have

$$|f(x) - r(x)| \le \Phi(x, \cdots, x), \quad \forall x \in X.$$

For each $i = 1, 2, \cdots, m$, replacing x_i by $\frac{x_i}{m^n}$ in (2.1) and dividing by m^n, we obtain

$$\begin{aligned}&\frac{1}{m^n}\left|f\left(\frac{1}{m^n}\left(\frac{\sum_{i=1}^m x_i}{m}\right)\right)-f\left(\frac{1}{m^n}\left(\sum_{i=1}^m x_i\right)\right)\right.\\ &\quad\left.-\frac{(m-1)\prod_{i=1}^m f(\frac{x_i}{m^n})}{\sum_{i=1}^m\prod_{k\ne i,1\le k\le m}f(\frac{x_k}{m^n})}\right|\\ &\le \frac{1}{m}\left(\frac{1}{m^{n-1}}\phi\left(\frac{x_1}{m^n},\frac{x_2}{m^n},\cdots.\frac{x_m}{m^n}\right)\right)\end{aligned}$$

for all $(x_1, x_2, \cdots, x_m) \in X^m$. Letting $n \to \infty$, we have

$$r\left(\frac{\sum_{i=1}^m x_i}{m}\right)-r\left(\sum_{i=1}^m x_i\right)=\frac{(m-1)\prod_{i=1}^m r(x_i)}{\sum_{i=1}^m\prod_{k\ne i,1\le k\le m}r(x_k)}$$

for all $(x_1, x_2, \cdots, x_m) \in X^m$. Now let $s : X \to Y$ be an another function which satisfies the equation (1.1) and $|f(x) - s(x)| \leq \Phi(x, x, \cdots, x)$ for all for all $x \in X$. Since $m^n s(x) = s(\frac{x}{m^n})$ and $m^n r(x) = r(\frac{x}{m^n})$, we have

$$\begin{aligned}|s(x) - r(x)| &= \frac{1}{m^n}\left|s(\frac{x}{m^n}) - r(\frac{x}{m^n})\right| \\ &\leq \frac{1}{m^n}\left(\left|s(\frac{x}{m^n}) - f(\frac{x}{m^n})\right| + \left|f(\frac{x}{m^n}) - r(\frac{x}{m^n})\right|\right) \\ &\leq 2\sum_{i=0}^{\infty}\frac{1}{m^{n+i}}\phi\left(\frac{x}{m^{n+i+1}}, \frac{x}{m^{n+i+1}}, \cdots, \frac{x}{m^{n+i+1}}\right)\end{aligned}$$

for all $x \in X$. Allowing $n \to \infty$ in the above inequality, we find that r is unique. This completes the proof of the theorem. □

Corollary 2.2. *Let X and Y be spaces of non-zero real numbers and $2 \leq m \in N$. Assume that $f : X \to Y$ satisfies the functional inequality*

$$\left|f\left(\frac{\sum_{i=1}^m x_i}{m}\right) - f\left(\sum_{i=1}^m x_i\right) - \frac{(m-1)\prod_{i=1}^m f(x_i)}{\sum_{i=1}^m \prod_{k\neq i, 1\leq k\leq m} f(x_k)}\right| \leq \delta$$

for all $x_1, x_2, \cdots, x_m \in X$. Then there exists a unique reciprocal difference mapping $r : X \to Y$ which satisfies (1.1) and

$$|f(x) - r(x)| \leq \frac{m}{m-1}\delta$$

for all $x \in X$.

Proof. Let $\phi(x_1, x_2, \cdots, x_n) = \delta$ for all $x_1, x_2, \cdots, x_m \in X^m$. Then, by Theorem 1, $\Phi(x, x, \cdots, x) = \sum_{i=0}^{\infty}\frac{\delta}{m^i} = \frac{m}{m-1}\delta$ for all $x \in X$. This completes the proof of the corollary. □

Theorem 2.3. *Let X and Y be spaces of non-zero real numbers and $2 \leq m \in N$. Assume that $f : X \to Y$ satisfies the functional inequality*

$$\begin{aligned}&\left|f\left(\frac{\sum_{i=1}^m x_i}{m}\right) - f\left(\sum_{i=1}^m x_i\right) - \frac{(m-1)\prod_{i=1}^m f(x_i)}{\sum_{i=1}^m \prod_{k\neq i, 1\leq k\leq m} f(x_k)}\right| \\ &\leq \psi\left(x_1, x_2, \cdots, x_m\right)\end{aligned} \tag{2.4}$$

for all $x_1, x_2, \cdots, x_m \in X$, where $\psi : X^m \to Y$ is a function such that for all $x_1, x_2, \cdots, x_m \in X$

$$\Psi(x_1, x_2, \cdots, x_m) := \sum_{i=0}^{\infty} m^{i+1}\psi\left(m^i x_1, m^i x_2, \cdots, m^i x_m\right) < \infty.$$

Then there exists a unique reciprocal adjoint mapping $r : X \to Y$ which satisfies (1.1) and the inequality

$$|f(x) - r(x)| \leq \Psi(x, x, \cdots, x)$$

for all $x \in X$.

Proof. Replacing $(x_1, x_2, \cdots, x_m)$ by $(x, x, \cdots, x)$ in (2.4) and multiplying by m, we obtain

$$|f(x) - mf(mx)| \leq m\psi(x, x, \cdots, x). \tag{2.5}$$

Again replacing x by mx in (2.5) and multiplying m we get

$$\left|f(x) - m^2 f\left(m^2 x\right)\right| \leq m\psi(x, x, \cdots, x) + m^2\psi(mx, mx, \cdots mx).$$

Again replacing x by mx in the above inequality and multiplying by m we get

$$\left|f(x) - m^3 f\left(m^3 x\right)\right| \leq \sum_{i=0}^{2} m^{i+1}\psi\left(m^i x, \cdots, m^i x\right).$$

Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} |f(x) - m^n f(m^n x)| &\leq \sum_{i=0}^{n-1} m^{i+1}\psi\left(m^i x, \cdots m^i x\right) \\ &\leq \sum_{i=0}^{\infty} m^{i+1}\psi\left(m^i x, \cdots, m^i x\right) \end{aligned} \tag{2.6}$$

for all $x \in X$. In order to prove the convergence of the sequence $\{m^n f(m^n x)\}$, replacing x by $m^p x$ in (2.6) and multiplying m^p , we find that for $n > p > 0$

$$\begin{aligned} \left|m^p f(m^p x) - m^{n+p} f(m^{n+p} x)\right| &= m^p \left|f(m^p x) - m^n f(m^{n+p} x)\right| \\ &\leq \sum_{i=0}^{n-1} m^{p+i+1}\psi\left(m^{p+i} x, \cdots, m^{p+i} x\right) \\ &\leq \sum_{i=p}^{\infty} m^{i+1}\psi\left(m^i x, \cdots, m^i x\right). \end{aligned}$$

Allow $p \to \infty$, the right-hand side of the above inequality tends to 0. Thus the sequence $\{m^n f(m^n x)\}$ is a Cauchy sequence. Thus we may define a mapping $r : X \to Y$ by

$$r(x) := \lim_{n\to\infty} m^n f(m^n x)$$

for all $x \in X$. By (2.6) with $n \to \infty$, we have

$$|f(x) - r(x)| \leq \Psi(x, x, \cdots, x)$$

for all $x \in X$. Replacing $(x_1, x_2, \cdots, x_m)$ by $(m^n x_1, m^n x_2, \cdots, m^n x_m)$ in (2.4) and multiplying by m^n we obtain

$$m^n \left| f\left(m^n(\frac{\sum_{i=1}^m x_i}{m})\right) - f\left(m^n(\sum_{i=1}^m x_i)\right) - \frac{(m-1)\prod_{i=1}^m f(m^n x_i)}{\sum_{i=1}^m \prod_{k\neq i, 1\leq k\leq m} f(m^n x_k)}\right|$$
$$\leq \frac{1}{m} m^{n+1} \psi(m^n x_1, m^n x_2, \cdots, m^n x_m)$$

for all $x_1, x_2, \cdots, x_m \in X$. Since $\Psi(x_1, x_2, \cdots, x_m)$ is finite, letting $n \to \infty$, we have

$$r\left(\frac{\sum_{i=1}^m x_i}{m}\right) - r\left(\sum_{i=1}^m x_i\right) = \frac{(m-1)\prod_{i=1}^m r(x_i)}{\sum_{i=1}^m \prod_{k\neq i, 1\leq k\leq m} r(x_k)}$$

for all $x_1, x_2, \cdots, x_m \in X$. Now let $s : X \to Y$ be another function which satisfies the equation (1.1) and $|f(x) - s(x)| \leq \Psi(x, x, \cdots, x)$ for all $x \in X$. Since $s(m^n x) = \frac{1}{m^n} s(x)$ and $r(m^n x) = \frac{1}{m^n} r(x)$, we have

$$\begin{aligned}|s(x) - r(x)| &= m^n \left|s(m^n x) - r(m^n x)\right| \\ &\leq m^n \left(|s(m^n x) - f(m^n x)| + |f(m^n x) - r(m^n x)|\right) \\ &\leq 2\sum_{i=0}^{\infty} m^{n+i+1} \psi\left(m^{n+i}x, m^{n+i}x, \cdots, m^{n+i}x\right)\end{aligned}$$

for all $x \in X$. Allowing $n \to \infty$ in the above inequality, we find that r is unique. This completes the proof of the theorem. □

THEOREM 2.4. *Let X and Y be spaces of non-zero real numbers and $2 \leq m \in N$. Assume that $f : X \to Y$ satisfies the functional inequality*

$$\left| f\left(\frac{\sum_{i=1}^m x_i}{m}\right) + f\left(\sum_{i=1}^m x_i\right) - \frac{(m+1)\prod_{i=1}^m f(x_i)}{\sum_{i=1}^m \prod_{k\neq i, 1\leq k\leq m} f(x_k)}\right| \leq \phi\left(x_1, x_2, \cdots, x_m\right) \tag{2.7}$$

for all $x_1, x_2, \cdots, x_m \in X$, where $\phi : X^m \to Y$ is a function such that

$$\Phi(x_1, x_2, \cdots, x_m) := \sum_{i=0}^{\infty} \frac{1}{m^i} \phi\left(\frac{x_1}{m^{i+1}}, \frac{x_2}{m^{i+1}}, \cdots, \frac{x_m}{m^{i+1}}\right) < \infty.$$

Then there exists a unique reciprocal adjoint mapping $r : X \to Y$ which satisfies (1.2) and

$$|f(x) - r(x)| \leq \Phi(x, x, \cdots, x)$$

for all $x \in X$.

Proof. Replacing $(x_1, x_2, \cdots, x_m)$ by $\left(\frac{x}{m}, \frac{x}{m}, \cdots, \frac{x}{m}\right)$ in (2.7), we obtain

$$\left|\frac{1}{m} f\left(\frac{x}{m}\right) - f(x)\right| \leq \phi\left(\frac{x}{m}, \frac{x}{m}, \cdots, \frac{x}{m}\right),$$

which states nothing else but (2.2) in Theorem 1. Hence, by the same method of proof in Theorem 1, the proof of Theorem is completed. □

Corollary 2.5. *Let* X *and* Y *be spaces of non-zero real numbers and* $2 \leq m \in N$. *Assume that* $f : X \to Y$ *satisfies the functional inequality*

$$\left| f\left(\frac{\sum_{i=1}^m x_i}{m}\right) + f\left(\sum_{i=1}^m x_i\right) - \frac{(m+1)\prod_{i=1}^m f(x_i)}{\sum_{i=1}^m \prod_{k\neq i, 1\leq k\leq m} f(x_k)}\right| \leq \delta$$

for all $x_1, x_2, \cdots, x_m \in X$. *Then there exists a unique reciprocal adjoint mapping* $r : X \to Y$ *which satisfies (1.2) and*

$$|f(x) - r(x)| \leq \frac{m}{m-1}\delta$$

for all $x \in X$.

Theorem 2.6. *Let* X *and* Y *be spaces of non-zero real numbers and* $2 \leq m \in N$. *Assume that* $f : X \to Y$ *satisfies the functional inequality*

$$\left| f\left(\frac{\sum_{i=1}^m x_i}{m}\right) + f\left(\sum_{i=1}^m x_i\right) - \frac{(m+1)\prod_{i=1}^m f(x_i)}{\sum_{i=1}^m \prod_{k\neq i, 1\leq k\leq m} f(x_k)}\right| \leq \psi(x_1, x_2, \cdots, x_m) \tag{2.8}$$

for all $x_1, x_2, \cdots, x_m \in X$, *where* $\psi : X^m \to Y$ *is a function such that for all* $x_1, x_2, \cdots, x_m \in X$

$$\Psi(x_1, x_2, \cdots, x_m) := \sum_{i=0}^{\infty} m^{i+1}\psi\left(m^i x_1, m^i x_2, \cdots, m^i x_m\right) < \infty.$$

Then there exists a unique reciprocal adjoint mapping $r : X \to Y$ *which satisfies (1.2) and the inequality*

$$|f(x) - r(x)| \leq \Psi(x, x, \cdots, x)$$

for all $x \in X$.

Proof. Replacing $(x_1, x_2, \cdots, x_m)$ by $(x, x, \cdots, x)$ in (2.8) and multiplying m, we obtain

$$|f(x) - mf(mx)| \leq m\psi(x, x, \cdots, x).$$

By the same method of proof in Theorem 2, we complete the proof. □

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