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CONVERGENCE FOR ARRAYS OF ROWWISE NEGATIVELY QUADRANT DEPENDENT RANDOM VARIABLES

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Abstract. In this paper we consider some results on convergence for arrays of rowwise and pairwise negatively quadrant dependent random variables.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variable defined on a fixed probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Lehmann(1966) introduced a simple and natural definition of bivariate dependence: A sequence $\{X_n, n \geq 1\}$ of random variables is said to be pairwise negatively quadrant dependent if for any r_i, r_j and $i \neq j$,

$$P(X_i > r_i, X_j > r_j) \leq P(X_i > r_i)P(X_j > r_j). \tag{1.1}$$

Since the concept of complete convergence was introduced by Hsu and Robbins(1947), there have been many authors who devote the study to complete convergence for sums and weighted sums of random variables. Li et al.(1995) proved the complete convergence for weighted sums of independent and identically distributed random variables, Liang and Su(1999) and Liang(2000) showed complete convergence for weighted sums of negatively associated random variables and Kuczmaszewska (2009) studied complete convergence for arrays of rowwise negatively associated random variables. There are few literature on complete convergence for pairwise negatively quadrant random variables. In this paper we study the complete convergence for arrays of rowwise and pairwise negatively quadrant dependent random variables.

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We close this section with introducing a few lemmas needed in the future part of this paper.

LEMMA 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise negatively quadrant dependent random variables and $\{f_n, n \geq 1\}$ be a sequence of nondecreasing functions. Then $\{f_n(X_n), n \geq 1\}$ is still a sequence of negatively quadrant dependent random variables.*

LEMMA 1.2 (Wu(2006)). *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise negatively quadrant dependent random variables with mean zero and finite second moment. Then*

$$E(\max_{1\leq k\leq n} |\sum_{j=1}^{k} X_j|)^2 \leq (\log_2 n)^2 \sum_{j=1}^{n} EX_j^2. \tag{1.2}$$

In this paper we give some results concerning complete convergence of weighted sums

$$\sum_{i=1}^{b_n} a_{ni}X_{ni},$$

where $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of constants(weighted), $\{X_{ni}, i \geq 1, n \geq 1\}$ is an array of rowwise and pairwise negatively quadrant dependent random variables and $\{b_n, n \geq 1\}$ is an increasing sequence of positive integers.

DEFINITION 1.3. A real valued function $h(x)$, positive and measurable on $[a, \infty)$ for some $a > 0$, is said to be slowly varying if

$$\lim_{x\to 0} \frac{h(\lambda x)}{h(x)} = 1 \text{ for each } \lambda > 0.$$

LEMMA 1.4 (Zhidong and Chan(1985)). *If $h(x) > 0$ is slowly varying function as $x \to \infty$, then*

(a) $\lim_{x\to 0} \frac{h(x+u)}{h(x)} = 1$ for each $u > 0$,

(b) $\lim_{k\to\infty} \sup_{2^k \leq x < 2^{k+1}} \frac{h(x)}{h(2^k)} = 1$,

(c) $c_1 2^{kr} h(\epsilon 2^k) \leq \sum_{j=1}^{k} 2^{jr} h(\epsilon 2^j) \leq c_2 2^{kr} h(\epsilon 2^k)$ *for every $r > 0, \epsilon > 0$, positive integer k and some positive constants c_1 and c_2,*

(d) $c_3 2^{kr} h(\epsilon 2^k) \leq \sum_{j=k}^{\infty} 2^{jr} h(\epsilon 2^j) \leq c_4 2^{kr} h(\epsilon 2^k)$ *for every $r < 0, \epsilon > 0$, positive integer k and some positive constants c_3 and c_4.)*

2. Results

THEOREM 2.1. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise and pairwise negatively quadrant dependent random variables with mean zero and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of positive numbers. Let $\{b_n, n \geq 1\}$ be a non decreasing sequence of positive integers and $\{c_n, n \geq 1\}$ be a sequence of positive numbers. Assume that for some $0 < t < 2$ and any $\epsilon > 0$*

$$\sum_{n=1}^{\infty} c_n (\log_2 b_n)^2 \sum_{j=1}^{b_n} P\{|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}\} < \infty \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} a_{nj}^2 E(X_{nj})^2 I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}] < \infty. \tag{2.2}$$

Then

$$\sum_{n=1}^{\infty} c_n \times \tag{2.3}$$

$$P\{\max_{1 \leq k \leq b_n} |\sum_{j=1}^{k} (a_{nj}X_{nj} - a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}])| > \epsilon b_n^{\frac{1}{t}}\} < \infty.$$

Proof. If the $\sum_{n=1}^{\infty} c_n$ is convergent, then (2.3) holds. Hence we will consider only the case when $\sum_{n=1}^{\infty} c_n$ is divergent. Let

$$\begin{aligned} \tilde{X_{nj}} &= X_{nj} I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}] \\ &\quad + \frac{\epsilon b_n^{\frac{1}{t}}}{a_{nj}} I[a_{nj}X_{nj} \geq \epsilon b_n^{\frac{1}{t}}] - \frac{\epsilon b_n^{\frac{1}{t}}}{a_{nj}} I[a_{nj}X_{nj} \leq -\epsilon b_n^{\frac{1}{t}}], \\ Y_{ni} &= \tilde{X_{ni}} - E\tilde{X_{ni}} \text{ and } T_{nk} = \sum_{i=1}^{k} a_{ni}Y_{ni}. \end{aligned}$$

Since $a_{nj}EX_{nj}I[|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}] = -a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}]$ it follows from (2.1) that, for sufficient large n we have

$$P\{\max_{1 \leq k \leq b_n} |\sum_{j=1}^{k} (a_{nj}X_{nj} - a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}])| > \epsilon b_n^{\frac{1}{t}}\} \tag{2.4}$$

$$\leq \sum_{j=1}^{b_n} P\{|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}\} + \epsilon^{-2} b_n^{-\frac{2}{t}} E(\max_{1\leq k\leq b_n} |T_{nk}|)^2.$$

We estimate

$$EY_{nj}^2 \leq EX_{nj}^2 I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}] + \frac{\epsilon^2 b_n^{\frac{2}{t}}}{a_{nj}^2} P\{|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}\}. \tag{2.5}$$

Thus by (2.4), (2.5) and Lemma 1.2 we get

$$\begin{aligned} P\{\max_{1\leq k\leq b_n} |\sum_{j=1}^{k} a_{nj}X_{nj} - a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}]| > \epsilon b_n^{\frac{1}{t}}\} \\ \leq \quad & \sum_{j=1}^{b_n} P\{|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}\} \\ & + b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \{\sum_{j=1}^{b_n} a_{nj}^2 EX_{nj}^2 I[|a_{nj}X_{nj}| < \epsilon b_n^{\frac{1}{t}}]\} \\ & + (\log_2 b_n)^2 \sum_{j=1}^{b_n} P\{|a_{nj}X_{nj}| > \epsilon b_n^{\frac{1}{t}}\}. \end{aligned} \tag{2.6}$$

It follows from (2.1) that

$$\sum_{n=1}^{\infty} c_n \sum_{j=1}^{b_n} P\{|a_{nj}X_{nj}| \geq \epsilon b_n^{\frac{1}{t}}\} < \infty. \tag{2.7}$$

Hence by (2.1), (2.2), (2.6) and (2.7) we obtain (2.3). □

Corollary 2.2. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise and pairwise negatively quadrant dependent random variables with $EX_{ni} = 0$ for all $i \geq 1$, $n \geq 1$ and $EX_{ni}^2 < \infty$ $1 \leq i \leq b_n$, $n \geq 1$, where $\{b_n, n \geq 1\}$ is a nondecreasing sequence of positive integers and let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of positive numbers. If, for some $\{c_n, n \geq 1\}$ of positive numbers and $0 < t < 2$*

$$\sum_{n=1}^{\infty} c_n b_n^{-\frac{2}{t}} (\log_2 b_n)^2 \sum_{j=1}^{b_n} a_{nj}^2 EX_{nj}^2 < \infty, \tag{2.8}$$

then for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\{\max_{1\leq k\leq b_n} |\sum_{j=1}^{k} a_{nj}X_{nj}| > \epsilon b_n^{\frac{1}{t}}\} < \infty. \tag{2.9}$$

Corollary 2.3. *Let* $\{X_{ni}, i \geq 1, n \geq 1\}$ *be an array of rowwise and pairwise negatively quadrant dependent random variables with mean zeros and finite variances. Let* $\{a_{ni}, i \geq 1, n \geq 1\}$ *be an array of positive numbers satisfying*

$$\sum_{j=1}^{n} a_{nj}^2 E X_{nj}^2 (\log_2 n)^2 = O(n^\delta) \text{ as } n \to \infty \tag{2.10}$$

for some $0 < \delta < 1$. *Then, for any* $\epsilon > 0$ *and* $\alpha \geq \frac{1}{2}$

$$\sum_{n=1}^{\infty} n^{2(\alpha-1)} P\{\max_{1 \leq k \leq n} |\sum_{j=1}^{k} a_{nj} X_{nj}| > \epsilon n^\alpha\} < \infty. \tag{2.11}$$

Proof. By Chebyshev's inequality and (2.10) we have

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{2(\alpha-1)} P\{\max_{1 \leq k \leq n} |\sum_{j=1}^{k} a_{nj} X_{nj}| > \epsilon n^\alpha\} \\ < \ & \epsilon^{-2} \sum_{n=1}^{\infty} n^{2(\alpha-1)} (\log_2 n)^2 \sum_{j=1}^{n} \frac{a_{nj}^2 E|X_{nj}|^2}{n^{2\alpha}} \\ < \ & C \sum_{n=1}^{\infty} n^{-2+\delta} < \infty. \end{aligned}$$

□

Corollary 2.4. *Let* $\{X_{ni}, i \geq 1, n \geq 1\}$ *be an array of rowwise and pairwise negatively quadrant dependent random variables with mean zeros. Assume that there exists a constant* D *such that, for all* $x \geq 0$, $i \geq 1$ *and* $n \geq 1$

$$P\{|X_{ni}| > x\} \geq D P\{D|X| > x\} \tag{2.12}$$

by a random variable X *and* $\{a_{ni}, i \geq 1, n \geq 1\}$ *is a array of constants such that*

$$\lim_{n \to \infty} a_{ni} = 0 \text{ for each } i \geq 1 \tag{2.13}$$

and

$$\sum_{i=1}^{n} |a_{ni}| \leq C \text{ for all } n \geq 1, \text{ and a positive constant C.} \tag{2.14}$$

If for some $0 < t < 2$, $\delta > \frac{1}{t}$

$$\sup_{i \geq 1} |a_{ni}| = O(n^{\frac{1}{t} - \delta}) \text{ and } E|X|^{1+\frac{1}{\delta}} < \infty, \tag{2.15}$$

then for any $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\{\max_{1\le k\le n} |\sum_{j=1}^{k} a_{nj}X_{nj}| > \epsilon n^{\frac{1}{t}}\} < \infty.$$

Proof. Let $c_n = 1,\ b_n = n$ for $n \ge 1$ in Theorem 2.1. Then, by (2.12), (2.13) and (2.15) we have

$$\begin{aligned}
& \sum_{n=1}^{\infty}\sum_{j=1}^{n} P\{|a_{nj}X_{nj}| \ge \epsilon n^{\frac{1}{t}}\} \\
\le\ & \sum_{n=1}^{\infty}\sum_{j=1}^{n} P\{|a_{nj}X| > \frac{\epsilon n^{\frac{1}{t}}}{D}\} \\
\le\ & \sum_{n=1}^{\infty}\sum_{j=1}^{n} P\{|X| \ge C\epsilon n^{\frac{1}{t}}\} \\
\le\ & \sum_{n=1}^{\infty} nP\{C\epsilon n^{\delta} \le |X| < C(n+1)^{\delta}\} \\
\le\ & CE|X|^{\frac{1}{\delta}} < \infty
\end{aligned}$$

and

$$\begin{aligned}
\le\ & C\sum_{n=1}^{\infty} n^{-\frac{2}{t}} \sum_{i=1}^{n} a_{ni}^2 (EX^2 I[|a_{ni}X| < \epsilon n^{\frac{1}{t}}] \\
& + \frac{n^{\frac{2}{t}}}{a_{ni}^2} P\{|a_{ni}X| \ge \epsilon n^{\frac{1}{t}}\}) \\
\le\ & C\sum_{n=1}^{\infty} n^{-\frac{1+\frac{1}{\delta}}{t}} \sum_{i=1}^{n} |a_{ni}|^{1+\frac{1}{\delta}} E|X|^{1+\frac{1}{\delta}} \\
& + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\{|a_{ni}X| \ge \epsilon n^{\frac{1}{t}}\} \\
\le\ & C\sum_{n=1}^{\infty} n^{-\frac{1}{t}-1} E|X|^{1+\frac{1}{\delta}} \sum_{i=1}^{n} |a_{ni}| + CE|X|^{\frac{1}{\delta}} \\
\le\ & C\sum_{n=1}^{\infty} n^{-\frac{1}{t}-1} + CE|X|^{\frac{1}{\delta}} < \infty.
\end{aligned}$$

For each $1 \leq i \leq n$ we have

$$\begin{aligned}
& n^{-\frac{1}{t}} \sum_{j=1}^{n} a_{nj} EX_{nj} I[|a_{nj}X_{nj}| < \epsilon n^{\frac{1}{t}}] \\
\leq \;& C(n^{-\frac{1}{t}} \sum_{j=1}^{n} |a_{nj}| E|X| + \sum_{j=1}^{n} P\{|a_{nj}X| \geq \epsilon n^{\frac{1}{t}}\}) \\
\leq \;& Cn^{-\frac{1}{t}} \to 0 \ as \ n \to \infty.
\end{aligned}$$

Hence the proof is complete by Theorem 2.1. □

COROLLARY 2.5. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise and pairwise negative quadrant dependent random variables and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers. Let $h(x) > 0$ be a slowly varying function as $x \to \infty$, $\alpha > \frac{1}{2}$ and $\alpha r \geq 1$. If for $0 < t < 2$ the following conditions hold for any $\epsilon > 0$*

$$\sum_{n=1}^{\infty} n^{\alpha r-2} (\log n)^2 h(n) \sum_{j=1}^{n} P\{|a_{nj}X_{nj}| \geq \epsilon n^{\frac{1}{t}}\} < \infty, \tag{2.16}$$

$$\sum_{n=1}^{\infty} n^{\alpha r-2-\frac{2}{t}} (\log_2 n)^2 h(n) \sum_{j=1}^{n} a_{nj}^2 EX_{nj}^2 I[|a_{nj}X_{nj}| < \epsilon n^{\frac{1}{t}}], \tag{2.17}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha r-2} h(n) \tag{2.18}$$

$$\times P\{\max_{1 \leq k \leq n} |\sum_{j=1}^{k} a_{nj}X_{nj} - a_{nj}EX_{nj}I[|a_{nj}X_{nj}| < \epsilon n^{\frac{1}{t}}]| > \epsilon n^{\frac{1}{t}}\}.$$

Proof. Let $c_n = n^{\alpha r-2}h(n)$ and $b_n = n$. Then by Theorem 2.1 (2.18) follows. □

COROLLARY 2.6. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise and identically distributed pairwise negative quadrant dependent random variables with $EX_{11} = 0$ and let $h(x) > 0$ be a slowly varying function as $x \to \infty$. If for $\alpha > \frac{1}{2}$, $\alpha r \geq 1$ and $0 < t < 2$, $E|X_{11}|^{\alpha rt} h(|X_{11}|^t) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha r-2} h(n) P\{\max_{1 \leq k \leq n} |\sum_{j=1}^{k} X_{nj}| \geq \epsilon n^{\frac{1}{t}}\} < \infty. \tag{2.19}$$

Proof. It is enough to show that under the assumptions of Corollary 2.6, for $a_{ni} = 1, i \geq 1, n \geq 1$, the conditions (2.16) and (2.17) of Corollary 2.5 hold. Indeed, we see that using Lemma 1.4 we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha r-1} h(n) P\{|X_{11}| \geq \epsilon n^{\frac{1}{t}}\} \\
\leq\ & C \sum_{k=1}^{\infty} (2^k)^{\alpha r} h(2^k) P\{|X_{11}| \geq \epsilon (2^k)^{\frac{1}{t}}\} \\
\leq\ & C \sum_{m=1}^{\infty} P\{\epsilon (2^m)^{\frac{1}{t}} \leq |X_{11}| < \epsilon (2^{m+1})^{\frac{1}{t}}\} \sum_{j=1}^{m} (2^j)^{\alpha r} h(2^j) \\
\leq\ & C \sum_{m=1}^{\infty} (2^m)^{\alpha r} h(2^m) P\{\epsilon (2^m)^{\frac{1}{t}} \leq |X_{11}| < \epsilon (2^{m+1})^{\frac{1}{t}}\} \\
\leq\ & CE|X_{11}|^{\alpha r t} l(|X_{11}|^t) < \infty,
\end{aligned}$$

from which (2.16) is satisfied.

To prove that (2.17) is fulfilled, we first note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha r-1-\frac{2}{t}} h(n) E|X_{11}|^2 I[|X_{11}| < \epsilon n^{\frac{1}{t}}] \\
\leq\ & C \sum_{k=1}^{\infty} (2^k)^{\alpha r-\frac{2}{t}} h(2^k) \int_0^{(2^k)^{\frac{1}{t}}} x^2 dF(x) \\
\leq\ & C \sum_{k=1}^{\infty} (2^k)^{\alpha r-\frac{2}{t}} h(2^k) \sum_{i=1}^{k} \int_{(2^{i-1})^{\frac{1}{t}}}^{(2^i)^{\frac{1}{t}}} x^2 dF(x) \\
\leq\ & C \sum_{m=1}^{\infty} (2^m)^{\alpha r-\frac{2}{t}} h(2^m) \int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}} x^2 dF(x) \\
=\ & C \sum_{m=1}^{\infty} (2^m)^{\alpha r-\frac{2}{t}} \int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}} \frac{h(2 \times 2^{m-1})}{h(|x|^t)} h(|x|^t) x^2 dF(x) \\
=\ & I.
\end{aligned}$$

But by Lemma 1.2, we see that for sufficiently large m

$$I \leq C \sum_{m=1}^{\infty} (2^m)^{\alpha r-\frac{2}{t}} \int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}} h(|x|^t) x^2 dF(x).$$

Then,

$$\begin{aligned}
&\sum_{m=1}^{\infty}(2^m)^{\alpha r-\frac{2}{t}}\int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}} h(|x|^t)x^2dF(x)\\
\leq\ & \sum_{m=1}^{\infty}\int_{(2^{m-1})^{\frac{1}{t}}}^{(2^m)^{\frac{1}{t}}}(|x|^t)^{\alpha r}h(|x|^t)x^2dF(x)\\
=\ & E|X_{11}|^{\alpha rt}h(|X_{11}|^t)<\infty.
\end{aligned}$$

Hence, (2.7) is satisfied. The proof will be completed if we show that for each $1\leq i\leq n$

$$n^{-\frac{1}{t}}i|EX_{11}I[|X_{11}|<\epsilon n^{\frac{1}{t}}]|\to 0 \ as \ n\to\infty.$$

If $\alpha rt<1$, then

$$n^{-\frac{1}{t}}i|EX_{11}I[|X_{11}|<\epsilon n^{\frac{1}{t}}]\leq(\epsilon)^{1-\alpha rt}n^{1-\alpha r}E|X_{11}|^{\alpha rt}\to 0 \ as \ n\to\infty$$

and in the case $\alpha rt\geq 1$, because $EX_{11}=0$

$$\begin{aligned}
& n^{-\frac{1}{t}}i|EX_{11}I[|X_{11}|<\epsilon n^{\frac{1}{t}}]\leq n^{1-\frac{1}{t}}|-EX_{11}I[|X_{11}|\geq\epsilon n^{\frac{1}{t}}]|\\
\leq\ & (\epsilon)^{1-\alpha rt}n^{1-\alpha r}E|X_{11}|^{\alpha rt}\to 0 \ as \ n\to\infty.
\end{aligned}$$

□

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