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AN INTEGRATION FORMULA FOR ANALOGUE OF WIENER MEASURE AND ITS APPLICATIONS

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Abstract. In this note, we will establish the integration formulae for functionals such as $F(x) = \prod_{j=1}^{n} x(s_j)^2$ and $G(x) = \exp\{\lambda \int_0^t x(s)^2 dm_L(s)\}$ in the analogue of Wiener measure space and using our formulae, we will derive some formulae for series.

1. Introduction

From Brown's finding for the moving of small particle in water and Einstein's suggestion for it, Wiener established the existence theorem of a probability measure m_w on $C_0[0,t]$, the space of all real-valued continuous functions on a closed interval $[0,t]$ that vanish at the origin, the so-called Wiener space in 1923 [11]. Since then, the many new problems in mathematics arose from the study of Wiener space and it was one of the starting points of surprising development in mathematics.

But, the theory of Wiener measure space is a theory of a single particle, merely. The author and Dr. Im hoped to make a measure theory on many particles moving along the law of Brownian motion, so we presented the definition and theories of analogue of Wiener measure space since 2002 [3, 4, 5, 6, 7, 8, 9, 10].

The researchers who studied either the Wiener measure space or the analogue of Wiener measure space, met frequently evaluation problems of integral for functionals on these spaces and sometimes, this calculation problems are not easy. In this note, we will derive the integration formulae for functionals such as $F(x) = \prod_{j=1}^{n} x(s_j)^2$ and $G(x) = \exp\{\lambda \int_0^t x(s)^2 dm_L(s)\}$ in the analogue of Wiener measure space and using our formulae, we will derive some formulae for series.

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2. Preliminaries

In this section, we give some notations, definitions and well-known facts which are needed to understand the next main section.

A. We let $(-1)!! = 1!! = 1$, $(2n)!! = (2n)(2n-2)\cdots 2$ and $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$ for a natural number n. Let $\prod_{p=k}^{n} c_p = c_k c_{k+1}\cdots c_n$ whenever $n \geq k$ and let $\prod_{p=k}^{n} c_p = 1$ whenever $n < k$.

B. By the elementary calculus for integral and the properties of Gamma functions, for a positive real number A and for a non-negative integer m, we have the following equality.

$$\begin{aligned}(2.1)\qquad &\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi A}} u^m \exp\{-\frac{(u-u_o)^2}{2A}\} dm_L(u) \\ &= \sum_{k=0}^{[\frac{m}{2}]} \binom{m}{2k} A^k (2k-1)!! u_0^{m-2k} \\ &= \sum_{k=0}^{[\frac{m}{2}]} \frac{m! A^k}{(m-2k)!(2k)!!} u_0^{m-2k}.\end{aligned}$$

Here $[\cdot]$ is the Gauss symbol.

C. Using Dirichilet's integral in [2] and the change of variables theorem, we can show the following equality.

$$\begin{aligned}(2.2)\qquad &\int_{\Delta_n^t} \prod_{j=1}^{n} (s_j - s_{j-1})^{k_j} d(\prod_{j=1}^{n} m_L)(s_1, s_2, \cdots, s_n) \\ &= t^{n+\sum_{j=1}^{n} k_j} \frac{\prod_{j=1}^{n} k_j!}{(n+\sum_{j=1}^{n} k_j)!}.\end{aligned}$$

where $k_1, k_2, \cdots, k_n$ are all non-negative integers, $\Delta_n^t = \{(s_1, s_2, \cdots, s_n) \,|\, 0 < s_1 < s_2 < \cdots < s_n \leq t\}$ and $s_0 = 0$.

D. For a natural number n, we let

$$\begin{aligned}(2.3)\qquad &\sum_{k,n}{}' p(k_1, k_2, \cdots, k_n) \\ &= \sum_{k_n=0}^{1} \sum_{k_{n-1}=0}^{2-k_n} \sum_{k_{n-2}=0}^{3-k_n-k_{n-1}} \cdots \sum_{k_1=0}^{n-\sum_{j=2}^{n} k_j} p(k_1, k_2, \cdots, k_n).\end{aligned}$$

For $1 \le u \le n-1$, k_{n-u} moves from 0 to $(u+1) - \sum_{p=1}^{u} k_{n-(p-1)}$, so $(u+2) - \sum_{p=1}^{u+1} k_{n-(p-1)} = [(u+1) - (\sum_{p=1}^{u} k_{n-(p-1)}) - k_{n-u}] + 1 \ge 1$. Hence $2 - k_n$, $3 - (k_n + k_{n-1})$, $\cdots, n - \sum_{p=2}^{n} k_p$ are all large than or equal 1 which implies that $\sum_{k,n}{}' p(k_1, k_2, \cdots, k_n)$ is well-defined.

E. For a positive real number t, let $C[0,t]$ be the space of all real-valued continuous functions on a closed bounded interval $[0,t]$ with the supremum norm $\| \cdot \|_\infty$. By the Stone-Weierstrass theorem, $(C[0,t], \| \cdot \|_\infty)$ is a real separable Banach space.

Let n be a non-negative integer. For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le t$, let $J_{\vec{t}} : C[0,t] \to \mathbb{R}^{n+1}$ be a function with

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n)) .$$

For B_j $(j = 0, 1, 2, \cdots, n)$ in $\mathcal{B}(\mathbb{R})$, the set of all Borel subsets of $\mathbb{R}$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all intervals. For a non-negative finite Borel measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we let

$$(2.4)\quad m_\varphi(J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)) = \int_{B_0} [\int_{\prod_{j=1}^{n} B_j} W(n+1; \vec{t}; u_0, u_1, \cdots, u_n)\, d\prod_{j=1}^{n} m_L(u_1, \cdots, u_n)]\, d\varphi(u_0)$$

where

$$W(n+1; \vec{t}; u_0, u_1, \cdots, u_n) = (\prod_{j=1}^{n} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}}) exp\{-\frac{1}{2} \sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\} .$$

Then $\mathcal{B}(C[0,t])$, the set of all Borel subsets in $C[0,t]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique positive measure ω_φ on $(C[0,t], \mathcal{B}(C[0,t]))$ such that $\omega_\varphi(I) = m_\varphi(I)$ for all I in $\mathcal{I}$.

Using the change of variables theorem, we can prove the following theorem.

Theorem 2.1. *(The Wiener integration formula) If $f : \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function, then the following equality holds.*

$$\begin{aligned}(2.5)\qquad &\int_{C[a,b]} f(x(t_0), x(t_1), \cdots, x(t_n))\ d\omega_\varphi(x) \\ &\overset{*}{=} \int_{\mathbb{R}^{n+1}} f(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, u_1, \cdots, u_n) \\ &\qquad d(\prod_{j=1}^{n} m_L \times \varphi)((u_1, u_2, \cdots, u_n), u_0)\end{aligned}$$

where $\overset{*}{=}$ means that if one side of 2.5 exists then both sides exist and the two values are equal.

In [4], we can find the following theorem.

Theorem 2.2. *(The translation theorem on $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$) Let h be in $C[0,t]$ and of bounded variation. Let α be in $\mathbb{R}$ and let $x_0(s) = \int_0^s h(u) dm_L(u) + \alpha$ for $0 \le s \le t$. Let $L : C[0,t] \to C[0,t]$ be a function with $L(x) = x + x_0$ and let φ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let φ_α be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\varphi_\alpha(B) = \varphi(B + \alpha)$ for B in $\mathcal{B}(\mathbb{R})$. Then if F is ω_φ-integrable then $F(x + x_0)$ is ω_φ-integrable of x and*

$$\begin{aligned}(2.6)\qquad &\int_{C[0,t]} F(y)\ d\omega_\varphi(y) \\ &= e^{-\frac{1}{2}\|h\|_2^2} \int_{C[0,t]} F(x + x_0)\ e^{-\int_0^t h(u)\ dx(u)} d\omega_{\varphi_\alpha}(x)\ .\end{aligned}$$

In [10], we can see the following theorem.

Theorem 2.3. *If $\alpha < \frac{1}{2}$ and $\int_{\mathbb{R}} \exp\{4\alpha u^2\} d\varphi(u)$ is finite then*

$$(2.7)\qquad \int_{C[0,t]} \exp\{\alpha \sup_{0 \le s \le t} x(s)^2\}\ dm_\varphi(y)$$

is finite.

F. In 1987 [1], Chiang, Chow and Lee presented nice evaluation for Wiener functional as follows ;

Theorem 2.4. *For any real number a,*

$$\begin{aligned}(2.8)\qquad &\int_{C_0[0,1]} \exp\{-\frac{a^2}{2} \int_0^1 x(s)^2 ds\}\ dm_w(x) \\ &= (cosh\ a)^{-\frac{1}{2}}\end{aligned}$$

is finite.

3. The main results

In this section, we investigate the integral of functionals such as $F(x) = (\int_0^t x(s)^2 dm_L(s))^n$ and $G(x) = \exp\{\lambda \int_0^t x(s)^2 dm_L(s)\}$ and we give some corollaries, follows from our results.

LEMMA 3.1. *Let $0 = s_0 < s_1 < s_2 < \cdots < s_n = t$. Suppose u_0^{2n} is φ-integrable. Then*

$$\int_{C[0,t]} \prod_{j=1}^n x(s_j)^2 \, d\omega_\varphi(x) \tag{3.1}$$

$$= \sum_{k,n}{}' \{ \frac{\prod_{l=2}^n \{(l - \sum_{j=n+2-l}^n k_j)(2l - 2\sum_{j=n+2-l}^n k_j - 1)\}}{(n - \sum_{j=1}^n k_j)!(2n - 2\sum_{j=1}^n k_j - 1)!! \prod_{j=1}^n k_j!}$$

$$\times \prod_{j=1}^n (s_j - s_{j-1})^{k_j} \} \int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^n k_j} \, d\varphi(u_0)$$

Proof. By the following equalities, our lemma is proved.

$$\int_{C[0,t]} \prod_{j=1}^n x(s_j)^2 \, d\omega_\varphi(x)$$

$$\overset{(1)}{=} \int_{\mathbb{R}} \{ \int \underset{\mathbb{R}^n}{\cdots} \int [\prod_{j=1}^n 2\pi(s_j - s_{j-1})]^{-\frac{1}{2}} [\prod_{j=1}^n u_j^2] \exp\{-\frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{s_j - s_{j-1}}\}$$

$$d(\prod_{j=1}^n m_L)(u_1, u_2, \cdots, u_n)\} d\varphi(u_0)$$

$$\overset{(2)}{=} \sum_{k_n=0}^{1} \sum_{k_{n-1}=0}^{2-k_n} \cdots \sum_{k_1=0}^{n-\sum_{j=2}^n k_j} \frac{2!}{(2-2k_n)!} \frac{(4-2k_n)!}{(4-2k_n-2k_{n-1})!}$$

$$\cdots \frac{(2n - 2\sum_{j=2}^n k_j)!}{(2n - 2\sum_{j=1}^n k_j)!} \frac{\prod_{j=1}^n (s_j - s_{j-1})^{k_j}}{\prod_{j=1}^n (2k_j)!} \int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^n k_j} \, d\varphi(u_0)$$

$$\overset{(3)}{=} \sum_{k,n}{}' 2! \frac{(4-2k_n)!}{(2-2k_n)!} \frac{(6-2k_n-2k_{n-1})!}{(4-2k_n-2k_{n-1})!} \cdots \frac{(2n - 2\sum_{j=2}^n k_j)!}{(2n-2-2\sum_{j=2}^n k_j)!}$$

$$\times \frac{\prod_{j=1}^n (s_j - s_{j-1})^{k_j}}{(2n - 2\sum_{j=1}^n k_j)! \prod_{j=1}^n (2k_j)!} \int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^n k_j} \, d\varphi(u_0)$$

$$\overset{(4)}{=} \sum_{k,n}{}' \{ \frac{\prod_{l=2}^{n} \{(l - \sum_{j=n+2-l}^{n} k_j)(2l - 2\sum_{j=n+2-l}^{n} k_j - 1)\}}{(n - \sum_{j=1}^{n} k_j)!(2n - 2\sum_{j=1}^{n} k_j - 1)!! \prod_{j=1}^{n} k_j!} \times \frac{\prod_{j=1}^{n}(s_j - s_{j-1})^{k_j}}{\prod_{j=1}^{n}(2k_j - 1)!!} \} \int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n} k_j} \, d\varphi(u_0).$$

Step (1) results from Theorem 2.2. By the equality (1) in B of section 2, we obtain Step (2). Step (3) follows from the notation **D** of section 2. From the elementary calculus, $(2l)! = 2^l l!(2l-1)!!$ and $(2l)!! = 2^l l!$, we have Step (4). □

THEOREM 3.2. *Let $F(x) = (\int_0^t x(s)^2 dm_L(s))^n$ on $C[0,t]$ where n is a natural number. Suppose u_0^{2n} is φ-integrable. Then*

$$(3.2) \quad \int_{C[0,t]} F(x) \, d\omega_\varphi(x) = n! \sum_{k,n}{}' \{ \frac{\prod_{l=2}^{n} \{(l - \sum_{j=n+2-l}^{n} k_j)(2l - 2\sum_{j=n+2-l}^{n} k_j - 1)\}}{(n + \sum_{j=1}^{n} k_j)!(n - \sum_{j=1}^{n} k_j)!(2n - 2\sum_{j=1}^{n} k_j - 1)!!} \times \frac{t^{n+\sum_{j=1}^{n} k_j}}{\prod_{j=1}^{n}(2k_j - 1)!!} \} \int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n} k_j} \, d\varphi(u_0).$$

Proof. For $1 \le i,j \le n$ with $i \ne j$, let $D_{i,j} = \{(s_1, s_2, \cdots, s_n) \in [0,t]^n | s_i = s_j\}$ and let $s_0 = 0$. Then by the Fubini's theorem $D_{i,j}$ is $\prod_{k=1}^{n} m_L$-null set. For a permutation σ on $\{1, 2, \cdots, n\}$, let $\Delta_\sigma = \{(s_1, s_2, \cdots, s_n) \in [0,t]^n | 0 < s_{\sigma(1)} < s_{\sigma(2)} < \cdots < s_{\sigma(n)} \le t\}$ and let $\Delta_n^t = \Delta_I$ where I is an identity permutation on $\{1, 2, \cdots, n\}$. Then $[0,t]^n = [\bigcup_\sigma \Delta_\sigma] \cup [\bigcup_{\substack{0 \le i,j \le n \\ i \ne j}} D_{i,j}]$ and for any permutation σ on $\{1, 2, \cdots, n\}$,

$$\int_{\Delta_n^t} \prod_{j=1}^{n} x(s_j)^2 \, d(\prod_{k=1}^{n} m_L)(s_1, s_2, \cdots, s_n) = \int_{\Delta_\sigma} \prod_{j=1}^{n} x(s_j)^2 \, d(\prod_{k=1}^{n} m_L)(s_1, s_2, \cdots, s_n)$$

holds for all x in $C[0,t]$.

From Tonelli's theorem, Theorem 2.1 and **C** in section 1, we have

$$\begin{aligned}
&\int_{C[0,t]} F(x)\ d\omega_\varphi(x)\\
&= \int_{C[0,t]} \int_{[0,t]^n} \prod_{j=1}^{n} x(s_j)^2\ d(\prod_{j=1}^{n} m_L)(s_1, s_2, \cdots, s_n) d\omega_\varphi(x)\\
&= \int_{C[0,t]} n! \int_{\Delta_n^t} \prod_{j=1}^{n} x(s_j)^2\ d(\prod_{j=1}^{n} m_L)(s_1, s_2, \cdots, s_n) d\omega_\varphi(x)\\
&= n! \int_{\Delta_n^t} \int_{C[0,t]} \prod_{j=1}^{n} x(s_j)^2\ d\omega_\varphi(x) d(\prod_{j=1}^{n} m_L)(s_1, s_2, \cdots, s_n)\\
&= n! \sum_{k,n}{}' \{ \frac{\prod_{l=2}^{n}\{(l - \sum_{j=n+2-l}^{n} k_j)(2l - 2\sum_{j=n+2-l}^{n} k_j - 1)\}}{(n + \sum_{j=1}^{n} k_j)!(n - \sum_{j=1}^{n} k_j)!(2n - 2\sum_{j=1}^{n} k_j - 1)!!}\\
&\quad \times \frac{t^{n+\sum_{j=1}^{n} k_j}}{\prod_{j=1}^{n}(2k_j - 1)!!} \} \int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n} k_j}\ d\varphi(u_0),
\end{aligned}$$

as desired. □

In Theorem 3.2, putting $t = 1$ and $\varphi = \delta_0$, the Dirac measure at the origin 0 in $\mathbb{R}$, ω_φ is the concrete Wiener measure on $C_0[0,t]$, that is, $\omega_\varphi = m_w$, $\int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n} k_j}\ d\varphi(u_0) = 0$ if $n \neq \sum_{j=1}^{n} k_j$ and $\int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n} k_j}\ d\varphi(u_0) = 1$ if $n = \sum_{j=1}^{n} k_j$. So, we have the following corollary.

Corollary 3.3. *Let* $F(x) = (\int_0^1 x(s) dm_L(s))^n$ *on* $C_0[0,1]$. *Then*

$$\begin{aligned}
(3.3)\qquad &\int_{C[0,1]} F(x)\ dm_w(x)\\
&= \frac{1}{2^n(2n-1)!!}\\
&\quad \times \sum_{k,n}{}' \frac{\prod_{l=2}^{n}\{(l - \sum_{j=n+2-l}^{n} k_j)(2l - 2\sum_{j=n+2-l}^{n} k_j - 1)\}}{\prod_{j=1}^{n}(2k_j - 1)!!}
\end{aligned}$$

Theorem 3.4. *Suppose* $\lambda t < \frac{1}{2}$ *and* $\exp\{u^{2n}\}$ *is* φ*-integrable on* $\mathbb{R}$ *for all natural number* n. *Let* $G(x) = \exp\{\lambda \int_0^t x(s)^2 dm_L(s)\}$ *on* $C[0,t]$.

Then $G(x)$ is ω_φ-integrable and

$$\begin{aligned}(3.4)\qquad &\int_{C[0,t]} G(x)\ d\omega_\varphi(x)\\ &= \varphi(\mathbb{R}) + \sum_{n=1}^{\infty} \lambda^n \sum_{k,n}{}' \{ \frac{\prod_{l=2}^{n}\{(l-\sum_{j=n+2-l}^{n} k_j)}{(n+\sum_{j=1}^{n} k_j)!(n-\sum_{j=1}^{n} k_j)!}\\ &\quad\times \frac{(2l-2\sum_{j=n+2-l}^{n} k_j - 1)\}}{(2n-2\sum_{j=1}^{n} k_j - 1)!!\prod_{j=1}^{n}(2k_j-1)!!} t^{n+\sum_{j=1}^{n} k_j} \}\\ &\quad\times \int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n} k_j}\ d\varphi(u_0)\end{aligned}$$

Proof. If $\lambda \le 0$ then clearly $G(x)$ is ω_φ-integrable. Suppose λ is a positive real number. Putting $\lambda t = \alpha$ in Theorem 2.4, since $\exp\{4\lambda t u^2\}$ is φ-integrable,

$$\begin{aligned}&\int_{C[0,t]} G(x)\ d\omega_\varphi(x)\\ &\le \int_{C[0,t]} \exp\{\lambda t \sup_{0\le s\le t} x(s)^2\}\ d\omega_\varphi(x)\end{aligned}$$

is finite, so G is ω_φ-integrable.

Moreover, by the dominated convergence theorem, Theorem3.2 and $\omega_\varphi(C[0,t]) = \varphi(\mathbb{R})$,

$$\begin{aligned}&\int_{C[0,t]} G(x)\ d\omega_\varphi(x)\\ &= \int_{C[0,t]} \sum_{n=0}^{\infty} \frac{1}{n!}(\int_0^t x(s)^2 dm_L(s))^n\ d\omega_\varphi(x)\\ &= \varphi(\mathbb{R}) + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_{C[0,t]} (\int_0^t x(s)^2 dm_L(s))^n\ d\omega_\varphi(x)\\ &= \varphi(\mathbb{R}) + \sum_{n=1}^{\infty} \lambda^n \sum_{k,n}{}' \{ \frac{\prod_{l=2}^{n}\{(l-\sum_{j=n+2-l}^{n} k_j)}{(n+\sum_{j=1}^{n} k_j)!(n-\sum_{j=1}^{n} k_j)!}\\ &\quad\times \frac{(2l-2\sum_{j=n+2-l}^{n} k_j - 1)\}}{(2n-2\sum_{j=1}^{n} k_j - 1)!!\prod_{j=1}^{n}(2k_j-1)!!} t^{n+\sum_{j=1}^{n} k_j} \}\\ &\quad\times \int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^{n} k_j}\ d\varphi(u_0),\end{aligned}$$

as desired. □

In Theorem 2.3 of section 2, putting $h(u)=0$ on $[0,t]$, if F is $\omega_{\varphi_{-\alpha}}$-integrable then $F(x+\alpha)$ is $\omega_{\varphi_{-\alpha}}$-integrable and

$$\int_{C[0,t]} F(x)\ d\omega_{\varphi_{-\alpha}}(x) = \int_{C[0,t]} F(x+\alpha)\ d\omega_{\varphi}(x).$$

Using this, we can prove the following corollary.

Corollary 3.5. *Suppose* $\lambda t < \frac{1}{2}$ *and* $\exp\{u^{2n}\}$ *is* $\varphi_{-\alpha}$*-integrable where* α *is a real number. Let* $G(x)=\exp\{\lambda\int_0^t (x(s)+\alpha)^2 dm_L(s)\}$ *on* $C[0,t]$*. Then* $G(x)$ *is* ω_φ*-integrable and*

$$\begin{aligned}(3.5)\qquad & \int_{C[0,t]} G(x)\ d\omega_\varphi(x) \\ &= \varphi(\mathbb{R}) + \sum_{n=1}^{\infty}\lambda^n {\sum_{k,n}}' \{ \frac{\prod_{l=2}^n\{(l-\sum_{j=n+2-l}^n k_j)}{(n+\sum_{j=1}^n k_j)!(n-\sum_{j=1}^n k_j)!} \\ &\times \frac{(2l-2\sum_{j=n+2-l}^n k_j - 1)\}}{(2n-2\sum_{j=1}^n k_j-1)!!\prod_{j=1}^n(2k_j-1)!!} t^{n+\sum_{j=1}^n k_j}\} \\ &\times \int_{\mathbb{R}} u_0^{2n-2\sum_{j=1}^n k_j}\ d\varphi_{-\alpha}(u_0)\end{aligned}$$

In Theorem3.4, putting $t=1$ and $\varphi=\delta_0$, we have the following corollary by Theorem2.4.

Corollary 3.6. *For any positive real number* λ*.*

$$\begin{aligned}&\int_{C_0[0,1]} \exp\{-\lambda\int_0^1 x(s)^2 dm_L(s)\}\ dm_w(x) \\ &= 1+\sum_{n=1}^{\infty}\frac{\lambda^n}{n!2^n(2n-1)!!} \\ &\times {\sum_{k,n}}' \frac{\prod_{l=2}^n\{(l-\sum_{j=n+2-l}^n k_j)(2l-2\sum_{j=n+2-l}^n k_j-1)\}}{\prod_{j=1}^n(2k_j-1)!!} \\ &= (cosh\sqrt{2\lambda})^{-\frac{1}{2}}.\end{aligned}$$

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