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TWO-SIDED BEST SIMULTANEOUS APPROXIMATION

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Abstract. Let $C_1(X)$ be a normed linear space over $\mathbb{R}^m$, and S be an n-dimensional subspace of $C_1(X)$ with spaned by $\{s_1, \cdots, s_n\}$. For each $\ell-$ tuple vectors F in $C_1(X)$, the two-sided best simultaneous approximation problem is

$$\min_{s\in S} \max_{i=1}^{\ell}\{||f_i - s||_1\}.$$

A $s \in S$ attaining the above minimum is called a two-sided best simultaneous approximation or a Chebyshev center for $F = \{f_1, \cdots, f_\ell\}$ from S. This paper is concerned with algorithm for calculating two-sided best simultaneous approximation, in the case of continuous functions.

1. Introduction

We assume that X is a compact subset of $\mathbb{R}^m$ satisfying $X = \overline{\text{int}X}$, S is an n-dimensional subspace of $C_1(X)$, and μ is any 'admissible' measure on X. For any $l-$tuple $f_1, \cdots, f_\ell$ in $C_1(X)$, we present a algorithm to the solution of our problem of finding a $s^* \in S$ such that

$$\max_{1\le i\le \ell} ||f_i - s^*||_1 \le \max_{1\le i\le \ell} ||f_i - s||_1$$

for any $s \in S$.

Definition 1.1. Suppose that K is a nonempty subset of a normed linear space W. Given any bounded subset $F \subset W$, define

$$d(F, K) := \inf_{k\in K} \sup_{f\in F} ||f - k||_1.$$

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An element $k^* \in K$ is said to be a two-sided best simultaneous approximation or a Chebyshev center for the set F from K if

$$d(F,K) = \sup_{f \in F} ||f - k^*||_1.$$

That mean that a ball of center k^* with radius $d(F,K)$ is the smallest circle containing the set F.

For each positive integer n, define the set

$$\bar{F_n} := \{(\bar{\lambda_n}, \bar{f_n}) : \bar{\lambda_n} = (\lambda_1, \cdots, \lambda_n), \bar{f_n} = (f_1, \cdots, f_n), f_i \in F,$$

$$\lambda_i \geq 0 \ (i = 1, \cdots, n), \ \Sigma_{i=1}^n \lambda_i = 1\}.$$

Then, for any compact set F, $\bar{F_n}$ is a compact set in the product space and we note that for any $k \in K$,

$$\max_{f \in F} ||f - k||_1 = \max_{\bar{F_n}} \Sigma_{i=1}^n \lambda_i ||f_i - k||_1.$$

So $d(F,K) = d(\mathrm{con}(F), K)$ where con(F) denotes the convex hull of F. Furthermore, as a consequence of the above equality we have a remark.

Remark 1.2. For any compact set F and K is a nonempty subset of a normed linear space W, k^* is a best simultaneous approximation for F from K if and only if k^* is a best simultaneous approximation for $\mathrm{con}(F)$ from K.

In the next theorem, we are able to determine criteria for when a finite dimensional subspace. It is the existence of a best simultaneous approximation. For a closed convex set, this problem is much more difficult.

Theorem 1.3. *[7] If K is a finite dimensional subspace of a normed linear space W, then for any compact subset $F \subset W$, there exists a best simultaneous approximation from K.*

Corollary 1.4. *[8] Suppose that K is a closed convex subset of a finite dimensional subspace of a normed linear space W. For any compact subset $F \subset W$, there exists a best simultaneous approximation from K.*

2. Two-sided best simultaneous approximation

This algorithm is based on a discretization of our problem. For each positive integer m, let $x_1^m, \cdots, x_m^m \in X$ and $\delta_1^m, \cdots, \delta_m^m$ be strictly positive numbers such that

$$\lim_{m\to\infty} \sum_{i=1}^{m} \delta_i^m f(x_i^m) = \int_X f d\mu$$

for each $f \in C_1(X)$ where μ is any 'admissible' measure on X. For convenience only, we assume that $\sum_{i=1}^{m} \delta_i^m = \mu(X)$ for all m. The points $\{x_i^m\}_{i=1}^m$ become dense in X. Before proving the convergence of the algorithm, we need to pursue some technical facts.

LEMMA 2.1. *[4] There exists a natural number M_1 where $\{x_1^m, \cdots, x_m^m\}$ is such that for all $m \geq M_1$ we have $dim S|_{\{x_1^m,\cdots,x_m^m\}} = n$.*

For any $f_1, \cdots, f_\ell$ in $C_1(X)$, let $F = (f_1, \cdots, f_\ell)$ and $||F||_1 = ||(f_1, \cdots, f_\ell)||_1 = d(F, \{0\})$. By the remark 1.0.2, we denote $||F||_1 = \max_{a\in A} ||\sum_{i=1}^{\ell} a_i f_i||_1$ where $A = \{a = (a_1, \cdots, a_\ell) : \sum_{i=1}^{\ell} a_i = 1, a_i \geq 0, i = 1, \cdots, \ell\}$. Throughout this article, we assume that $F = (f_1, \cdots, f_\ell)$ in $C_1(X)$ are given and S is an $n-$dimensional subspace of $C_1(X)$. We want to consider the problem of approximating there functions simultaneously by elements in S. In other words, we want to find $s^* \in S$ to minimize

$$||(f_1 - s^*, \cdots, f_\ell - s^*)||_1.$$

For each positive integer m, we denote by

$$\sigma_m := \min_{s\in S} \max_{j=1}^{\ell} \sum_{i=1}^{m} \delta_i^m |f_j(x_i^m) - s(x_i^m)|.$$

Let $s^m \in S$ be a solution to satisfy the above equality.

We first prove the following lemma.

LEMMA 2.2. *There exists a positive integer M_2 such that the set $\{s^m\}_{m\geq M_2}$ are uniformly bounded.*

Proof. By the lemma 2.0.5, there exists a natural number M_1, for all $m \geq M_1$, $\dim S|_{\{x_1^m,\cdots,x_m^m\}} = n$. Since, for each $f \in C_1(X)$, $\lim_{m\to\infty} \sum_{i=1}^{m} \delta_i^m f(x_i^m) = \int_X f d\mu$, and $||F||_1 = ||(f_1, \cdots, f_\ell)||_1 = \max_{a\in A} ||\sum_{i=1}^{\ell} a_i f_i||_1 = \max_{1\leq j\leq \ell} ||f_j||_1$ there exists a natural number M_2, for all $m \geq M_2$,

$$\max_{1\leq j\leq \ell} \sum_{i=1}^{m} \delta_i^m |f_j(x_i^m)| \leq ||F||_1 + 1.$$

Let $M = max\{M_1, M_2\}$. Then, for all $m \geq M$ and any $j \in \{1, \cdots, \ell\}$,

$$\sum_{i=1}^{m} \delta_i^m |s^m(x_i^m)| \leq \sum_{i=1}^{m} \delta_i^m |f_j(x_i^m) - s^m(x_i^m)| + \sum_{i=1}^{m} \delta_i^m |f_j(x_i^m)| \leq 2||F||_1 + 2,$$

since

$$\max_{1 \leq j \leq \ell} \sum_{i=1}^{m} \delta_i^m |f_j(x_i^m) - s^m(x_i^m)| \leq \max_{1 \leq j \leq \ell} \sum_{i=1}^{m} \delta_i^m |f_j(x_i^m) - s(x_i^m)|$$

for all $s \in S$, that is

$$\max_{1 \leq j \leq \ell} \sum_{i=1}^{m} \delta_i^m |f_j(x_i^m) - s^m(x_i^m)| \leq \max_{1 \leq j \leq \ell} \sum_{i=1}^{m} \delta_i^m |f_j(x_i^m)|.$$

Hence there exists a constant $C > 0$ such that

$$\sum_{i=1}^{m} \delta_i^m |s^m(x_i^m)| \leq C$$

for all $m \geq M$.

Let $s^m = \sum_{h=1}^{n} a_h^m s_h$, where $s_1, \cdots, s_n$ is a basis for S. If the $\{s^m\}$ are not uniformly bounded, there exists a subsequence $\{m_k\}$ on which

$$(a) \ |a_r^{m_k}| = \max\{|a_h^{m_k}| : h = 1, \cdots, n\},$$
$$(b) \ \lim_{k \to \infty} |a_r^{m_k}| = \infty.$$

Let $b_h^{m_k} = a_h^{m_k} / a_r^{m_k}$, $h = 1, \cdots, n$. Then the sequence $\{b_h^{m_k}\}$ is bounded, so there exists a subsequence of $\{m_k\}$, again denoted by $\{m_k\}$, on which

$$\lim_{k \to \infty} b_h^{m_k} = b_h, \ h = 1, \cdots, n.$$

Set $\sum_{h=1}^{n} b_h s_h = v$. Since $|b_h^{m_k}| \leq 1 = |b_r^{m_k}|$, $h = 1, \cdots, n$, $v \neq 0$. Thus

$$\begin{aligned} \sum_{i=1}^{m_k} \delta_i^{m_k} |\sum_{h=1}^{n} b_h^{m_k} s_h(x_i^{m_k})| &= \sum_{i=1}^{m_k} \delta_i^{m_k} |\sum_{h=1}^{n} \frac{a_h^{m_k}}{a_r^{m_k}} s_h(x_i^{m_k})| \\ &= \frac{1}{|a_r^{m_k}|} \sum_{i=1}^{m_k} \delta_i^{m_k} |\sum_{h=1}^{n} a_h^{m_k} s_h(x_i^{m_k})| \\ &= \frac{1}{|a_r^{m_k}|} \sum_{i=1}^{m_k} \delta_i^{m_k} |s^{m_k}(x_i^{m_k})| \\ &\leq \frac{C}{|a_r^{m_k}|} \end{aligned}$$

for all $m_k \geq M$. Let $k \to \infty$. The right hand side of the inequality tends to zero, but the left hand side of the inequality tends to

$\int_X |\sum_{h=1}^n b_h s_h| d\mu$. Since $\sum_{h=1}^n b_h s_h \neq 0$, it is a contradiction, which implies the desired result. □

The algorithm is of course given by finding σ^m and each solution s^m at each step and letting m tend to infinity. Next, we prove a main theorem.

THEOREM 2.3. *Let*

$$\sigma_0 = \min_{s \in S}\{\max_{j=1}^{\ell} ||f_j - s||_1\}.$$

Then $\lim_{m\to\infty} \sigma_m = \sigma_0$. *Furthermore, every convergent subsequence of the* $\{s^m\}$ *converges to a* $s^* \in S$ *satisfying* $\max_{j=1}^{\ell} ||f_j - s^*||_1 = \sigma_0$.

Proof. By the theorem 1.0.3, there exists a $s^o \in S$ satisfy $\max_{1\le j\le \ell} ||f_j - s^0||_1 = d(F, S) = \sigma_0$. Then

$$\max_{1\le j\le\ell} \sum_{i=1}^{m} \delta_i^m |f_j(x_i^m) - s^0(x_i^m)| \ge \max_{1\le j\le\ell} \sum_{i=1}^{m} \delta_i^m |f_j(x_i^m) - s^m(x_i^m)| = \sigma_m.$$

Furthermore

$$\lim_{m\to\infty} \max_{1\le j\le\ell} \sum_{i=1}^{m} \delta_i^m |f_j(x_i^m) - s^0(x_i^m)| = \max_{1\le j\le\ell} ||f_j - s^0||_1 = \sigma_0.$$

Therefore $\overline{\lim}_{m\to\infty}\sigma_m \le \sigma_0$. Assume that a subsequence $\{s^{m_k}\}$ satisfy $\lim_{k\to\infty} s^{m_k} = s^*$.

<u>claim</u> $\max_{1\le j\le\ell} ||f_j - s^*||_1 = \sigma_0$.

It is clear, by definition, $\max_{1\le j\le\ell} ||f_j - s^*||_1 \ge \sigma_0$. Since

$$\begin{aligned}\lim_{k\to\infty} \sigma_{m_k} &= \lim_{k\to\infty} \max_{1\le j\le\ell} \{\sum_{i=1}^{m_k} \delta_i^{m_k} |f_j(x_i^{m_k}) - s^{m_k}(x_i^{m_k})|\} \\ &= \max_{1\le j\le\ell} ||f_j - s^*||_1.\end{aligned}$$

Thus $\max_{1\le j\le\ell} ||f_j - s^*||_1 \le \sigma_0$. So $\max_{1\le j\le\ell} ||f_j - s^*||_1 = \sigma_0$.

By claim, s^* is a best simultaneous approximation to F from S. Thus every subsequence $\{m_k\}$ for which $\{s^{m_k}\}$ converge, $\lim_{k\to\infty} \sigma_{m_k} = \sigma_0$. Hence, $\lim_{m\to\infty} \sigma_m = \sigma_0$. We have therefore proven. □

Note that s^* is unique, then

$$\lim_{m\to\infty} s^m = s^*$$

and the convergence is uniform.

Since the best approximation problem is an almost totally general form of a linear programming problem. Many different algorithms exist for solving linear programing problems. The main theorem is bases on discretization to solve a two-sided best simultaneous approximation. Now we will extend the algorithm that involves absolutely no discretization, the study of this problem leads us to the important concepts of gradients and subgradients.

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