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DECOMPOSITION FORMULAS AND INTEGRAL REPRESENTATIONS FOR THE KAMPÉ DE FÉRIET FUNCTION $F^{0:3;3}_{2:0;0}[x,y]$

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Abstract. By developing and using certain operators like those initiated by Burchnall-Chaundy, the authors aim at investigating several decomposition formulas associated with the Kampé de Fériet function $F^{0:3;3}_{2:0;0}[x,y]$. For this purpose, many operator identities involving inverse pairs of symbolic operators are constructed. By employing their decomposition formulas, they also present a new group of integral representations of Eulerian type for the Kampé de Fériet function $F^{0:3;3}_{2:0;0}[x,y]$, some of which include several hypergeometric functions such as ${}_2F_1$, ${}_3F_2$, an Appell function F_3, and the Kampé de Fériet functions $F^{0:3;3}_{2:0;0}$ and $F^{0:2;3}_{1:0;1}$.

1. Introduction and preliminaries

A great interest in the theory of multiple hypergeometric functions (that is, hypergeometric functions of several variables) is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for example, [13], [14], and [17]). In fact, the energy absorbed by some nonferromagnetic conductor sphere included in an internal magnetic field was shown to be calculated with the help of such functions [12]. Multiple hypergeometric functions are used in physical applications as well (cf. [5] and [16]).

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Especially, many problems in gas dynamics lead to solution of degenerate second-order partial differential equations which are then solvable in terms of hypergeometric functions of several variables. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow (see [7]). It is noted that Riemann's functions and the fundamental solutions of the degenerate second-order partial differential equations are expressible by means of multiple hypergeometric functions [8]. In investigation of the boundary-value problems for these partial differential equations, we need decompositions for hypergeometric functions of several variables in terms of simpler hypergeometric functions of (for example) the Gauss and Appell types. Burchnall and Chaundy [2, 3], and Chaundy [4] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the following inverse pairs of symbolic operators:

$$\nabla (h) := \frac{\Gamma (h)\, \Gamma (\delta_1 + \delta_2 + h)}{\Gamma (\delta_1 + h)\, \Gamma (\delta_2 + h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k\, (-\delta_2)_k}{(h)_k\, k!}, \tag{1.1}$$

$$\begin{aligned} \Delta (h) &:= \frac{\Gamma (\delta_1 + h)\, \Gamma (\delta_2 + h)}{\Gamma (h)\, \Gamma (\delta_1 + \delta_2 + h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k\, (-\delta_2)_k}{(1 - h - \delta_1 - \delta_2)_k\, k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\, (h)_{2k}\, (-\delta_1)_k\, (-\delta_2)_k}{(h + k - 1)_k\, (h + \delta_1)_k\, (h + \delta_2)_k\, k!}, \end{aligned} \tag{1.2}$$

and

$$\left(\delta_1 := x \frac{\partial}{\partial x};\ \delta_2 := y \frac{\partial}{\partial y} \right). \tag{1.3}$$

Indeed, as already observed by Srivastava and Karlsson [17], the aforementioned method of Burchnall and Chaundy (cf. [2, 3], see also [4]) was subsequently applied mutatis mutandis by Pandey [10] and Srivastava [15] in order to derive the corresponding expansion and decomposition formulas for the triple hypergeometric functions $F_A^{(3)}$, F_E, F_K, F_M, F_N, F_P, F_T, H_A, H_C respectively (see, for definitions, [17, 19, 20]), and by Singhal and Bhati [13] for deriving analogous multiple-series expansions associated with several multivariable hypergeometric functions. Subsequently, by making use of the Laplace and inverse Laplace transform techniques in conjunction with the principle of multidimensional mathematical induction, Srivastava established several general families of expansion and decomposition formulas for Kampé de Fériet's double hypergeometric function [1, 17]:

$$F_{l:m;n}^{p:q;k}\left[\begin{array}{c}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{array}x,y\right]$$

$$=\sum_{r,s=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_{r+s}\prod\limits_{j=1}^{q}(b_j)_r\prod\limits_{j=1}^{k}(c_j)_s}{\prod\limits_{j=1}^{l}(\alpha_j)_{r+s}\prod\limits_{j=1}^{m}(\beta_j)_r\prod\limits_{j=1}^{n}(\gamma_j)_s\,r!s!}x^ry^s, \tag{1.4}$$

where, for convergence,

(I) $p+q<l+m+1$ or $p+k<l+n+1$

$$|x|<\infty \quad \text{and} \quad |y|<\infty;$$

(II) $p+q=l+m+1$ or $p+k=l+n+1$

$$\begin{cases}|x|^{\frac{1}{p-l}}+|y|^{\frac{1}{p-l}}<1, & \text{if} \quad p>l,\\ \max\{|x|,|y|\}<1, & \text{if} \quad p\leq l.\end{cases}$$

Some other closely related results involving Kampé de Fériet's double hypergeometric function can also be found in the works by Ragab [12] and Verma [20]. Here, by developing and using certain operators like those initiated by Burchnall-Chaundy, the authors aim at investigating several decomposition formulas associated with the Kampé de Fériet function $F_{2:0;0}^{0:3;3}[x,y]$. For this purpose, many operator identities involving inverse pairs of symbolic operators are constructed. By employing their decomposition formulas, they also present a new group of integral representations of Eulerian type for the Kampé de Fériet function $F_{2:0;0}^{0:3;3}[x,y]$, some of which include several hypergeometric functions such as ${}_2F_1$, ${}_3F_2$, an Appell function F_3, and the Kampé de Fériet functions $F_{2:0;0}^{0:3;3}$ and $F_{1:0;1}^{0:2;3}$.

2. A set of operator identities

By applying the pairs of symbolic operators in (1.1) to (1.3), we find the following set of operator identities involving Kampé de Fériet's double hypergeometric functions $F_{l:m;n}^{p:q;k}[x,y]$ (cf. [1, p. 150, Eq. (29)]

and [18, Section 1.3]):

$$F^{0:3;3}_{2:0;0}\left[\begin{array}{rrr} -; & b_1, b_2, b_3; & c_1, c_2, c_3; \\ d_1, d_2; & -; & -; \end{array} x, y\right] = \Delta(d_1)\,\Delta(d_2)\,F^{0:3;3}_{0:2;2}\left[\begin{array}{rrr} -; & b_1, b_2, b_3; & c_1, c_2, c_3; \\ -; & d_1, d_2; & d_1, d_2; \end{array} x, y\right]; \tag{2.1}$$

$$F^{0:3;3}_{2:0;0}\left[\begin{array}{rrr} -; & b_1, b_2, b_3; & c_1, c_2, c_3; \\ d, d_1; & -; & -; \end{array} x, y\right] = \Delta(d_1)\,F^{0:3;3}_{1:1;1}\left[\begin{array}{rrr} -; & b_1, b_2, b_3; & c_1, c_2, c_3; \\ d; & d_1; & d_1; \end{array} x, y\right]; \tag{2.2}$$

$$F^{0:3;3}_{2:0;0}\left[\begin{array}{rrr} -; & b_1, b_2, a; & c_1, c_2, a; \\ d_1, d_2; & -; & -; \end{array} x, y\right] = \Delta(a)\,\Delta(d_1)\,\Delta(d_2)\,F^{1:2;2}_{0:2;2}\left[\begin{array}{rrr} a; & b_1, b_2; & c_1, c_2; \\ -; & d_1, d_2; & d_1, d_2; \end{array} x, y\right]; \tag{2.3}$$

$$F^{0:3;3}_{2:0;0}\left[\begin{array}{rrr} -; & b_1, b_2, a; & c_1, c_2, a; \\ d_1, d_2; & -; & -; \end{array} x, y\right] = \Delta(a)\,\Delta(d_2)\,F^{1:2;2}_{1:1;1}\left[\begin{array}{rrr} a; & b_1, b_2; & c_1, c_2; \\ d_1; & d_2; & d_2; \end{array} x, y\right]; \tag{2.4}$$

$$F^{0:3;3}_{2:0;0}\left[\begin{array}{rrr} -; & b_1, b_2, a; & c_1, c_2, a; \\ d_1, d_2; & -; & -; \end{array} x, y\right] = \Delta(a)\,F^{1:2;2}_{2:0;0}\left[\begin{array}{rrr} a; & b_1, b_2; & c_1, c_2; \\ d_1, d_2; & -; & -; \end{array} x, y\right]; \tag{2.5}$$

$$F^{0:3;3}_{2:0;0}\left[\begin{array}{rrr} -; & b_1, b_2, b; & b_1, b_2, c; \\ d_1, d_2; & -; & -; \end{array} x, y\right] = \Delta(b_1)\,\Delta(b_2)\,\Delta(d_1)\,\Delta(d_2)\,F^{2:1;1}_{0:2;2}\left[\begin{array}{rrr} b_1, b_2; & b; & c; \\ -; & d_1, d_2; & d_1, d_2; \end{array} x, y\right]; \tag{2.6}$$

$$F^{0:3;3}_{2:0;0}\left[\begin{array}{rrr} -; & b_1, b_2, b; & b_1, b_2, c; \\ d_1, d_2; & -; & -; \end{array} x, y\right] = \Delta(b_1)\,\Delta(b_2)\,\Delta(d_2)\,F^{2:1;1}_{1:1;1}\left[\begin{array}{rrr} b_1, b_2; & b; & c; \\ d_1; & d_2; & d_2; \end{array} x, y\right]; \tag{2.7}$$

$$F^{0:3;3}_{2:0;0}\left[\begin{array}{rrr} -; & b_1, b_2, b; & b_1, b_2, c; \\ d_1, d_2; & -; & -; \end{array} x, y\right] = \Delta(b_1)\,\Delta(b_2)\,F^{2:1;1}_{2:0;0}\left[\begin{array}{rrr} b_1, b_2; & b; & c; \\ d_1, d_2; & -; & -; \end{array} x, y\right]; \tag{2.8}$$

$$F_{2:0;0}^{0:3;3}\left[\begin{matrix} -; & b_1,b_2,b_3; & b_1,b_2,b_3; \\ d_1,d_2; & -; & -; \end{matrix} x,y\right]$$
$$= \Delta(b_1)\,\Delta(b_2)\,\Delta(b_3)\,\Delta(d_1)\,\Delta(d_2) \tag{2.9}$$
$$\cdot F_{0:2;2}^{3:0;0}\left[\begin{matrix} b_1,b_2,b_3; & -; & -; \\ -; & d_1,d_2; & d_1,d_2; \end{matrix} x,y\right];$$

$$F_{2:0;0}^{0:3;3}\left[\begin{matrix} -; & b_1,b_2,b_3; & b_1,b_2,b_3; \\ d_1,d_2; & -; & -; \end{matrix} x,y\right]$$
$$= \Delta(b_1)\,\Delta(b_2)\,\Delta(b_3)\,\Delta(d_2)\,F_{1:1;1}^{3:0;0}\left[\begin{matrix} b_1,b_2,b_3; & -; & -; \\ d_1; & d_2; & d_2; \end{matrix} x,y\right]; \tag{2.10}$$

$$F_{2:0;0}^{0:3;3}\left[\begin{matrix} -; & b_1,b_2,b_3; & b_1,b_2,b_3; \\ d_1,d_2; & -; & -; \end{matrix} x,y\right]$$
$$= \Delta(b_1)\,\Delta(b_2)\,\Delta(b_3)\,F_{2:0;0}^{3:0;0}\left[\begin{matrix} b_1,b_2,b_3; & -; & -; \\ d_1,d_2; & -; & -; \end{matrix} x,y\right]. \tag{2.11}$$

In view of the known Mellin-Barnes contour integral representations for the Kampé de Fériet's double hypergeometric function $F_{l:m;n}^{p:q;k}[x,y]$, it is not difficult to give alternative proofs of the operator identities (2.1) to (2.11) above by using the Mellin and the inverse Mellin transformations [1, 8]. The details involved in these alternative derivations of the operator identities (2.1) to (2.11) are being omitted here. It should be noted that Mellin-Barnes integrals have their early history bound up in the study of hypergeometric functions of the late nineteenth and early twentieth centuries. It seems good to introduce a rather recent comprehensive book [11] to describe the theory of these integrals and to illustrate their power and usefulness in asymptotic analysis.

3. Decomposition formulas for Kampé de Fériet function $F_{2:0;0}^{0:3;3}[x,y]$

Making use of the principle of superposition of operators, in particular, the operator identities (2.1) to (2.11), we derive the following seven decomposition formulas for the Kampé de Fériet function$F_{2:0;0}^{0:3;3}[x,y]$:

$$F^{0:3;3}_{2:0;0}\left[\begin{matrix} -; & b_1,b_2,b_3; & c_1,c_2,c_3; \\ d,d_1; & -; & -; \end{matrix}\ x,y\right]$$
$$= \sum_{i=0}^{\infty} \frac{(-1)^i (b_1)_i (b_2)_i (b_3)_i (c_1)_i (c_2)_i (c_3)_i}{(d_1+i-1)_i \left[(d_1)_{2i}\right]^2 i!} x^i y^i \tag{3.1}$$
$$\cdot F^{0:3;3}_{1:1;1}\left[\begin{matrix} -; & b_1+i,b_2+i,b_3+i; & c_1+i,c_2+i,c_3+i; \\ d+2i; & d_1+2i; & d_1+2i; \end{matrix}\ x,y\right];$$

$$F^{0:3;3}_{2:0;0}\left[\begin{matrix} -; & b_1,b_2,a; & c_1,c_2,a; \\ d_1,d_2; & -; & -; \end{matrix}\ x,y\right]$$
$$= \sum_{i=0}^{\infty} \frac{(-1)^{-i-j} (a)_i (a)_{i+j} (b_1)_{i+j} (b_2)_{i+j} (c_1)_{i+j} (c_2)_{i+j} (d_2)_{2i}}{(d_1)_{2i+2j} (d_2+i-1)_i \left[(d_2)_{2i+j}\right]^2 i!j!} x^{i+j} y^{i+j} \tag{3.2}$$
$$\cdot F^{1:2;2}_{1:1;1}\left[\begin{matrix} a+i+j; & b_1+i+j,b_2+i+j; & c_1+i+j,c_2+i+j; \\ d_1+2i+2j; & d_2+2i+j; & d_2+2i+j; \end{matrix}\ x,y\right];$$

$$F^{0:3;3}_{2:0;0}\left[\begin{matrix} -; & b_1,b_2,a; & c_1,c_2,a; \\ d_1,d_2; & -; & -; \end{matrix}\ x,y\right] = \sum_{i=0}^{\infty} \frac{(-1)^i (a)_i (b_1)_i (b_2)_i (c_1)_i (c_2)_i}{(d_1)_{2i} (d_2)_{2i}\, i!} x^i y^i$$
$$\cdot F^{1:2;2}_{2:0;0}\left[\begin{matrix} a+i; & b_1+i,b_2+i; & c_1+i,c_2+i; \\ d_1+2i,d_2+2i; & -; & -; \end{matrix}\ x,y\right]; \tag{3.3}$$

$$F^{0:3;3}_{2:0;0}\left[\begin{matrix} -; & b_1,b_2,b; & b_1,b_2,c; \\ d_1,d_2; & -; & -; \end{matrix}\ x,y\right]$$
$$= \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k} (b_1)_i (b_1)_{i+j+k} (b_1)_k (b_2)_{i+j} (b_2)_{i+j+k} (b)_{i+j+k} (c)_{i+j+k} (d_2)_{2i}}{(d_2+i-1)_i (d_1)_{2i+2j+2k} \left[(d_2)_{2i+j+k}\right]^2 i!j!k!} x^{i+j+k}$$
$$\cdot y^{i+j+k} F^{2:1;1}_{1:1;1}\left[\begin{matrix} b_1+i+j+k,b_2+i+j+k; & b+i+j+k; & c+i+j+k; \\ d_1+2i+2j+2k; & d_2+2i+j+k; & d_2+2i+j+k; \end{matrix}\ x,y\right]; \tag{3.4}$$

$$F^{0:3;3}_{2:0;0}\left[\begin{matrix} -; & b_1,b_2,b; & b_1,b_2,c; \\ d_1,d_2; & -; & -; \end{matrix}\ x,y\right]$$
$$= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} (b_1)_i (b_1)_{i+j} (b_2)_{i+2j} (b)_{i+j} (c)_{i+j}}{(d_1)_{2i+2j} (d_2)_{2i+2j}\, i!j!} x^{i+j} y^{i+j} \tag{3.5}$$
$$\cdot F^{2:1;1}_{2:0;0}\left[\begin{matrix} b_1+i+j,b_2+i+2j; & b+i+j; & c+i+j; \\ d_1+2i+2j,d_2+2i+2j; & -; & -; \end{matrix}\ x,y\right];$$

$$F^{0:3;3}_{2:0;0}\left[\begin{array}{ccc} -; & b_1,b_2,b_3; & b_1,b_2,b_3; \\ d_1,d_2; & -; & -; \end{array} x,y\right]$$
$$=\sum_{i,j,k,l=0}^{\infty}\frac{(-1)^{i+j+k+l}(b_1)_i(b_1)_{i+j+2k+2l}(b_2)_{i+j}(b_2)_{i+j+k+2l}(b_3)_{i+j+k}(b_3)_{i+j+k+l}(d_2)_{2i}}{(d_1)_{2i+2j+2k+2l}(d_2+i-1)_i\left[(d_2)_{2i+j+k+l}\right]^2 i!j!k!l!}$$
$$\cdot x^{i+j+k+l}y^{i+j+k+l}$$
$$\cdot F^{3:0;0}_{1:1;1}\left[\begin{array}{ccccc} b_1+i+j+2k+2l, & b_2+i+j+k+2l, & b_3+i+j+k+l; & -; & -; \\ d_1+2i+2j+2k+2l; & d_2+2i+j+k+l; & d_2+2i+j+k+l; & & \end{array} x,\,y\right]; \tag{3.6}$$

$$F^{3:0;0}_{2:0;0}\left[\begin{array}{ccc} b_1+i+2j+2k,b_2+i+j+2k,b_3+i+j+k; & -; & -; \\ d_1+2i+2j+2k,d_2+2i+2j+2k; & -; & -; \end{array} x,y\right]$$
$$=\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}(b_1)_{i+2j+2k}(b_2)_i(b_2)_{i+2j+2k}(b_3)_{i+j}(b_3)_{i+j+k}}{(d_1)_{2i+2j+2k}(d_2)_{2i+2j+2k}i!j!k!}x^{i+j+k}y^{i+j+k}$$
$$\cdot {}_3F_2\left(b_1+i+2j+2k,b_2+i+j+2k,b_3+i+j+k;\; d_1+2i+2j+2k,d_2+2i+2j+2k;x+y\right). \tag{3.7}$$

Our operational derivations of the decomposition formulas (3.1) to (3.7) would indeed run parallel to those presented in the earlier works [1, 17], which we have already cited in the preceding sections. In addition to the various operator expressions and operator identities listed in Sections 1 and 2, we also make use of the following operator identities [9]:

$$(\delta+\alpha)_n\{f(\xi)\}=\xi^{1-\alpha}\frac{d^n}{d\xi^n}\left\{\xi^{\alpha+n-1}f(\xi)\right\},$$

$$\left(\delta:=\xi\frac{d}{d\xi};\ \alpha\in C;\ n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\ \mathbb{N}:=\{1,2,3,...\}\right)$$

and

$$(-\delta)_n\{f(\xi)\}=(-\xi)^n\frac{d^n}{d\xi^n}\{f(\xi)\},\ \left(\delta:=\xi\frac{d}{d\xi};\ n\in\mathbb{N}_0\right)$$

for every analytic function $f(\xi)$.

4. Integral representations for the Kampé de Fériet $F^{0:3;3}_{2:0;0}[x,y]$

Each of the following integral representations of Eulerian type for the Kampé de Fériet hypergeometric function $F^{0:3;3}_{2:0;0}[x,y]$ holds true:

$$
\begin{aligned}
&F_{2:0;0}^{0:3;3}\left[\begin{array}{rrrr} -; & b_1,b_2,b_3; & c_1,c_2,c_3; & x,y \\ d_1,d_2; & -; & -; & \end{array}\right] \\
&\quad = \frac{\Gamma(b)}{\Gamma(b_1)\,\Gamma(b-b_1)} \int_0^1 \xi^{b_1-1}(1-\xi)^{b-b_1-1} \\
&\qquad\quad \cdot F_{2:0;0}^{0:3;3}\left[\begin{array}{rrrr} -; & b,b_2,b_3; & c_1,c_2,c_3; & x\xi,y \\ d_1,d_2; & -; & -; & \end{array}\right] d\xi \\
&\quad (\Re(b_i)>0;\ \Re(c_i)>0;\ \Re(d_i)>\Re(b_i+c_i)>0;\ i=1,\,2)\,;
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
&F_{2:0;0}^{0:3;3}\left[\begin{array}{rrrr} -; & b_1,b_2,b_3; & c_1,c_2,c_3; & x,y \\ d_1,d_2; & -; & -; & \end{array}\right] \\
&\quad = \frac{\Gamma(d_1)\,\Gamma(d_2)}{\Gamma(b_1)\,\Gamma(b_2)\,\Gamma(c_1)\,\Gamma(c_2)\,\Gamma(d_1-b_1-c_1)\,\Gamma(d_2-b_2-c_2)} \\
&\quad \cdot \int_0^1 \cdots \int_0^1 t_1^{b_1-1} t_2^{b_2-1} t_3^{c_1-1} t_4^{c_2-1} (1-t_1)^{d_1-b_1-c_1-1} \\
&\quad \cdot (1-t_2)^{d_2-b_2-c_2-1}(1-t_3)^{d_1-c_1-1}(1-t_4)^{d_2-c_2-1} \\
&\quad \cdot [1-xt_1t_2(1-t_3)(1-t_4)]^{-b_3}(1-t_3t_4y)^{-c_3}\,dt_1dt_2dt_3dt_4 \\
&\quad (\Re(b_i)>0;\ \Re(c_i)>0;\ \Re(d_i)>\Re(b_i+c_i)>0;\ i=1,\,2)\,;
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
&F_{2:0;0}^{0:3;3}\left[\begin{array}{rrrr} -; & b_1,b_2,b_3; & c_1,c_2,c_3; & x,y \\ d_1,d_2; & -; & -; & \end{array}\right] \\
&\quad = \frac{\Gamma(d_1)\,\Gamma(d_2)}{\Gamma(b_1)\,\Gamma(c_1)\,\Gamma(d_1-b_1)\,\Gamma(d_2-c_1)} \\
&\quad \cdot \int_0^1 \int_0^1 \xi^{b_1-1}\eta^{c_1-1}(1-\xi)^{d_1-b_1-1}(1-\eta)^{d_2-c_1-1} \\
&\quad \cdot F(b_2,b_3;d_2-c_1;x\xi(1-\eta))\,F(c_2,c_3;d_1-b_1;y\eta(1-\xi))\,d\xi d\eta \\
&\qquad\qquad (\Re(d_1)>\Re(a_1)>0;\ \Re(d_2)>\Re(a_2)>0)\,;
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
&F_{2:0;0}^{0:3;3}\left[\begin{array}{rrrr} -; & b_1,b_2,b_3; & c_1,c_2,c_3; & x,y \\ d_1,d_2; & -; & -; & \end{array}\right] \\
&\quad = \frac{\Gamma(d_1)}{\Gamma(b_1)\,\Gamma(d_1-b_1)} \\
&\quad \cdot \int_0^1 \xi^{b_1-1}(1-\xi)^{d_1-b_1-1} F_{1:0;1}^{0:2;3}\left[\begin{array}{rrrr} -; & b_2,b_3; & c_1,c_2,c_3; & x\xi,y(1-\xi) \\ d_2; & -; & d_1-b_1; & \end{array}\right] d\xi \\
&\qquad\qquad (\Re(d_1)>\Re(b_1)>0)\,;
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
&F^{0:3;3}_{2:0;0}\left[\begin{array}{rrr} -; & b_1,b_2,b_3; & c_1,c_2,c_3; \\ d_1,d_2; & ; & ; \end{array} x,y\right] \\
&= \frac{\Gamma(d_1)\Gamma(d_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1)\Gamma(c_2)\Gamma(d_1-b_1-c_1)\Gamma(d_2-b_2-c_2)} \\
&\cdot \int_0^1 \cdots \int_0^1 t_1^{b_1-1} t_2^{b_2-1} t_3^{c_1-1} t_4^{c_2-1} (1-t_1)^{d_1-b_1-c_1-1} \\
&\cdot (1-t_2)^{d_2-b_2-c_2-1} (1-t_3)^{d_1-c_1-1} (1-t_4)^{d_2-c_2-1} \\
&\cdot [1-xt_1t_2(1-t_3)(1-t_4)]^{-b_3} (1-t_3t_4y)^{-c_3} \\
&\cdot {}_2F_1\left(b_3,c_3;d_2;\frac{t_1t_2t_3t_4(1-t_3)(1-t_4)xy}{[1-xt_1t_2(1-t_3)(1-t_4)](1-t_3t_4y)}\right) dt_1dt_2dt_3dt_4
\end{aligned} \tag{4.5}
$$

$$
(\Re(b_i) > 0;\ \Re(c_i) > 0;\ \Re(d_i) > \Re(b_i + c_i) > 0;\ i = 1,\, 2)\,;
$$

$$
\begin{aligned}
&F^{0:3;3}_{2:0;0}\left[\begin{array}{rrr} -; & b_1,b_2,b_3; & c_1,c_2,c_3; \\ d_1,d_2; & -; & -; \end{array} x,y\right] \\
&= \frac{\Gamma(d_1)\Gamma(d_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(d_1-b_1)\Gamma(d_2-b_2)} \\
&\cdot \int_0^1 \int_0^1 \xi^{b_1-1}\eta^{b_2-1}(1-\xi)^{d_1-b_1-1}(1-\eta)^{d_2-b_2-1}(1-x\xi\eta)^{-b_3} \\
&\cdot {}_3F_2(c_1,c_2,c_3;d_1-b_1,d_2-b_2;y(1-\xi)(1-\eta))\,d\xi d\eta
\end{aligned} \tag{4.6}
$$

$$
(\Re(d_i) > \Re(b_i) > 0;\ i = 1,\, 2)\,;
$$

$$
\begin{aligned}
&F^{0:3;3}_{2:0;0}\left[\begin{array}{rrr} -; & b_1,b_2,b_3; & c_1,c_2,c_3; \\ b_1+c_1,d_2; & -; & -; \end{array} x,y\right] \\
&= \frac{\Gamma(b_1+c_1)}{\Gamma(b_1)\Gamma(c_1)} \int_0^1 \xi^{b_1-1}(1-\xi)^{c_1-1} F_3(b_2,c_2,b_3,c_3;d_2;x\xi,y(1-\xi))\,d\xi
\end{aligned} \tag{4.7}
$$

$$
(\Re(b_1) > 0;\ \Re(c_1) > 0)\,.
$$

Proof. Each of the integral representations (4.1) to (4.7) can be proved mainly by expressing the series definition of the involved special function in each integrand and changing the order of the integral sign and the summation, and finally using the following well-known relationship between the Beta function $B(\alpha,\beta)$ and the Gamma function Γ:

$$B(\alpha,\,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha)>0;\ \Re(\beta)>0),\\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-),\end{cases} \tag{4.8}$$

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ denote the set of complex numbers and the set of nonpositive integers, respectively. □

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