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A GENERALIZED SINGULAR FUNCTION

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Abstract. We study a singular function which we call a generalized cylinder convex(concave) function induced from different generalized dyadic expansion systems on the unit interval. We show that the generalized cylinder convex(concave)function is a singular function and the length of its graph is 2. Using a local dimension set in the unit interval, we give some characterization of the distribution set using its derivative, which leads to that this singular function is nowhere differentiable in the sense of topological magnitude.

1. Introduction

P. Billingsley([2]) makes us to arrive at an idea to generalize his singular function from a mono base to a multiple base. In mono base case, he proved that the so-called Takács function is a singular function using that the function is increasing and has non-negative finite derivatives almost everywhere and the positive finite derivative cannot happen. We will use his idea to prove our new definition of a generalized singular function induced from a convex(concave) measure on the unit interval having a generalized dyadic expansion with a multiple base. We also study a local dimension on the unit interval having a generalized dyadic expansion with a multiple base. We apply the results regarding dimensions of distribution sets of unit interval having a generalized dyadic expansion with a multiple base to finding the derivatives of our generalized singular function induced from a convex(concave) measure.

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2. Preliminaries

Let $\mathbb{N}$ be the set of natural numbers. We define F_Z (cf. [1]) *the unit interval* $[0,1]$ *having a generalized dyadic expansion with a base* $Z = \{c_{i_1,\dots,i_n}\}$ where $i_j \in \{0,1\}$ with $j \in \mathbb{N}$ and $0 < c_{i_1,\dots,i_n} < 1$. We note that $c_{i_1,\dots,i_n,1} = 1 - c_{i_1,\dots,i_n,0}$ where $i_j \in \{0,1\}$. Let $I_0 = [0, c_0)$ and $I_1 = [c_0, 1]$. $I_0 = I_{00} \cup I_{01}$ with $I_{00} = [0, c_0c_{00})$ and $I_{01} = [c_0c_{00}, c_0)$. $I_1 = I_{10} \cup I_{11}$ with $I_{10} = [c_0, c_0 + c_1c_{10})$ and $I_{11} = [c_0 + c_1c_{10}, 1]$ with $c_{i_1,\dots,i_n,1} = 1 - c_{i_1,\dots,i_n,0}$. Continuing these processes, we obtain the fundamental intervals $I_{i_1i_2\cdots i_k}$ where $i_j \in \{0,1\}$ and $1 \le j \le k$ for $k \in \mathbb{N}$.

If $x \in F_Z = [0,1]$, then there is a unique code $\sigma \in \{0,1\}^{\mathbb{N}}$ such that $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1i_2\cdots i_k$ where $\sigma = i_1i_2\cdots i_ki_{k+1}\cdots$). We call a code $\sigma \in \{0,1\}^{\mathbb{N}}$ where $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$ *a generalized dyadic expansion with a base* $\{c_{i_1,\dots,i_n}\}$ of x and identify x with the code σ. We sometimes write $[\sigma]_{\{c_{i_1,\dots,i_n}\}}$ for the generalized dyadic expansion with a base $\{c_{i_1,\dots,i_n}\}$ of x to avoid some confusion. We adopt the terminating expansion in the sense that $[0,1]$ is a proper subset of $\{0,1\}^{\mathbb{N}}$. We note that $F_{\frac{1}{2}}$ is the unit interval $[0,1]$ having the dyadic expansion(cf. [3]).

If $x \in F_Z = [0,1]$ and $x \in I_\tau$ where $\tau \in \{0,1\}^k$, a cylinder $c_k(x)$ denotes the fundamental interval I_τ and $|c_k(x)|$ denotes the diameter of $c_k(x)$ for each $k = 0, 1, 2, \dots$.

We define *a base transformed identical code function* $f : F_X \to F_Y$ such that $f([\sigma]_X) = [\sigma]_Y$. More precisely, f carries $x(= [\sigma]_X \in [0,1])$ which has the generalized dyadic expansion σ with a base $X = \{x_{i_1,\dots,i_n}\}$ to $y(= [\sigma]_Y \in [0,1])$ which has the same code σ as its generalized dyadic expansion with a base $Y = \{y_{i_1,\dots,i_n}\}$. Then we easily see that it is a continuous strictly increasing function from $[0,1]$ onto $[0,1]$.

3. Main results

LEMMA 3.1. *Assume that*

$$\limsup_{n\to\infty} \frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < 1$$

and

$$\liminf_{n\to\infty} \frac{y_{i_1,\dots,i_n,1}}{x_{i_1,\dots,i_n,1}} > 1$$

for each $i_1, \dots, i_n, \dots \in \{0,1\}^{\mathbb{N}}$. *Then*

$$0 < \liminf_{n\to\infty} x_{i_1,\dots,i_n,0} \le \limsup_{n\to\infty} x_{i_1,\dots,i_n,0} < 1.$$

Similarly

$$0 < \liminf_{n\to\infty} x_{i_1,\dots,i_n,1} \le \limsup_{n\to\infty} x_{i_1,\dots,i_n,1} < 1.$$

Proof. Assume that $\limsup_{n\to\infty} \frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < 1$. Then there exists real numbers B and C such that

$$\limsup_{n\to\infty} \frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < B < 1 < C < \liminf_{n\to\infty} \frac{y_{i_1,\dots,i_n,1}}{x_{i_1,\dots,i_n,1}}.$$

Then there exists a positive integer N such that

$$\frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < B < 1 < C < \frac{y_{i_1,\dots,i_n,1}}{x_{i_1,\dots,i_n,1}}$$

for all $n \ge N$. Suppose that $\liminf_{n\to\infty} x_{i_1,\dots,i_n,0} = 0$. Then $\sigma = i_1, \dots, i_n, \dots$ such that $\lim_{n\to\infty} x_{i_1,\dots,i_{k_n},0} = 0$. Then $\lim_{n\to\infty} \frac{y_{i_1,\dots,i_{k_n},1}}{x_{i_1,\dots,i_{k_n},1}} = 1$, which is a contradiction. $\limsup_{n\to\infty} x_{i_1,\dots,i_n,0} < 1$ follows from the same arguments. Clearly

$$0 < \liminf_{n\to\infty} x_{i_1,\dots,i_n,0} \le \limsup_{n\to\infty} x_{i_1,\dots,i_n,0} < 1$$

implies $0 < \liminf_{n\to\infty} x_{i_1,\dots,i_n,1} \le \limsup_{n\to\infty} x_{i_1,\dots,i_n,1} < 1$. □

THEOREM 3.2. *Let $f : F_X \to F_Y$ be a base transformed identical code function such that $f([\sigma]_X) = [\sigma]_Y$ where F_X is the unit interval $[0, 1]$ having a generalized dyadic expansion with a base $X = \{x_{i_1,\dots,i_n}\}$ and F_Y is the unit interval $[0, 1]$ having a generalized dyadic expansion with a base $Y = \{y_{i_1,\dots,i_n}\}$. Assume that for each $i_1, \dots, i_n, \dots \in \{0, 1\}^{\mathbb{N}}$, $\frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < 1$ with $\limsup_{n\to\infty} \frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < 1$ and $\liminf_{n\to\infty} \frac{y_{i_1,\dots,i_n,1}}{x_{i_1,\dots,i_n,1}} > 1$. Then $f'(x) = 0$ almost everywhere.*

Proof. We note that $f(x)$ is a strictly increasing function on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. We clearly see that $\frac{y_{i_1,\dots,i_n,1}}{x_{i_1,\dots,i_n,1}} > 1$ from $\frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < 1$. Assume that $0 < f'(x) < \infty$. Then clearly we see that

$$0 < \lim_{n\to\infty} \frac{|f(c_n(x))|}{|c_n(x)|} = f'(x) < \infty.$$

Let $x = [\sigma]_X = x_{i_1,\dots,i_n,\dots}$. Then $f(x) = [\sigma]_Y = y_{i_1,\dots,i_n,\dots}$. Further we see that $|c_{n+1}(x)| = x_{i_1,\dots,i_n,0}|c_n(x)|$ or $|c_{n+1}(x)| = x_{i_1,\dots,i_n,1}|c_n(x)|$ with

$|f(c_{n+1}(x))| = y_{i_1,\dots,i_n,0}|f(c_n(x))|$ or $|f(c_{n+1}(x))| = y_{i_1,\dots,i_n,1}|f(c_n(x))|$ respectively. From the above fact that $0 < f'(x) < \infty$, we see that

$$|\frac{|f(c_{n+1}(x))|}{|f(c_n(x))|} - x_{i_1,\dots,i_n,i_{n-1}}| \to 0$$

as $n \to \infty$. Clearly we see that for each $n \in \mathbb{N}$,

$$\frac{|f(c_{n+1}(x))|}{|f(c_n(x))|} = y_{i_1,\dots,i_n,i_{n+1}},$$

which gives

$$|y_{i_1,\dots,i_n,i_{n+1}} - x_{i_1,\dots,i_n,i_{n+1}}| \to 0$$

as $n \to \infty$. By the above Lemma, we easily see that

$$|\frac{y_{i_1,\dots,i_n,i_{n-1}}}{x_{i_1,\dots,i_n,i_{n-1}}} - 1| \to 0$$

as $n \to \infty$. Since $\limsup_{n\to\infty} \frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < 1$ and $\liminf_{n\to\infty} \frac{y_{i_1,\dots,i_n,1}}{x_{i_1,\dots,i_n,1}} > 1$, there exist real numbers B and C such that

$$\limsup_{n\to\infty} \frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < B < 1 < C < \liminf_{n\to\infty} \frac{y_{i_1,\dots,i_n,1}}{x_{i_1,\dots,i_n,1}}.$$

Then there exists a positive integer N such that

$$\frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < B < 1 < C < \frac{y_{i_1,\dots,i_n,1}}{x_{i_1,\dots,i_n,1}}$$

for all $n \geq N$, which gives a contradiction. It follows from the fact that every increasing function $f(x)$ has a derivatives $f'(x)$ for almost every $x \in [0,1]$([5]). $\square$

We obtain the Riez-Nágy singular function([4]) or the Takács function from the above singular function when we put $X = \{x_{i_1,\dots,i_n}\} = \{\frac{1}{2}\}$ and $Y = \{y_{i_1,\dots,i_n}\} = \{p\}$ where $p < \frac{1}{2}$. We call such an above singular function a generalized cylinder convex function induced from different generalized dyadic expansion systems on the unit interval. The above Theorem holds even though we do not assume that $\frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < 1$.

Corollary 3.3. *Let $f : F_X \to F_Y$ be a base transformed identical code function such that $f([\sigma]_X) = [\sigma]_Y$ where F_X is the unit interval $[0,1]$ having a generalized dyadic expansion with a base $X = \{x_{i_1,\dots,i_n}\}$ and F_Y is the unit interval $[0,1]$ having a generalized dyadic expansion with a base $Y = \{y_{i_1,\dots,i_n}\}$. Assume that for each $i_1,\dots,i_n,\dots \in \{0,1\}^{\mathbb{N}}$, $\limsup_{n\to\infty} \frac{y_{i_1,\dots,i_n,0}}{x_{i_1,\dots,i_n,0}} < 1$ and $\liminf_{n\to\infty} \frac{y_{i_1,\dots,i_n,1}}{x_{i_1,\dots,i_n,1}} > 1$. Then $f'(x) = 0$ almost everywhere. Further the length of its graph is 2.*

Proof. It follows the same arguments of the proof of the above Theorem. The length of its graph is 2 since $f(x)$ is a strictly increasing singular function on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$ from 31.11 of [2]. □

We call such an above singular function a generalized local convex function induced from different generalized dyadic expansion systems on the unit interval. We can also consider a generalized local concave function induced from different generalized dyadic expansion systems on the unit interval and similar results with respect to a generalized local concave function in a parallel way. The following example is a peculiar example to be considered.

EXAMPLE 3.4. *Let* $x_{i_1,\ldots,i_n,0} = \frac{1}{2} + \frac{1}{n+3}$ *and* $y_{i_1,\ldots,i_n,0} = \frac{1}{2} + \frac{1}{2n+4}$ *for any integer integer* $n \geq 0$. *Then* $x_{i_1,\ldots,i_n,1} = \frac{1}{2} - \frac{1}{n+3}$ *and* $y_{i_1,\ldots,i_n,1} = \frac{1}{2} - \frac{1}{2n+4}$ *for any integer integer* $n \geq 0$. *Then the contraction ratios satisfy the condition* $\frac{y_{i_1,\ldots,i_n,0}}{x_{i_1,\ldots,i_n,0}} < 1 < \frac{y_{i_1,\ldots,i_n,1}}{x_{i_1,\ldots,i_n,1}}$. *We clearly see that*

$$\lim_{n\to\infty} y_{i_1,\ldots,i_n,i_{n-1}} = \lim_{n\to\infty} x_{i_1,\ldots,i_n,i_{n+1}} = \frac{1}{2}.$$

REMARK 3.5. If f satisfies local convex or local concave condition for each point in the unit interval, then it is a singular function, so the length of its graph is 2.

References

[1] I. S. Baek, *Derivative of the Riesz-Nágy-Takács function*, Bull. Kor. Math. Soc., to appear.

[2] P. Billingsley, Probabilitity and Measure, *John Wiley and Sons*, 1995.

[3] L. Olsen. *Extremely non-normal numbers*, Math. Proc. Camb. Phil. Soc. **137** (2004), 43-53.

[4] J.Paradís, P. Viader and L. Bilibiloni. *Riez-Nágy singular functions revisited*, J. Math. Anal. Appl. **329** (2007), 592-602.

[5] H. L. Royden, Real Analysis, *Macmillan Publishing Company*, 1988.

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