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DIFFEOMORPHISMS WITH THE STABLY ASYMPTOTIC AVERAGE SHADOWING PROPERTY

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Abstract. Let p be a hyperbolic periodic point of f, and let $\Lambda(p)$ be a closed set which containing p. In this paper, we show that C^1-generically, if $f|_{\Lambda(p)}$ has the C^1-stably asymptotic average shadowing property, then it admits a dominated splitting.

1. Introduction

The notion of the pseudo-orbits very often appears in several branches of the modern theory of dynamical system. For instance, the pseudo-orbit property(shadowing property) usually plays an important role in stability theory. The asymptotic average shadowing property is a special version of the shadowing property. In this paper, we study dominated splitting structure and the asymptotic average shadowing property. In differentiable dynamical system, dominated splitting and partially hyperbolic are investigated by many authors ([2, 3, 5, 7, 8]). Let M be a closed n-dimensional smooth manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \mathrm{Diff}(M)$. For $\delta > 0$, a sequence $\{x_i\}_{i=-\infty}^{\infty}$ in M is called a *δ-average pseudo orbit* of $f \in \mathrm{Diff}(M)$ if there is a natural number $N = N(\delta) > 0$ such that for all $n \geq N$, and $k \in \mathbb{Z}$,

$$\frac{1}{n}\sum_{i=1}^{n} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

It is easy to see that a δ-pseudo orbit is always a δ-average pseudo orbit. We say that f has the *average-shadowing property* or *is average*

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shadowable if for every $\epsilon > 0$, there is a $\delta > 0$ such that every δ-average pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$ is ϵ-shadowed in average by some $z \in M$, that is,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(f^i(z), x_i) < \epsilon.$$

A sequence $\{x_i\}_{i=-\infty}^{\infty}$ in M is called an *asymptotic average pseudo orbit* of f if for $k \in \mathbb{Z}$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{k+i}), x_{k+i+1}) = 0.$$

A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is said to be *asymptotically shadowed in average by the point z in M* if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) = 0.$$

Let Λ be a closed f-invariant set. We say that $f_{|\Lambda}$ has the asymptotic average shadowing property if for every $\epsilon > 0$, there is $\delta > 0$ such that for any asymptotic average pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ of f, ϵ-shadowed in asymptotic average by some $z \in M$.

Let $f \in \mathrm{Diff}(M)$, and let Λ be a closed f-invariant set. A closed f-invariant set $\Lambda \subset M$ is said to be *chain transitive* if for any points $x, y \in \Lambda$ and $\delta > 0$, there exists a δ-pseudo orbit $\{x_i\}_{i=a_\delta}^{b_\delta} \subset \Lambda(a_\delta < b_\delta)$ of f such that $x_{a_\delta} = x$ and $x_{b_\delta} = y$. We say that a chain transitive set is nontrivial if it is not a single periodic orbit.

For any points $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=a_\delta}^{b_\delta}(a_\delta < b_\delta)$ of f such that $x_{a_\delta} = x$ and $x_{b_\delta} = y$. The set $\{x \in M : x \rightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. If we denote the set of periodic points of f by $P(f)$ then $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$. Here $\Omega(f)$ is non-wandering set of f.

Define a relation $\sim$ on $\mathcal{R}(f)$ by $x \sim y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. It is clear that $\sim$ is an equivalent relation on $\mathcal{R}(f)$. The equivalence classes are called the *chain components* of f. Clearly every chain components is a maximal chain transitive set.

We say that Λ is *locally maximal* if there is a compact neighborhood U of Λ such that $\bigcap_{n\in\mathbb{N}} f^n(U) = \Lambda$.

Let $\Lambda \subset M$ be a f-invariant closed set. We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$

such that

$$\|D_xf^n|_{E(x)}\| \cdot \|D_xf^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

We introduce a notion of C^1-stably the asymptotic average shadowing property on a closed set.

DEFINITION 1.1. Let Λ be a closed set of $f \in \mathrm{Diff}(M)$. We say that f has the *C^1-stably asymptotic average shadowing property* on Λ, (or Λ is *C^1-stably asymptotic average shadowabe* of f) if there are a compact neighborhood U of f and a C^1-neighborhood $\mathcal{U}(f)$ of f such that

- $\Lambda(U) = \bigcap_{n\in\mathbb{Z}} f^n(U)$,
- for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g(U)}$ has the asymptotic average shadowing property, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is the *continuation* of Λ.

The following remark gives an equivalent definition of dominated splitting.

REMARK 1.2. Let Λ be a closed f-invariant set. A splitting $T_\Lambda M = E \oplus F$ is called a l-dominated splitting for a positive integer l if E and F are Df-invariant and

$$\|Df^l|_{E(x)}\|/m(Df^l|_{F(x)}) \leq \frac{1}{2},$$

for all $x \in \Lambda$, where $m(A) = \inf\{\|Av\| : \|v\| = 1\}$ denotes the minimum norm of a linear map A.

Let p be a hyperbolic periodic point of f, and let $\Lambda(p)$ be a closed f-invariant set which contains p.

THEOREM 1.3. *Let $\Lambda(p)$ be as the above. C^1-generically, if $f_{|\Lambda(p)}$ has the C^1-stably asymptotic average shadowing property then it admits a dominated splitting.*

2. Proof of Theorem 1.3.

Let M be as before, and let $f \in \mathrm{Diff}(M)$. In this section, we will use the notation of *pre-sink (resp. pre-source)*. A periodic point p is called a *pre-sink (resp. pre-source)* if $Df^{\pi(p)}(p)$ has an multiplicity one eigenvalue equal to $+1$ or -1 and the other eigenvalues has norm less than 1(resp. bigger than 1)

In this section, we show that if a closed set which contains hyperbolic periodic point is asymptotic average shadowable then C^1-generically, it is the homoclinic class. Moreover, we show that it admits a dominated splitting.

Let M be as before, and let $f \in \text{Diff}(M)$. It is well known that if p is a hyperbolic periodic point f with period k then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \text{ and}$$

$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M.

A point $x \in W^s(p) \overline{\pitchfork} W^u(p)$ is called a *homoclinic point* of f. The closure of the homoclinic points of f associated to p is called the *homoclinic class* of f and it is denoted by $H_f(p)$.

LEMMA 2.1. [4, 6] *Let Λ be a closed set of $f \in \text{Diff}(M)$. If $f_{|_\Lambda}$ has the asymptotic average shadowing property then Λ is a chain transitive set.*

Let p be a hyperbolic periodic point of f, and let $C_f(p)$ denote the chain component of f containing p.

LEMMA 2.2. *Let $\Lambda(p)$ be a closed set of f with containing p. If $f_{|_{\Lambda(p)}}$ has the asymptotic average shadowing property then $\Lambda(p)$ is $C_f(p)$.*

Proof. Let p be a hyperbolic periodic point of f, and let $\Lambda(p)$ be a closed set which containing p. It is sufficient to prove that $x \rightsquigarrow p$; i.e., for $\epsilon > 0$, and $x \in \Lambda(p)$ there is a ϵ-chain of f from x to p (for given $\epsilon > 0$, we say that a finite ϵ-pseudo orbit $\{x_i\}_{i=0}^n$ of f is a ϵ-chain from $x_0 = x$ to $x_n = p$.)

For simplicity, assume that p is a fixed point of f. Let $l = \text{diam}(\Lambda(p))$.

At first, we construct asymptotic average pseudo orbit of f. For any $x \in \Lambda(p)$,

$$\begin{aligned}
&x_0 = x, x_1 = x, x_2, = x, x_3 = p,\\
&x_4 = x, x_5 = f(x), x_6 = p, x_7 = p,\\
&x_8 = x, x_9 = f(x), \ldots, x_{11} = f^3(x), x_{12} = p, x_{13} = p, \ldots, x_{15} = p,\\
&\ldots,\\
&x_{2^k} = x, x_{2^k+1} = f(x), \ldots, x_{2^{k+1}-2} = p, x_{2^{k+1}-1} = p,\\
&\ldots
\end{aligned}$$

Then we get an asymptotic average pseudo orbit ξ,

$$\xi = \{x, x, x, p, x, f(x), p, p, x, f(x), f^2(x), f^3(x), p, p, p, p, x, \ldots\}.$$

Thus for $k \in \mathbb{Z}, 2^k \leq n < 2^{k+1}$, and $j \in \mathbb{Z}$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+j}), x_{i+j+1}) < \frac{lk}{2^k}$$

Hence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+j}), x_{i+j+1}) = 0.$$

Since $f_{|_{\Lambda(p)}}$ has the asymptotic average shadowing property, we can choose a point $z \in M$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) = 0.$$

Let $\epsilon > 0$ be as above. Since f is a continuous, choose a $0 < \delta < \epsilon$ such that for any $x, y \in \Lambda(p)$, $d(f(x), f(y)) < \epsilon$ whenever $d(x, y) < \delta$. Then we show that the following: there are infinitely many $k \in \mathbb{N}$ such that $x_{n_k} \in \{x, f(x), f^2(x), \ldots, f^{2^k-1}(x)\}$ and $d(f^{n_k}(z), x_{n_k}) < \delta$. Also, there are infinitely many $j \in \mathbb{N}$ such that $x_{n_j} = p$, and $d(f^{n_j}(z), x_{n_j}) < \delta$.

To show this, assume that there is a $m > 0$ such that for all $k > m$, we know that $d(f^i(z), x_i) \geq \delta$, whenever $x_i \in \{x, f(x), \cdots, f^{2^k-1}(x)\}$.

From the above facts, we can get the following that

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) \geq \delta/2.$$

This is a contradiction by f has the asymptotic average shadowing property on $\Lambda(p)$. Other cases we do the same. The proof is completed.

Therefore, we know that there are infinitely many $k \in \mathbb{N}$ such that $x_{n_k} \in \{x, f(x), f^2(x), \ldots, f^{2^k-1}(x)\}$ and $d(f^{n_k}(z), x_{n_k}) < \delta$. Also, there are infinitely many $j \in \mathbb{N}$ such that $x_{n_j} = p$, and $d(f^{n_j}(z), x_{n_j}) < \delta$.

Thus we can choose i_1 and $i_2 > 0$ such that $n_{i_1} < n_{i_2}$ and $x_{n_{i_1}} \in \{x, f(x), f^2(x), \ldots, f^{2^{i_1}-1}(x)\}$ and $d(f^{n_{i_1}}(z), x_{n_{i_1}}) < \delta$. Also, $x_{n_{i_2}} = p$, and $d(f^{n_{i_2}}(z), p) < \delta$.

By the above arguments, we can choose $k_1 > 0$ and $k_2 > 0$ such that

$$x_{n_{i_1}} = f^{k_1}(x), \text{ and } x_{n_{i_2}} = f^{k_2}(x) = p.$$

We get the ϵ-chain from x to p; i.e.,

$$x, f(x), f^2(x), \ldots, f^{n_{i_1}}(x) = x_{n_{i_1}}, x_{n_{i_2}} = p.$$

Thus if a closed set $\Lambda(p)$ which containing a hyperbolic periodic point p and $f_{|_{\Lambda(p)}}$ has the asymptotic average shadowing property then it is a chain transitive which containing hyperbolic periodic point p, that is, $x \rightsquigarrow p$. Similarly, we show that $p \rightsquigarrow x$. Thus $\Lambda(p) = C_f(p)$. Here $\Lambda(p)$ is the closed set of f containing p. The proof of Lemma 2.2 is completed. □

Let p be a hyperbolic periodic point of f.

LEMMA 2.3. [1] *There exists a residual set $\mathcal{G} \subset \mathrm{Diff}(M)$ such that if $f \in \mathcal{G}$, the chain component $C_f(p)$ is the homoclinic class $H_f(p)$.*

In [5], the authors proved that the number of sinks and sources of a C^1-stably weakly shadowing diffeomorphism f is uniformly bounded in a C^1-neighborhood of f. Then it is a consequence of the main result of [2]. Every C^1-stably asymptotic average shadowing diffeomorphism $f \in \mathrm{Diff}(M)$ has neither pre-sinks nor pre-sources. Therefore, for a C^1-stably asymptotic average shadowing diffeomorphism $g(C^1$-near by $f)$, we show that the number of sinks and sources is bounded from above in a C^1-neighborhood of g.

Let Λ be a closed f-invariant set, and suppose that $f_{|\Lambda}$ is C^1-stably asymptotic average shadowing in U, where U is as in the definition of C^1-stably asymptotic average shadowing property.

LEMMA 2.4. (Lemma 4.1. [5]) *Under the above notation and hypothesis, there are a C^1-neighborhood $\mathcal{U}_0(f)$ of f, $K > 0, \lambda \in (0,1)$ and $m > 0$ such that for any $g \in \mathcal{U}_0(f)$,*

$$\prod_{i=0}^{k-1} \|D_{g^{im}(p)} g^m\| \le K\lambda^k \left(\textit{resp.} \prod_{i=0}^{k-1} \|D_{g^{-im}(p)} g^{-m}\| \le K\lambda^k\right)$$

for any sink (resp. source) $p \in \Lambda_g(U) \cap P(g)$ with $\pi(p) > m$, where $k = [\pi(p)/m]$.

LEMMA 2.5. (Proposition 1. [5]) *Under the notation and hypothesis of Lemma 2.4, let $\mathcal{U}_0(f)$ be given by Lemma 2.4. Then there exists a neighborhood $\mathcal{V}(f) \subset \mathcal{U}_0(f)$ of f such that the number of sinks and sources in $\Lambda_g(U)(g \in \mathcal{V}(f))$ is bounded from above. More precisely, for any $g \in \mathcal{V}(f)$,*

$$\sup\{\sharp \mathrm{sink}(P(g) \cap \Lambda_g(U)\} < \infty, \text{ and}$$

$$\sup\{\sharp \mathrm{source}(P(g) \cap \Lambda_g(U)\} < \infty,$$

where $sink(P(g) \cap \Lambda_g(U)$ (resp. $source(P(g) \cap \Lambda_g(U))$ is the set of sinks(resp. sources) of $g_{|\Lambda_g(U)}$.

The assertion of Theorem 1.3 is directly obtained by the following remarkable result proved in [2].

LEMMA 2.6. (Theorem 1. [2]) *Let $f \in \mathrm{Diff}(M)$, and let $p \in P(f)$ be a hyperbolic saddle. Then*

- *either the homoclinic class $H_f(p)$ of p admits a dominated splitting,*
- *or for any neighborhood U of $H_f(p)$, any $k \in \mathbb{N}$ and any $\mathcal{U}(f)$, there exists $g \in \mathcal{U}(f)$ such that $\sharp sink(P(g) \cap \Lambda_g(U)) \geq k$ or $\sharp source(P(g) \cap \Lambda_g(U)) \geq k$.*

End of the proof of Theorem. Let $f \in \mathcal{G}$. Suppose $f_{|\Lambda}$ is C^1-stably asymptotic average shadowing in U. Then by Lemma 2.1 and 2.2, Λ is a chain transitive set and if Λ having a hyperbolic periodic point p then it is a chain component, say, $C_f(p)$. Since $f \in \mathcal{G}$, $H_f(p) = C_f(p)$. Also, by the Franks' Lemma for any $g \in \mathcal{V}(f) \subset \mathcal{U}(f)$, g has neither pre-sink nor pre-source. By Lemma 2.5 and 2.6, $H_f(p)$ admits a dominated splitting.

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