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FIXED POINT THEOREMS FOR SET–VALUED MAPS IN QUASI–METRIC SPACES

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Abstract. In this paper, we introduce the concept of generalized weak contractivity for set-valued maps defined on quasi metric spaces. We analyze the existence of fixed points for generalized weakly contractive set-valued maps. And we have Nadler's fixed point theorem and Banach's fixed point theorem in quasi metric spaces. We investigate the convergene of iterate schem of the form $x_{n+1} \in Fx_n$ with error estimates.

1. Introduction and Preliminaries

In [1], the authors introduced the concept of weakly contractive maps for single valued maps on Hilbert spaces. This notion is one of generalizations of contractions. They proved the existence of fixed points and confined their theorems to Hilbert spaces. Rhoades [9] extended some of their theorems to arbitrary Banach spaces. In fact, weakly contractive maps are closely related to maps of Boyd and Wong type ones [6] and Reich type ones [9].

Recently, Bae [3] gave the notion of weak contractivity for set-valued maps defined on metric spaces and gave some fixed point theorems for these maps with inwardness or weakly inwardness conditions.

In [4], the authors proved the existence of coincidence points and common fixed points for two single valued maps satisfying generalized weakly contractive conditions. They also constructed modified Mann and Ishikawa iterative scheme which converge to the common fixed points of the two single valued maps mentioned before.

In this paper we give the notions of generalized weakly contractive set-valued maps in quasi-metric spaces, and we give a new fixed point theorems

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for these maps. And then we obtain a fixed point theorems for weakly contractive set-valued maps, and we have Nadler's fixed point theorem and Banach's fixed point theorem in quasi metric spaces. Also, we investigate the convergene of iterate schem of the form $x_{n+1} \in Fx_n$ with error estimates, where F is a weakly contractive set-valued map.

For the convenience, recall the following well known definition of a quasi-metric space.

Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a *quasi-metric* on X if the following are satisfied:
$(m1)$ $d(x, y) \geq 0$ for all $x, y \in X$;
$(m2)$ $d(x, y) = 0$ if and only if $x = y$;
$(m3)$ $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A nonempty set X together with a quasi-metric d is called a *quasi-metric space* and it is denoted by (X, d). Note that the notion of a quasi-metric space is a generalization of the notion of a metric space.

Throughout the paper, unless otherwise specified, X is assumed to be a quasi-metric space with the quasi-metric d.

We know that each quasi-metric d on X generartes a T_0 topology on X. For a quasi-metric d on X, the conjugate quasi metric d^{-1} on X of d is defined by $d^{-1}(x, y) = d(y, x)$. We denote by d^u the metric $d \vee d^{-1}$, that is, $d^u(x, y) = max\{d(x, y), d(y, x)\}$, for all $x, y \in X$.

We denote by $K(X)$ the family of nonempty compact subsets of (X, d^u) and by $C(X)$ the family of nonempty closed subsets of (X, d). Let H_d on $C(X)$ be defined by

$$H_d(A, B) = max\{sup_{a \in A} d(a, B), sup_{b \in B} d(A, b)\}, \quad A, B \in C(X),$$

where $d(a, B) = inf\{d(a, b) : b \in B\}$ and $d(A, b) = inf\{d(a, b) : a \in A\}$. We say that H_d on $K(X)$ is the Hausdorff quasi-pseudometric(see [5,8]).

Let $D_d(A, B) = sup_{a \in A} d(a, B)$. Then obviousely $D_d(A, B) \leq H_d(A, B)$.

A sequence $\{x_n\}$ in X is called *left K-Cauchy* [10] if, for each $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \in \mathbb{N}$ such that

$m \geq n \geq n_0$. A sequence $\{x_n\}$ in X *converges* to some point $x \in X$ if, for each $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $d(x, x_n) < \epsilon$, for all $n \geq n_0$.

X is called *left K-complete* [10,12] if every left K-Cauchy sequence in X is convergent with respect to d. X is called *Smyth-complete* [7,13] if every left K-Cauchy sequence in X is convergent with respect to d^u. Obviousely, We know that every Smyth-complete quasi-metric space is left K-complete. In general, it is known that the converse implication does not hold.

LEMMA 1.1. *Let $A \subset X$. If A is a compact subset of (X, d^u), then it is a closed subset of (X, d). That is, $K(X) \subset C(X)$.*

Proof. Let $\{x_n\}$ be a sequence in A such that $\lim_{n\to\infty} d(x, x_n) = 0$ for some $x \in X$. Since A is a compact subset of (X, d^u), there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $z \in A$ such that $\lim_{k\to\infty} d^u(z, x_{n_k}) = 0$. Thus we have $\lim_{k\to\infty} d(x_{n_k}, z) = 0$. From $(m3)$ we have

$$d(x, z) \leq d(x, x_{n_k}) + d(x_{n_k}, z).$$

Letting $k \to \infty$ in above inequality, we get $x = z$ and $x \in A$. Thus A is a closed subset of (X, d). □

From now on, let $\varphi : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that
(φ_1) $\varphi(0) = 0$,
(φ_2) $0 < \varphi(t) < t$ for each $t > 0$,
(φ_3) for any sequence $\{t_n\}$ of $(0, \infty)$, $\sum_{n=1}^{\infty} \varphi(t_n) < \infty$ implies
$\sum_{n=1}^{\infty} t_n < \infty$.

A set-valued map $F : X \to 2^X$ is called *weakly contractive* [3] if, for each $x, y \in X$,

$$H_d(Fx, Fy) \leq d(x, y) - \varphi(d(x, y)).$$

A single valued map $f : X \to X$ is called *weakly contractive* [11] if, for each $x, y \in X$,

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)).$$

2. Fixed point theorems

A set-valued map $F : X \to 2^X$ is called *generalized weakly contractive* if, for each $x, y \in X$ and $u \in Fx$, there exists a $v \in Fy$ such that

$$d(u,v) \leq d(x,y) - \varphi(d(x,y)). \tag{2.1}$$

In this section, we give a new fixed point theorem for a generalized weakly contractive set-valued map. And then we have a fixed point theorem for a weakly contractive set-valued map and Nadler's fixed point theorem in quasi-metric space.

THEOREM 2.1. *Let (X,d) be a Smyth-complete quasi-metric space. If $F : X \to C(X)$ is a generalized weakly contractive set-valued map, then F has a fixed point in X.*

Proof. Let $x_0 \in X$ and $x_1 \in Fx_0$. By (2.1), there exists a $x_2 \in Fx_1$ such that

$$d(x_1,x_2) \leq d(x_0,x_1) - \varphi(d(x_0,x_1)).$$

Again by (2.1), there exists an $x_3 \in Fx_2$ such that

$$d(x_2,x_3) \leq d(x_1,x_2) - \varphi(d(x_1,x_2)).$$

Continuing this process, we can find a sequence $\{x_n\}$ in X such that for $n = 0, 1, 2 \cdots$

$$x_{n+1} \in Fx_n \text{ and } d(x_{n+1},x_{n+2}) \leq d(x_n,x_{n+1}) - \varphi(d(x_n,x_{n+1})).$$

Thus the sequence $\{d(x_n,x_{n+1})\}$ is nonincreasing and so $\lim_{n\to\infty} d(x_n,x_{n+1}) = l$ for some $l \geq 0$. We now show that $l = 0$. Suppose $l > 0$. Then we have

$$d(x_n,x_{n+1}) \leq d(x_{n-1},x_n) - \varphi(d(x_{n-1},x_n)) \leq d(x_{n-1},x_n) - \varphi(l),$$

and so

$$d(x_{n+N},x_{n+N+1}) \leq d(x_{n-1},x_n) - N\varphi(l),$$

which is a contradiction for N large enough. Thus we have

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

For $m \in \mathbb{N}$ with $m \geq 3$, we have

$$\begin{aligned} & d(x_{m-1}, x_m) \\ & \leq d(x_{m-2}, x_{m-1}) - \varphi(d(x_{m-2}, x_{m-1})) \cdots \\ & \leq d(x_1, x_2) - \varphi(d(x_1, x_2)) - \cdots - \varphi(d(x_{m-2}, x_{m-1})). \end{aligned}$$

Hence we have

$$\sum_{k=1}^{m-2} \varphi(d(x_k, x_{k+1})) \leq d(x_1, x_2) - d(x_{m-1}, x_m).$$

Letting $m \to \infty$ in above inequality, we have

$$\sum_{n=1}^{\infty} \varphi(d(x_n, x_{n+1})) \leq d(x_1, x_2) < \infty$$

which implies

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty \quad \text{by} \quad (\varphi_3).$$

Thus $\{x_n\}$ is a left K-Cauchy sequence in (X, d). Since the space (X, d) is Smyth-complete, there exists a $p \in X$ such that

$$\lim_{n \to \infty} d^u(x_n, p) = 0.$$

Thus we have

$$\lim_{n \to \infty} d(x_n, p) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(p, x_n) = 0.$$

We now show that p is a fixed point of F.

From (2.1) there exists $z_n \in Fp$ such that

$$d(x_{n+1}, z_n) \leq d(x_n, p) - \varphi(d(x_n, p)).$$

Letting $n \to \infty$ in above inequality, we have $\lim_{n \to \infty} d(x_{n+1}, z_n) = 0$. Thus we have

$$d(p, z_n) \leq d(p, x_{n+1}) + d(x_{n+1}, z_n).$$

Letting $n \to \infty$ in above inequality, we have

$$\lim_{n \to \infty} d(p, z_n) = 0.$$

Therefore, $p \in Fp$ because $Fp \in C(X)$. □

EXAMPLE 2.1. Let $X = \{\frac{1}{2^n} : n = 0, 1, 2 \cdots\} \cup \{0\}$ and let $d(x, y) = \begin{cases} y - x & (y \geq x) \\ 2(x - y) & (x > y) \end{cases}$ for all $x, y \in X$. Then (X, d) is a Smyth complete quasi-metric space.

Let $\varphi(t) = \frac{1}{2}t$ for all $t \geq 0$ and let $F : X \to C(X)$ be a set-valued map defined as

$$Fx = \begin{cases} \{\frac{1}{2^{n+1}}, 0\} & (x = \frac{1}{2^n}, n = 0, 1, 2, \cdots), \\ \{0\} & (x = 0). \end{cases}$$

We now show that F satisfies condition (2.1).

Case 1. $x = 0$ and $y = \frac{1}{2^n}(n = 0, 1, 2, \cdots)$.

For $u = 0 \in Fx$ there exists $v = 0 \in Fy$ such that

$$d(u, v) = d(0, 0) = 0 \leq d(x, y) - \varphi(d(x, y)).$$

Case 2. $x = \frac{1}{2^n}(n = 0, 1, 2, \cdots)$ and $y = 0$.

For $u = 0 \in Fx$ there exists $v = 0 \in Fy$ such that

$$d(u, v) = d(0, 0) = 0 \leq d(x, y) - \varphi(d(x, y)).$$

For $u = \frac{1}{2^{n+1}} \in Fx$ there exists $v = 0 \in Fy$ such that

$$d(u, v) = d(\frac{1}{2^{n+1}}, 0) = \frac{1}{2^n} \leq d(x, y) - \varphi(d(x, y)).$$

Case 3. $x = \frac{1}{2^n}$ and $y = \frac{1}{2^m}(m > n)$.

For $u = 0 \in Fx$ there exists $v = 0 \in Fy$ such that

$$d(u, v) = d(0, 0) = 0 \leq d(x, y) - \varphi(d(x, y)).$$

For $u = \frac{1}{2^{n+1}} \in Fx$ there exists $v = \frac{1}{2^{m+1}} \in Fy$ such that

$$d(u, v) = d(\frac{1}{2^{n+1}}, \frac{1}{2^{m+1}}) = \frac{2^m - 2^n}{2^{n+m}} \leq d(x, y) - \varphi(d(x, y)).$$

Case 4. $x = \frac{1}{2^m}$ and $y = \frac{1}{2^n}(m > n)$.

For $u = 0 \in Fx$ there exists $v = 0 \in Fy$ such that

$$d(u, v) = d(0, 0) = 0 \leq d(x, y) - \varphi(d(x, y)).$$

For $u = \frac{1}{2^{m+1}} \in Fx$ there exists $v = \frac{1}{2^{n+1}} \in Fy$ such that

$$d(u, v) = d(\frac{1}{2^{m+1}}, \frac{1}{2^{n+1}}) = \frac{2^m - 2^n}{2^{n+m+1}} \leq d(x, y) - \varphi(d(x, y)).$$

Thus F is a generalized weakly contractive set-valued map and $0 \in F0$.

From Lemma 1.1 we have the next Corollary.

Corollary 2.2. *Let* (X,d) *be a Smyth-complete quasi-metric space. If* $F : X \to K(X)$ *is a generalized weakly contractive set-valued map, then* F *has a fixed point in* X.

Note that the condition (2.2) of the next Corollary 2.3 imply (2.1). Thus from Theorem 2.1(Corollary 2.2) we have the following corollary.

Corollary 2.3. *Let* (X,d) *be a Smyth-complete quasi-metric space. If* $F : X \to C(X)$*[or* $F : X \to K(X)$*] is a set-valued map satisfying for each* $x, y \in X$

$$D_d(Fx, Fy) \le d(x,y) - \varphi(d(x,y)), \tag{2.2}$$

then F *has a fixed point in* X.

Corollary 2.4. *Let* (X,d) *be a Smyth-complete quasi-metric space. If* $F : X \to C(X)$*[or* $F : X \to K(X)$*] is a weakly contractive set-valued map, then* F *has a fixed point in* X.

Corollary 2.5. *Let* (X,d) *be a Smyth-complete quasi-metric space. If* $f : X \to X$ *is a weakly contractive map, then* f *has a unique fixed point in* X.

Proof. From Corollary 2.4 there exists a point $p \in X$ such that $p = fp$. We show the uniqueness of the fixed point p of f. Let $z \in X$ be such that $z = fz$. If $z \neq p$, then $d(p,z) > 0$ and $d(p,z) = d(fp, fz) \le d(p,z) - \varphi(d(p,z)) < d(p,z)$ which is a contradiction. Thus we have $p = z$. □

Remark 2.1. In Theorem 2.1 $\sim$ Corollary 2.4, if the map is single valued then it has a unique fixed point.

Let $\phi : [0,\infty) \to [0,\infty)$ be a function satisfying the following conditions:
(ϕ_1) $\phi(0) = 0$ and $0 < \phi(t) < t$ for each $t > 0$,
(ϕ_2) $t \le s$ implies $\phi(s) - \phi(t) \le s - t$,
(ϕ_3) for any sequence $\{t_n\}$ of $(0,\infty)$, $\sum_{n=1}^{\infty}(t_n - \phi(t_n)) < \infty$ implies $\sum_{n=1}^{\infty} t_n < \infty$.

Let $\varphi(t) = t - \phi(t)$. Then $\varphi : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $(\varphi_1) \sim (\varphi_3)$. Thus from Theorem 2.1 we have the next corollary.

Corollary 2.6. *Let (X, d) be a Smyth-complete quasi-metric space and let $F : X \to C(X)$ be set-valued map satisfying*

for each $x, y \in X$ and $u \in Fx$, there exists an $v \in Fy$ such that

$$d(u, v) \leq \phi(d(x, y)).$$

Then F has a fixed point in X.

By Corollary 2.5, we have the following corollary.

Corollary 2.7. *Let (X, d) be a Smyth-complete quasi-metric space. If $f : X \to X$ is a map satisfying*

for each $x, y \in X$,

$$d(fx, fy) \leq \phi(d(x, y)),$$

then f has a unique fixed point in X.

Remark 2.2. In Corollary 2.2 $\sim$ Corollary 2.5, if we have $\varphi(t) = t - \phi(t)$ then conclusions are still satisfied.

In particular, if we have $\varphi(t) = t - kt$ for some $0 \leq k < 1$ in Corollary 2.4[resp. Corollary 2.5] then we have Nadler's fixed point theorem(Corollary 2.8)[resp. Banach's fixed point theorem(Corollary 2.9)] in quasi metric spaces.

Corollary 2.8. *Let (X, d) be a Smyth-complete quasi-metric space and let $F : X \to C(X)$ be a set-valued map. If there exists a $k \in [0, 1)$ such that for each $x, y \in X$*

$$H_d(Fx, Fy) \leq kd(x, y),$$

then F has a fixed point in X.

Corollary 2.9. *Let (X, d) be a Smyth-complete quasi-metric space and let $f : X \to X$ be a map. If there exists a $k \in [0, 1)$ such that, for each $x, y \in X$*

$$d(fx, fy) \leq kd(x, y),$$

then f has a unique fixed point, $p \in X$ and for each $x \in X$

$\lim_{n\to\infty} d(p, f^n(x)) = 0.$

Theorem 2.10. *Let (X, d) be a Smyth-complete quasi-metric space and $F : X \to K(X)$ be a weakly contractive set-valued map and let $p \in Fp$. If $x_{n+1} \in Fx_n$ with $d(p, x_{n+1}) = d(p, Fx_n)$, then $\lim_{n\to\infty} d(p, x_n) = 0$ with the following error estimate:*

$$d(p, x_{n+1}) \leq d(p, x_1) - \sum_{k=1}^{n} \varphi(d(p, x_k)).$$

Proof. By assumption, we have

$$\begin{aligned}
&d(p, x_{n+1})\\
&= d(p, Fx_n)\\
&\leq H_d(Fp, Fx_n)\\
&\leq d(p, x_n) - \varphi(d(p, x_n))\\
&\leq d(p, x_{n-1}) - \varphi(d(p, x_{n-1})) - \varphi(d(p, x_n))\\
&\dots\\
&\leq d(p, x_1) - \sum_{k=1}^{n} \varphi(d(p, x_k)).
\end{aligned}$$

We now show that $\lim_{n\to\infty} d(p, x_n) = 0$.

Since the sequence $\{d(p, x_n)\}$ is nonincreasing, there exists an $l \geq 0$ such that

$$\lim_{n\to\infty} d(p, x_n) = l.$$

If $l > 0$ then we have

$$d(p, x_{n+1}) \leq d(p, x_n) - \varphi(d(p, x_n)) \leq d(p, x_n) - \varphi(l)$$

and so

$$d(p, x_{n+N+1}) \leq d(p, x_n) - N\varphi(l),$$

which is a contradiction for N large enough. Thus $l = 0$ and $\lim_{n\to\infty} d(p, x_n) = 0$. □

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