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THE STRUCTURE OF THE RADICAL OF THE NON SEMISIMPLE GROUP RINGS

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Abstract. It is well known that the group ring $K[G]$ has the non-trivial Jacobson radical if K is a field of characteristic p and G is a finite group of which order is divided by a prime p. This paper is concerned with the structure of the Jacobson radical of such a group ring.

1. Introduction

By Maschke's theorem [4], the group ring $K[G]$ of a finite group G over a field K is semisimple if and only if K is of characteristic 0 or the characteristic of K is p and G is a finite group of which order is not divided by p, where p is a prime. Thus, if K is a field of characteristic p and the order of a group G is divided by p, then the Jacobson radical of the group ring $K[G]$ contains a non-zero element. The purpose of this paper is to determine the structure of the Jacobson radical of such a group ring. The following is the main theorem.

Theorem. *Let G be a group of order $p^a b$ and $(p,b) = 1$ and let K be a field of characteristic p. Assume that G has a normal Sylow $p-$subgroup H. Then $JK[G] = \sum_{x\in H-\{1\}} K[G](x-1)$ and $\dim_K JK[G] = b(p^a - 1)$, where $JK[G]$ is the Jacobson radical of the group ring $K[G]$.*

2. Preliminaries

Let R denote a ring, and let $\circ$ be a binary operation on R defined by $a \circ b = a + b - ab$. An element a of R is said to be right quasi-

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regular if there exists an element b of R such that $a \circ b = 0$. A right ideal of R is said to be right quasi-regular if each of its elements is right quasi-regular. The Jacobson radical $J(R)$ is the set of all element a of R such that aR is right quasi-regular. If $J(R) = \{0\}$ then R is called a semisimple ring.

PROPOSITION 2.1. *([5]) Let R be a ring. The Jacobson radical $J(R)$ of R contains every nil right (or left) ideals of R.*

PROPOSITION 2.2. *Let R and S be rings. If $f : R \to S$ is a ring epimorphism then $f\big(J(R)\big) \subseteq J(S)$.*

Proof. Let $a \in J(R)$ and $s \in S$. Then there exists $r \in R$ such that $s = f(r)$. On the other hand, $ar \circ b = 0$ for some $b \in R$ since $ar \in aR$ and aR is right quasi- regular. Hence

$$f(a)s \circ f(b) = f(a)f(r) \circ f(b) = f(ar \circ b) = f(0) = 0,$$

and so $f(a)s$ is right quasi-regular. Because s is an arbitrary element of S, $f(a)S$ is right quasi-regular. Thus $f\big(J(R)\big) \subseteq J(S)$. □

A ring R is said to be right [resp. left] Artinian if any non empty set of right [resp. left] ideals of R has a minimal element. R is said to be Artinian if R is both right and left Artinian.

In the Artinian case, the following propositions are hold.

PROPOSITION 2.3. *([1]) Let R be an Artinian ring. Then R is semisimple if and only if every submodule of R_R is a direct summand, where R_R is a right R-module R.*

PROPOSITION 2.4. *([2]) Let R be an Artinian ring. Then*
(1) $J(R)$ is a nilpotent ideal of R.
(2) Any nil right (or left) ideal of R is nilpotent.

3. The structure of the Jacobson radical of group rings

Let K be a field and let G be a multiplicative group. Then the group ring $K[G]$ is an associative K-algebra with the elements of G as a basis and with addition and multiplication defined by

$$\alpha + \beta = \textstyle\sum_{g \in G}(a_g + b_g)g, \quad \alpha\beta = \sum_{g,h \in G} a_g b_h gh,$$

respectively, where $\alpha = \sum_{g\in G} a_g g$ and $\beta = \sum_{h\in G} b_h h$ $(a_g, b_h \in K,\ g, h \in G)$ are elements of $K[G]$.

From now on, assume that K is a field of characteristic p and G is a finite group of which order is divided by p, where p is a prime. Then the Jacobson radical $JK[G]$ of group ring $K[G]$ is non-trivial by Maschke's theorem [4]. In fact, the following theorem holds.

THEOREM 3.1. *Let G be a finite group and K be a field of characteristic p. If the order of G is divided by p, then*

$$K[G]\alpha \subseteq JK[G], \qquad 1 \leq dim_K JK[G] \leq |\, G\,| -1,$$

where $\alpha = \sum_{g\in G} g$.

Proof. Let $|\, G\,| = n$. Then $\alpha \neq 0$ and $\alpha g = \alpha = g\alpha$ for all $g \in G$. Hence α is a non-zero central element of $K[G]$ and

$$\alpha^2 = (\textstyle\sum_{g\in G} g)\alpha = n\alpha = 0,$$

so that $K[G]\alpha = \alpha K[G]$ is a non-trivial nil ideal of $K[G]$. Thus, by Proposition 2.1, $K[G]\alpha \subseteq JK[G]$.

Moreover, since $K[G] = \oplus_{g\in G} Kg$ and $g\alpha = \alpha$ for all $g \in G$, we have

$$K[G]\alpha = \oplus_{g\in G} Kg\alpha = K\alpha.$$

Thus $\dim_K K[G]\alpha = 1$, and so $1 \leq \dim_K JK[G] \leq |\, G\,| -1$. □

Let $\rho : K[G] \to K$ be the K-algebra homomorphism defined by $\rho(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g$. The kernel

$$\omega(K[G]) = \{\, \textstyle\sum_{g\in G} a_g g \in K[G] \mid \sum_{g\in G} a_g = 0\}$$

of ρ is called the augmentation ideal of $K[G]$. Since ρ is a K-algebra epimorphism, $K[G]/\omega(K[G])$ is isomorphic to K as a K-algebra and so $\dim_K \omega(K[G]) = |\, G\,| -1$. In fact $\omega(K[G])$ has a K-basis $\{x - 1 \mid x \in G,\ x \neq 1\}$. Thus

$$\omega(K[G]) = \oplus_{x\in G-\{1\}} K(x-1).$$

More generally, suppose H is a normal subgroup of a group G. Then the map $\rho_H : K[G] \to K[G/H]$ defined by

$$\rho_H(\textstyle\sum_{g\in G} a_g g) = \sum_{g\in G} a_g \bar{g},$$

where $\bar{g} = gH$ in G/H, is a K-algebra homomorphism and it is easy to show that

$$\ker\, \rho_H = K[G]\omega(K[G]).$$

In the following theorem, we explicitly determine the structure of the Jacobson radical of the group ring concerned with a finite p-group, and this result will be generalized in Theorem 3.5.

THEOREM 3.2. *Let G be a finite p-group with $\mid G \mid = p^a$, where $a \geq 1$, and let K be a field of characteristic p. Then*

$$JK[G] = \omega(K[G]) = \textstyle\sum_{x\in G-\{1\}} K(x-1), \qquad dim_K JK[G] = p^a - 1.$$

Proof. Since $K[G]/\omega(K[G])$ is isomorphic to K as a K-algebra, the augmentation ideal $\omega(K[G])$ is maximal in $K[G]$. Hence $JK[G] \subseteq \omega(K[G])$.

To prove the reverse inclusion $\omega(K[G]) \subseteq JK[G]$, by Proposition 2.1, it suffices to show that $\omega(K[G])$ is a nil ideal of $K[G]$. We will prove this by induction on a.

If $\mid G \mid = p$ then it is easy to show that $\omega(K[G])^p = 0$, and so $\omega(K[G])$ is a nil ideal of $K[G]$.

Assume that the assertion holds for a group of order p^a, and let G be a p-group with $\mid G \mid = p^{a+1}$. Now let H be the subgroup of G with $\mid H \mid = p$ which is contained in the center of G, and let $\rho_H : K[G] \to K[G/H]$ be the K-algebra homomorphism defined by

$$\rho_H(\textstyle\sum_{g\in G} a_g g) = \sum_{g\in G} a_g \bar{g}.$$

Then, $\rho_H\big(\omega(K[G])\big) \subseteq \omega(K[G/H])$ and $\omega(K[G/H])$ is a nil ideal of $K[G/H]$ by inductive hypothesis.

Suppose that α is any element of $\omega(K[G])$. Then $\rho_H(a^m) = \rho_H(\alpha)^m = 0$ for some $m > 0$ since $\omega(K[G])$ is a nil ideal, which implies that α^m is contained in the kernel $K[G]\omega(K[H])$ of ρ_H. Since $\omega(K[G])$ is nilpotent by the above and since it is naturally central in $K[G]$, we see that $K[G]\omega(K[H])$ is a nil ideal, so that α is a nilpotent element. Thus $\omega(K[G])$ is a nil ideal, the former insistence follows.

Clearly,

$$\omega(K[G]) = \textstyle\sum_{x\in G-\{1\}} K(x-1),$$
$$\dim_K JK[G] = \dim_K \omega(K[G]) = p^a - 1. \qquad \square$$

The following two lemmas are crucial tools for the proof of our main theorem.

Lemma 3.3. *Let G be a finite group which contains normal Sylow p-subgroup H and let K be a field of characteristic p. Then*

$$K[G]JK[H] = JK[H]K[G],$$

and thus $K[G]JK[H]$ is a nilpotent ideal of $K[G]$.

Proof. Certainly, $JK[H]K[G]$ is closed under left multiplication by K. Thus we will show that it is closed under left multiplication by G.

But if $g \in G$, then $gJK[H]K[G] = gJK[H]g^{-1}gK[G] = g\omega(K[G])g^{-1}gK[G]$

$\subseteq \omega(K[H])gK[G] = JK[H]K[G]$ since $JK[H] = \omega(K[H])$ by Theorem 3.2 and since H is normal in g. Thus we have $K[G]JK[H] \subseteq JK[H]K[G]$. By symmetry, the reverse inclusion also holds. Hence $K[G]JK[H] = JK[H]K[G]$ and it is an ideal of $K[G]$. Furthermore, since $JK[H]$ is nilpotent by Proposition 2.4(1), it follows that $K[G]JK[H]$ is nilpotent. □

Lemma 3.4. *Let G be a group with $\mid G \mid = p^a b$ and $(p,b) = 1$ and let K be a field of characteristic p. If G has a normal Sylow p-subgroup H, then $K[G/H]$ is a semisimple ring.*

Proof. By Proposition 2.3, it is sufficient to prove that if $M \neq \{0\}$ is a right ideal of $K[G/H]$ then $K[G/H] = M \oplus N$ as a $K[G/H]$-module for some right ideal N of $K[G/H]$.

Since M is a K-subspace of $K[G/H]$ and since $\dim_K M < \infty$, there exists a K-subspace N of $K[G/H]$ such that $K[G/H] = M \oplus N$ as a K-space. Thus, there is, of course, the canonical projection μ of $K[G/H]$ onto M, which is a K-homomorphism.

Define $\mu^* : K[G/H] \longrightarrow K[G/H]$ by

$$\mu^*(\alpha) = \frac{1}{b} \textstyle\sum_{\bar{x} \in G/H} \mu(\alpha \cdot \bar{x}) \cdot \bar{x}^{-1},$$

where $\bar{x} = xH$ in G/H, which is meaningful since $(p,b) = 1$. Then, clearly, μ^* is a K-homomorphism. Moreover, for any $\bar{y} \in G/H$, $\alpha \in K[G/H]$,

$$\begin{aligned} \mu^*(\alpha \cdot \bar{y}) &= \frac{1}{b} \textstyle\sum_{\bar{x} \in G/H} \mu\big((\alpha \cdot \bar{y}) \cdot \bar{x}\big) \cdot \bar{x}^{-1} \\ &= \frac{1}{b} \textstyle\sum_{\bar{x} \in G/H} \mu\big(\alpha \cdot (\bar{y}\bar{x})\big) \cdot (\bar{y}\bar{x})^{-1}\bar{y} \\ &= \big\{\frac{1}{b} \textstyle\sum_{\bar{y}\bar{x} \in \bar{y}(G/H) = G/H} \mu\big(\alpha \cdot (\bar{y}\bar{x})\big) \cdot (\bar{y}\bar{x})^{-1}\big\} \cdot \bar{y} \end{aligned}$$

$= \mu^*(\alpha) \cdot \bar{y}$, which shows that μ^* is a $K[G/H]$-module homomorphism.

Now, suppose that $\alpha \in M$. Then $\alpha \cdot \bar{x} \in M$ for all $\bar{x} \in G/H$ since M is a right ideal of $K[G/H]$. This yields

$$\begin{aligned} \mu^*(\alpha) &= \frac{1}{b} \textstyle\sum_{\bar{x} \in G/H} \mu(\alpha \cdot \bar{x}) \cdot \bar{x}^{-1} \\ &= \frac{1}{b} \textstyle\sum_{\bar{x} \in G/H} (\alpha \cdot \bar{x}) \cdot \bar{x}^{-1} \\ &= \frac{1}{b} \textstyle\sum_{\bar{x} \in G/H} \alpha = \alpha, \end{aligned}$$

so that $\mu^* \mid_M = 1_M$. Since Img $\mu^* \subseteq M$, it therefore follows that $\mu^* \circ \mu^* = \mu^*$. Hence we deduce that $K[G/H] = \text{Img } \mu^* \oplus \ker \mu^* = M \oplus N$ as a $K[G/H]$-module, the result follows. □

We now state and prove our main theorem.

THEOREM 3.5. *Let G be a group of order $p^a b$ and $(p, b) = 1$ and let K be a field of characteristic p. Assume that G has a normal Sylow p-subgroup H. Then*

$$JK[G] = K[G]\omega(K[H]) = \sum_{x \in H - \{1\}} K[G](x - 1),$$

$$dim_K JK[G] = b(p^a - 1).$$

Proof. Because it follows immediately from Theorem 3.2 that

$$K[G]\omega(K[H]) = \textstyle\sum_{x \in H - \{1\}} K[G](x - 1),$$

we will show that $JK[G] = K[G]\omega(K[H])$.

Since H is a p-group, $\omega(K[H]) = JK[H]$ by Theorem 3.2. Hence, by Lemma 3.3, $K[G]\omega(K[H]) = K[G]JK[H]$ is a nilpotent ideal of $K[G]$ and hence it follows that $K[G]\omega(K[H]) \subseteq JK[G]$. To show the reverse inclusion, consider the K-algebra homomorphism $\rho_H : K[G] \to K[G/H]$ defined by

$$\rho_H(\textstyle\sum_{g \in G} a_g g) = \sum_{g \in G} a_g \bar{g}.$$

Since ρ_H is a ring epimorphism, $\rho_H(JK[G]) \subseteq JK[G/H]$ by Proposition 2.2. But then, since $JK[G/H] = \{0\}$ by Lemma 3.4, $JK[G]$ is contained in the kernel $K[G]\omega(K[H])$ of ρ_H. Thus we have $JK[G] = K[G]\omega(K[H])$.

It remains to show $\dim_K JK[G] = b(p^a - 1)$. By the above result,

$$K[G]/JK[G]=K[G]/K[G]\omega(K[H]).$$

Furthermore, since $K[G]/K[G]\omega(K[H])\cong K[G/H]$ as a K-space and $\dim_K K[G/H]=b$, we have

$$\dim_K JK[G]=\dim_K K[G]\omega(K[H])=b(p^a-1).\qquad\square$$

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