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A HOMOMORPHISM OF MINIMAL SETS AND ITS REGULARIZER

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Abstract. In this paper we give some results on homomorphisms of flows. In particular, we investigate the sufficient conditions for the homomorphism of flows to be its own regularizer.

1. Introduction

The regular minimal sets were first studied by Auslander in [2]. These sets may be described as minimal subsets of enveloping semi-groups. In [6], Shoenfeld introduced the regular homomorphisms which are defined by extending regular minimal sets to homomorphisms with minimal range.

Given a homomorphism $\pi : X \longrightarrow Y$ with Y minimal, Shoenfeld constructed the homomorphism $\overline{\pi} : N \longrightarrow Y$ the regularizer of π and obtained the fact that the regular homomorphism of minimal sets is its own regularizer.

The purpose of this paper is to give some results on homomorphisms of flows and investigate the sufficient conditions for the homomorphism of flows to be its own regularizer.

2. Preliminaries

A *transformation group*, or *flow*, (X, T) , will consist of a jointly continuous action of the topological group T on the compact Hausdorff space X. The group T, with identity e, is assumed to be topologically

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discrete and remain fixed throughout this paper, so we may write X instead of (X,T).

A *point transitive flow*, (X,x) consists of a flow X with a distinguished point x which has dense orbit. A flow is said to be *minimal* if every point has dense orbit. Minimal flows are also referred to as minimal sets.

A *homomorphism* of flows is a continuous, equivariant map. A homomorphism whose domain is point transitive is determined by its value at a single point.

We say that (X,T) is an *extension* of (Y,T) if there exists a homomorphism of X onto Y.

A homomorphism $\pi : X \longrightarrow Y$ is said to be *proximal* if whenever $x_1, x_2 \in \pi^{-1}(y)$ then x_1 and x_2 are proximal. A homomorphism $\pi : X \longrightarrow Y$ is said to be *distal* if whenever $x_1, x_2 \in \pi^{-1}(y)$ then x_1 and x_2 are distal. We say the homomorphism $\pi : X \longrightarrow Y$ is *almost one to one* if there exists a point $y_0 \in Y$ such that $\pi^{-1}(y_0)$ is a singleton.

Given a flow (X,T), we may regard T as a set of self-homeomorphisms of X. We define $E(X)$, the *enveloping semigroup* of X to be the closure of T in X^X, taken with the product topology. $E(X)$ is at once a flow and a sub-semigroup of X^X. The minimal right ideals of $E(X)$, considered as a semigroup, coincide with the minimal sets of $E(X)$.

If E is some enveloping semigroup, and there exists a homomorphism $\theta : (E,e) \longrightarrow (E(X),e)$ we say that E is an *enveloping semigroup for* X. If such a homomorphism exists, it must be unique, and, given $x \in X$ and $p \in E$ we may write xp to mean $x\theta(p)$ unambiguously.

Lemma 2.1. *([6]) If (X,x) and (Y,y) are point transitive flows, and E is an enveloping semigroup for X and Y, there exists a (unique) homomorphism $\psi : (X,x) \longrightarrow (Y,y)$ if and only if $xp = xq$ for $p,q \in E$ implies $yp = yq$.*

Lemma 2.2. *([3]) Let $\pi : (X,T) \longrightarrow (Y,T)$ be an epimorphism(onto homomorphism). Then there exists a unique epimorphism $\psi : E(X) \longrightarrow E(Y)$ such that $\pi(x)\psi(p) = \pi(xp)$ for all $p \in E(X)$.*

Auslander and Glasner proved the following lemma :

Lemma 2.3. *([1]. [5]).*

(1) *A distal extension of a distal flow is distal.*
(2) *A distal extension of a minimal flow is a disjoint union of minimal sets.*

(3) *A proximal extension of a minimal flow contains a unique minimal set.*

LEMMA 2.4. *The following are true :*

(1) *A distal extension of a pointwise almost periodic flow is pointwise almost periodic.*
(2) *A proximal extension of a proximal flow is proximal .*
(3) *If the proximal extension of a minimal flow is pointwise almost periodic, then it is minimal.*

Proof. (1) Let $\pi : (X,T) \longrightarrow (Y,T)$ be a distal epimorphism and $x \in X$. By Lemma 2.2 we can find the unique epimorphism $\psi : E(X) \longrightarrow E(Y)$ such that $\pi(xp) = \pi(x)\psi(p)$ for all $p \in E(X)$. Since $\pi(x)$ is an almost periodic point, there exists an idempotent $v \in E(Y)$ with $\pi(x)v = \pi(x)$. Then there exists an idempotent $u \in E(X)$ with $\psi(u) = v$ such that $\pi(xu) = \pi(x)$. This implies that x and xu are distal. Clearly they are also proximal and hence $xu = x$. Thus x is an almost periodic point of (X,T).

(2) Let $\pi : (X,T) \longrightarrow (Y,T)$ be a proximal epimorphism and let $x_1, x_2 \in X$. Since (Y,T) is proximal, there exists $q \in E(Y)$ with $\pi(x_1)q = \pi(x_2)q$ whence $\pi(x_1 p) = \pi(x_2 p)$. But since π is proximal, there exists $r \in E(X)$ with $(x_1 p)r = (x_2 p)r$. Also, $x_1(pr) = x_2(pr)$ shows that x_1 and x_2 are proximal.

(3) Let $\pi : (X,T) \longrightarrow (Y,T)$ be a proximal homomorphism, Y minimal, and X pointwise almost periodic. Since every point of X is an almost periodic point, it follows that $\{\overline{xT} \mid x \in X\}$ is a partition of X consisting of minimal sets. The fact that X is minimal follows from (3) of Lemma 2.3. □

REMARK 2.5. *Let Y be a minimal set and $\pi : (X,T) \longrightarrow (Y,T)$ a proximal and distal homomorphism. Then X is a minimal set. This follows from Lemma 2.3. Indeed π is an isomorphism since π whose range is minimal is always onto.*

3. Some results on homomorphisms and related regularizers

Let $\pi : X \longrightarrow Y$ be a fixed epimorphism with Y pointwise almost periodic and let $y \in Y$.

Then $X^{\pi^{-1}(y)}$ is a flow whose elements are functions from $\pi^{-1}(y)$ to X.

Definition 3.1. *([6]) Define* $z_y \in X^{\pi^{-1}(y)}$ *by* $z_y(x) = x$ *for all* $x \in \pi^{-1}(y)$. *Let* $E(\pi, y)$ *be the orbit closure of* z_y, *i.e.,* $E(\pi, y) = \overline{z_y T} \subset X^{\pi^{-1}(y)}$.

Remark 3.2. (1) *If* Y *is a singleton* $\{y\}$, *then* $E(\pi, y) = E(X)$.
(2) *For each* $y \in Y$, $E(X)$ *is an enveloping semigroup for* $E(\pi, y)$.

Definition 3.3. *([6]) Let* $\overline{\pi}_y : E(\pi, y) \longrightarrow \overline{yT}$ *be the unique homomorphism with* $\overline{\pi}_y(z_y) = y$.

Lemma 3.4. *([6]) Let* $y \in Y$ *and let* N *and* N' *be minimal subsets of* $E(\pi, y)$. *Then there is an isomorphism* $\varphi : N \longrightarrow N'$ *such that* $(\overline{\pi}_y|_{N'}) \circ \varphi = \overline{\pi}_y|_N$.

The next lemmas follow easily from Theorem 2.1.5 and Theorem 2.1.6 in [6].

Lemma 3.5. *Let* $y \in Y$ *and* $y' \in \overline{yT}$. *Then there exist minimal sets* $N \subset E(\pi, y)$, $N' \subset E(\pi, y')$, *and an isomorphism* $\psi : N \longrightarrow N'$ *such that* $(\overline{\pi}_{y'}|_{N'}) \circ \psi = \overline{\pi}_y|_N$.

Lemma 3.6. *Let* $y \in Y$, $y' \in \overline{yT}$, *and* N *and* N' *minimal subsets of* $E(\pi, y)$ *and* $E(\pi, y')$ *respectively. Then there exists an isomorphism* $\psi : N \longrightarrow N'$ *such that* $(\overline{\pi}_{y'}|_{N'}) \circ \psi = \overline{\pi}_y|_N$.

Theorem 3.7. *(Shoenfeld [6])*
Let Y *be a minimal set and* $y, y' \in Y$. *Suppose* N *and* N' *are minimal subsets of* $E(\pi, y)$ *and* $E(\pi, y')$ *respectively. Then there exists an isomorphism* $\psi : N \longrightarrow N'$ *such that* $(\overline{\pi}_{y'}|_{N'}) \circ \psi = \overline{\pi}_y|_N$.

Remark 3.8. *Theorem 3.7 defines a minimal set* N *and an essentially (up to isomorphism) unique homomorphism which we call* $\overline{\pi} : N \longrightarrow Y$.

Definition 3.9. *([6]) Given a homomorphism* $\pi : X \longrightarrow Y$ *with* Y *minimal, we call the homomorphism* $\overline{\pi} : N \longrightarrow Y$ *the* regularizer *of* π *which is denoted by Reg*(π).

Lemma 3.10. *([6]) Given a homomorphism* $\pi : X \longrightarrow Y$ *with* X *and* Y *minimal, the following are equivalent :*

(1) *For any two points* $x, x' \in X$ *with* $\pi(x) = \pi(x')$ *there exists a homomorphism* $\theta : X \longrightarrow X$ *such that* $\theta(x)$ *and* x' *are proximal and* $\pi \circ \theta = \pi$.

(2) π *is its own regularizer, i.e., Reg(π) = π.*

LEMMA 3.11. *An almost one to one extension of a proximal and minimal flow is proximal.*

Proof. Let $\pi : (X,T) \longrightarrow (Y,T)$ be an almost one to one epimorphism, (Y,T) minimal and proximal, and $x_1, x_2 \in X$. Then there exists $y_0 \in Y$ such that $\pi^{-1}(\{y_0\}) = \{x_0\}$. Since Y is proximal we can find $q \in E(Y)$ with $\pi(x_1)q = \pi(x_2)q$. But since Y is minimal there exists $r \in E(Y)$ such that $y_0 = (\pi(x_1)q)r = \pi(x_1)(qr)$. Also there exists $p \in E(X)$ such that $\psi(p) = qr$, where $\psi : E(X) \longrightarrow E(Y)$ is the unique epimorphism induced by π. Now $\pi(x_1p) = \pi(x_1)\psi(p) = \pi(x_1)qr = \pi(x_2)qr = \pi(x_2)\psi(p) = \pi(x_2p)$ and thus $x_1p = x_2p$. This shows that X is proximal. $\square$

Let βT denote the Stone-Cěch compactification of T and let I be a minimal right ideal in βT. Then $(\beta T, e)$ is a universal point transitive flow and (I,T) is a universal minimal set. It is also clear that βT is an enveloping semigroup for X, whenever X is a flow with acting group T.

We choose a distinguished idempotent $u \in I$, and denote by G the group Iu. Given a minimal flow X, we choose a point $x_0 \in Xu$. Under the canonical map $(\beta T, e) \longrightarrow (X, x_0)$, I is mapped onto X and u onto x_0.

Let (X, x_0) be a pointed minimal flow. We define the Ellis group of (X, x_0) to be

$G(X, x_0) = \{\alpha \in G \mid x_0\alpha = x_0\}$.

Clearly $G(X, x_0)$ is a subgroup of G.

LEMMA 3.12. *([5]) Let $\pi : (X, x_0) \longrightarrow (Y, y_0)$ be a homomorphism of pointed minimal flows. The following are true :*

(1) $G(X, x_0) \subset G(Y, y_0)$.

(2) $G(X, x_0) = G(Y, y_0)$ *if and only if π is proximal.*

THEOREM 3.13. *Let Y be a minimal set and $\pi : (X,T) \longrightarrow (Y,T)$ a proximal homomorphism. If X is pointwise almost periodic, then it is minimal and Reg(π) = π.*

Proof. The fact that X is minimal follows from (3) of Lemma 2.4. If we take θ by the identity homomorphism, we have from Lemma 3.10 that Reg$(\pi) = \pi$. $\square$

Remark 3.14. *Let Y be a minimal set and $\pi : (X,T) \longrightarrow (Y,T)$ a proximal and distal homomorphism. It follows immediately from Remark 2.5 and Lemma 3.10 that Reg(π) $= \pi$.*

Theorem 3.15. *Suppose that $\pi : (X,T) \longrightarrow (Y,T)$ is a proximal and almost one to one homomorphism and that Y is minimal. The following are true :*

(1) *There is only one minimal subset M in X.*
(2) *Let $y_0 \in Y$ with $\pi^{-1}(\{y_0\}) = \{x_0\}$. Then $x_0 \in M$.*
(3) *Let I be a minimal right ideal in βT and $u \in I$ the distinguished idempotent with $y_0 u = y_0$. Then $G(M, x_0) = G(Y, y_0)$.*
(4) *Reg($\pi|_M$) $= \pi|_M$.*

Proof. (1) This follows from (3) of Lemma 2.3.
(2) Since $\pi(M)$ is closed, non-vacuous and invariant, it follows that $\pi(M) = Y$. Also $\pi^{-1}(\{y_0\}) = \{x_0\}$. This shows that $x_0 \in M$.
(3) Since Y is minimal, we have the distinguished idempotent $u \in I$ with $y_0 u = y_0$. Then $\pi(x_0 u) = \pi(x_0)u = y_0 u = y_0$. Since $\pi^{-1}(\{y_0\}) = \{x_0\}$ this implies that $x_0 u = x_0$ whence $u \in G(M, x_0)$. Thus we have $G(M, x_0) = G(Y, y_0)$ by (2) of Lemma 3.12 and the proximality of π.
(4) This follows from Lemma 3.10.

□

Theorem 3.16. *Suppose that $\pi : (X,T) \longrightarrow (Y,T)$ is an almost one to one homomorphism and that Y is proximal and minimal. The following are true :*

(1) *X is proximal.*
(2) *There is only one minimal subset M in X.*
(3) *Let $y_0 \in Y$ with $\pi^{-1}(\{y_0\}) = \{x_0\}$. Then $x_0 \in M$.*
(4) *Given $x \in X$, there exists $p \in E(X)$ such that $xp \in M$.*
(5) *Let $y_0 \in Y$ with $\pi^{-1}(\{y_0\}) = \{x_0\}$. Then x_0 is contained the orbit closure of x for all $x \in X$.*
(6) *Let I be a minimal right ideal in βT and $u \in I$ the distinguished idempotent with $y_0 u = y_0$. Then $G(M, x_0) = G(Y, y_0)$.*
(7) *Reg($\pi|_M$) $= \pi|_M$.*

Proof. (1) This follows from Lemma 3.11.
(2) This follows from (3) of Lemma 2.3 and the fact that if X is proximal, then π is also proximal.

(4) Since (X,T) is proximal and M is minimal, there exists $p \in E(X)$ such that $xp = x_0p \in M$.
(5) Let $x \in X$. Then as in (4) there exists $p \in E(X)$ such that $xp \in M$. Since $x_0 \in M$ and M is minimal, it follows that $x_0 = (xp)r = x(pr) \in \overline{xT}$ for some $r \in E(X)$.

The proof of the statements (3), (6) and (7) is identical to that of Theorem 3.15. □

THEOREM 3.17. *Suppose that $\pi : (X,T) \longrightarrow (Y,T)$ is a distal and almost one to one homomorphism and that Y is proximal and minimal. The following are true :*

(1) *X is proximal and minimal.*
(2) *Let $y_0 \in Y$ with $\pi^{-1}(\{y_0\}) = \{x_0\}$. Then $G(X,x_0) = G(Y,y_0)$.*
(3) *Reg(π) $= \pi$.*

Proof. (1) Lemma 3.11 shows that (X,T) is proximal. Also, (2) of Lemma 2.3 shows that (X,T) is pointwise almost periodic. Since X is proximal this implies that π is proximal whence X is minimal by (3) of Lemma 2.4.

The proof of the statements (2) and (3) is completely analogous to that of Theorem 3.15. □

THEOREM 3.18. *Suppose that $\pi : (X,T) \longrightarrow (Y,T)$ is a distal and almost one to one homomorphism and that Y is distal and minimal. The following are true :*

(1) *X is distal.*
(2) *X is a disjoint union of minimal sets.*
(3) *Let $y_0 \in Y$ with $\pi^{-1}(\{y_0\}) = \{x_0\}$, I a minimal right ideal in βT, and $u \in I$ the distinguished idempotent with $y_0u = y_0$. Then $x_0u = x_0$.*
(4) *Let M be a minimal set with $x_0 \in M$. Then $\pi|_M : (M,x_0) \longrightarrow (Y,y_0)$ is a unique epimorphism.*
(5) *$G(M,x_0) \subset G(Y,y_0)$.*

Proof. (1) This follows from (1) of Lemma 2.3.
(2) This follows from (2) of Lemma 2.3.
(3) Let $y \in Y$ with $\pi^{-1}(\{y_0\}) = \{x_0\}$ and let I be a minimal right ideal in βT. Since Y is minimal, there exists the distinguished idempotent $u \in I$ with $y_0u = y_0$. Now $\pi(x_0u) = \pi(x_0)u = y_0u = y_0$. Hence $x_0u = x_0$.

(4) Let M be a minimal set with $x_0 \in M$ and let $y \in Y$. Then, by the minimality of Y, there exists $p \in \beta T$ with $y = y_0 p = \pi(x_0 p)$. Since X is minimal, it follows that $x_0 p \in M$. This implies that $Y = \pi(M)$ whence $\pi|_M : (M, x_0) \longrightarrow (Y, y_0)$ is an epimorphism. The uniqueness of π follows immediately from Lemma 2.1 and the fact that $\pi^{-1}(\{y_0\}) = \{x_0\}$.

(5) This follows from (1) of Lemma 3.12 and the above statements (3) and (4).

□

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