Korean J. Math. **18** (2010), No. 1, pp. 53–61

AN APPLICATION OF A LINKING METHOD TO A GENERAL ELLIPTIC SYSTEM

Hyewon Nam

Abstract. In this work, we consider an elliptic system of three equations in dimension greater than one. We prove that the system has at least three nontrivial solutions by applying a linking theorem.

1. Introduction and background

Presently there are many significant results with respect to the elliptic system

$$\begin{cases} -\triangle u = \lambda u + \delta v + h_1(x, u, v), \\ -\triangle v = \theta u + \nu v + h_2(x, u, v), \end{cases}$$

in Ω, where $\Omega \subset R^n$ is the bounded smooth domain, subject to Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$, h_i, $i = 1, 2$ are real valued functions and λ, δ, ν and θ are real numbers.[[5], [6]]

In this paper we prove the existence of three nontrivial solutions for a general elliptic system. We use a variational approach and look for critical points of a suitable functional I on a Hilbert space H. Since the functional is strongly indefinite, it is convenient to use the notion of a linking theorem. In Section 2, we find a suitable functional I on a Hilbert space H. In Section 3, we prove the suitable version of the Palais-Smale condition for the topological method. In Section 4, we apply the three critical points theorem.

We recall some basic theorem and set up some terminology. Let H be a Hilbert space and V a C^2 complete connected Finsler manifold.

Received January 10, 2010. Revised March 7, 2010. Accepted March 10, 2010.
2000 Mathematics Subject Classification: 35J55, 49J35.
Key words and phrases: elliptic system, linking inequality.

Definition 1.1. *The cuplength of a space V, denoted cuplength(V), is the maximum number m of positive degree cohomology classes $[\omega_1]$, $[\omega_2]$, $\cdots$, $[\omega_m]$ such that $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_m \neq 0$ on V.*

Suppose $H = H_0 \oplus H_1 \oplus H_2 \oplus H_3$ and let $H_n = H_{0n} \oplus H_{1n} \oplus H_{2n} \oplus H_{3n}$ be a sequence of closed subspaces of H such that

$$H_{in} \subset H_i, \quad 1 \leq \dim H_{in} < +\infty \quad \text{for each} \quad i = 0, \cdots \quad \text{and} \quad n \in N$$

Moreover suppose that there exist $e_1 \in \cap_{n=1}^{\infty} H_{1n}$, and $e_2 \in \cap_{n=1}^{\infty} H_{2n}$, with $\|e_1\| = \|e_2\| = 1$.

For any Y subspace of H, consider $B_\rho(Y) := \{u \in Y | \|u\| \leq \rho\}$ and denote by $\partial B_\rho(Y)$ the boundary of $B_\rho(Y)$ relative to Y. Furthermore define, for any $e \in H$,

$$Q_R(Y, e) := \{u + ae \in Y \oplus [e] | u \in Y, a \geq 0, \|u + ad\| \leq R\}$$

and denote by $\partial Q_R(Y, e)$ its boundary relative to $Y \oplus [e]$, and denote by $X = H \times V$.

We recall the three critical points theorem in [3].

Theorem 1.1. *Suppose that f satisfies the $(PS)^*$ condition with respect to H_n. In addition assume that there exist ρ_i, R_i, $i = 1, 2$, such that $0 < \rho_i < R_i$ and*

$$\sup_{\partial Q_{R_1}(H_2 \oplus H_3, e_1) \times V} f < \inf_{\partial B_{\rho_1}(H_0 \oplus H_1) \times V} f,$$
$$\sup_{Q_{R_1}(H_2 \oplus H_3, e_1) \times V} f < +\infty, \qquad \inf_{B_{\rho_1}(H_0 \oplus H_1) \times V} f < -\infty,$$
$$\sup_{\partial Q_{R_2}(H_3, e_2) \times V} f < \inf_{\partial B_{\rho_2}(H_0 \oplus H_1 \oplus H_2) \times V} f,$$
$$\sup_{Q_{R_2}(H_3, e_2) \times V} f < +\infty, \qquad \inf_{B_{\rho_2}(H_0 \oplus H_1 \oplus H_2) \times V} f < -\infty.$$

If $R_2 < R_1$, then there exist at least 3 critical levels of f. Moreover the critical levels satisfy the following inequalities

$$\begin{aligned} \inf_{B_{\rho_2}(H_0 \oplus H_1 \oplus H_2) \times V} f &\leq c_1 \leq \sup_{\partial Q_{R_2}(H_3, e_2) \times V} f < \inf_{\partial B_{\rho_2}(H_0 \oplus H_1 \oplus H_2) \times V} f \leq c_2 \\ &\leq \sup_{Q_{R_2}(H_3, e_2) \times V} f \leq \sup_{\partial Q_{R_1}(H_2 \oplus H_3, e_1) \times V} f \\ &< \inf_{\partial B_{\rho_1}(H_1 \oplus H_2) \times V} f \leq c_3 \leq \sup_{Q_{R_1}(H_2 \oplus H_3, e_1) \times V} f, . \end{aligned}$$

and there exist at least $3 + 3$ cuplength(V) critical points of f.

2. Notations and main result

Let $\Omega \subset R^N$ be a bounded domain with the smooth boundary and $H = W_0^{1,p}(\Omega)$, the usual Sobolev space with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$.

In this paper, we consider the existence of nontrivial solutions to the elliptic system

$$(1) \qquad \begin{cases} -\triangle u = au + \delta u^+ + f_1(x,u,v,w) & \text{in } \Omega, \\ -\triangle v = bv + \eta v^- + f_2(x,u,v,w) & \text{in } \Omega, \\ -\triangle w = cw + f_3(x,u,v,w) & \text{in } \Omega, \\ u = v = w = 0 & \text{on } \partial\Omega. \end{cases}$$

And there exists a function $F : \bar{\Omega} \times R^3 \to R$ such that $\frac{\partial F}{\partial u} = f_1$, $\frac{\partial F}{\partial v} = f_2$, and $\frac{\partial F}{\partial w} = f_3$ without loss of generality, we set

$$\begin{aligned} &F(x,u,v,w) \\ &= \int_{(0,0,0)}^{(u,v,w)} f_1(x,u,v,w)du + f_2(x,u,v,w)dv + f_3(x,u,v,w)dw. \end{aligned}$$

Then $F \in C^1(\bar{\Omega} \times R^3, R)$.

We consider the following assumptions.

(F1) There exist $M > 0$ and $\alpha > 2$ such that

$$0 < \alpha F(x,u,v,w) \le uF_u(x,u,v,w) + vF_v(x,u,v,w) + wF_w(x,u,v,w)$$

for all $(x,u,v,w) \in \bar{\Omega} \times R^3$ with $u^2 + v^2 + w^2 > M^2$.

(F2) There exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$|F_u(x,u,v,w)| + |F_v(x,u,v,w)| + |F_w(x,u,v,w)| \le a_1 + a_2(|u|^r + |v|^r + |w|^r)$$

where $1 \le r < (N+2)/(N-2)$ if $N > 2$, $1 \le r < \infty$ otherwise.

(F3) For $(0,v,w) \to (0,0,0)$,

$$\frac{F(x,0,v,w)}{v^2 + w^2} \to 0.$$

Remark 2.1. *The condition (F1) shows that there exist constants $b_1 > 0$ and b_2 such that(cf. [2])*

$$F(x,u,v,w) \ge b_1(|u|^\alpha + |v|^\alpha + |w|^\alpha) - b_2.$$

Let λ_k denote the eigenvalues and e_k the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, with Dirichlet boundary condition, where each eigenvalue λ_k is respected as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots, \lambda_i \to +\infty$ and that $e_1 > 0$ for all $x \in \Omega$. Then $H = \text{span}\{e_i | i \in N\}$.

Let $e_i^1 = (e_i, 0, 0)$, $e_i^2 = (0, e_i, 0)$, and $e_i^3 = (0, 0, e_i)$. We define $H_j = \text{span}\{e_i^j | i \in N\}$, for $j = 1, 2, 3$ and $E = H_1 \oplus H_2 \oplus H_3$ with the norm $\|(u,v,w)\|_E^2 = \|u\|^2 + \|v\|^2 + \|w\|^2$.

We define the energy functional associated to (1) as

$$\begin{aligned} I(u,v,w) = {} & \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla u|^2 + |\nabla w|^2)dx - \frac{1}{2}\int_\Omega (au^2 + bv^2 + cw^2)dx \\ & -\frac{\delta}{2}\int_\Omega (u^+)^2 dx - \frac{\eta}{2}\int_\Omega (v^-)^2 dx - \int_\Omega F(x,u,v,w)dx \end{aligned} \tag{2}$$

It is easy to see that $I \in C^1(E, R)$ and thus it makes sense to lock for solutions to (1) in weak sense as critical points for I $i.e.(u,v,w) \in E$ such that $I'(u,v,w) = 0$, where

$$\begin{aligned} I'(u,v,w) \quad \cdot \quad (\phi,\psi,\sigma) &= \int_\Omega (\nabla u \nabla \phi + \nabla v \nabla \psi + \nabla w \nabla \sigma)dx \\ - & \int_\Omega (au\phi + bv\psi + cw\sigma)dx - \delta \int_\Omega u^+ \phi dx - \eta \int_\Omega v^- \psi dx \\ - & \int_\Omega (f_1(x,u,v,w)\phi + f_2(x,u,v,w)\psi + f_3(x,u,v,w)\sigma)dx. \end{aligned}$$

We will prove the following theorem.

THEOREM 2.1. *Assume F satisfies (F1), (F2) and (F3) with $\alpha = r+1$. If a, b, c, δ, and η are positive with $a + \delta < \lambda_1$, $b + \eta < \lambda_1$ and $c < \lambda_1$ then system (1) has at least three nontrivial solutions.*

3. The Palais Smale star condition

In [1] the following definition is given.

DEFINITION 3.1. *We say that I verifies the Palais Smale star condition at level c ($(PS)_c^*$) with respect to (E_n), if for any sequence (u_n) in E such that $u_n \in E_n$, $I(u_n) \to c$ and $I_n'(u_n) \to 0$ there exists a subsequence of (u_n) which converges to a critical point for I.*

DEFINITION 3.2. *A sequence $(u_n) \subset E$ is said to be a $(PS)_c^*$ sequence if $u_n \in E_n$, $I(u_n) \to c$, $I_n'(u_n) \to 0$ as $n \to \infty$.*

REMARK 3.1. *If any $(PS)_c^*$ sequence has a convergent subsequence, then we say that I satisfies the $(PS)_c^*$ condition.*

In this section we will prove the $(PS)_c^*$ condition which was required for the application of Theorem 1.1. In the following, we consider the following sequence of subspaces of E :

$$E_n = \text{span}\{e_i^j | i = 1, \cdots, n \quad \text{and} \quad j = 1, 2, 3\}, \qquad \text{for } n \geq 1.$$

LEMMA 3.1. *Assume F satisfies (F1) and (F2) with $\alpha = r + 1$. If $a + \delta < \lambda_1$, $b + \eta < \lambda_1$ and $c < \lambda_1$, then any $(PS)_c^*$ sequence is bounded.*

Proof. Let $\{(u_n, v_n, w_n)\} \subset E$ be a sequence such that

$$(u_n, v_n, w_n) \in E_n, \quad I(u_n, v_n, w_n) \to c, \quad I_n'(u_n, v_n, w_n) \to 0 \quad \text{as} \quad n \to \infty$$

In the following we denote different constants by C_1, C_2 etc. (F1) and Remark imply that

$$\begin{aligned} C_1 \quad &+ \quad \frac{1}{2} o(1)(\|u_n\| + \|v_n\| + \|w_n\|) \\ &\geq \quad I(u_n, v_n, w_n) - \frac{1}{2} I_n'(u_n, v_n, w_n) \cdot (u_n, v_n, w_n) \\ &= \quad \frac{1}{2} \int_\Omega (u_n f_1 + v_n f_2 + w_n f_3) dx - \int_\Omega F dx \\ &\geq \quad (\frac{\alpha}{2} - 1) \int_\Omega F(x, u_n, v_n, w_n) dx \\ &\geq \quad (\frac{\alpha}{2} - 1) b_1 \int_\Omega (|u_n|^\alpha + |v_n|^\alpha + |w_n|^\alpha) dx - C_2 \\ &\geq \quad (\frac{\alpha}{2} - 1) b_1 (\|u_n\|_{L^\alpha}^\alpha + \|v_n\|_{L^\alpha}^\alpha + \|w_n\|_{L^\alpha}^\alpha) - C_2 \qquad (3) \end{aligned}$$

On the other hand,

$$\begin{aligned} o(1)\|u_n\| \quad &\geq \quad I_n'(u_n, v_n, w_n) \cdot (u_n, 0, 0) \\ &= \quad \|u_n\|^2 - a \int_\Omega u_n^2 dx - \delta \int_\Omega (u_n^+)^2 dx - \int_\Omega f_1(x, u_n, v_n, w_n) u_n dx, \\ o(1)\|v_n\| \quad &\geq \quad I_n'(u_n, v_n, w_n) \cdot (0, v_n, 0) \\ &= \quad \|v_n\|^2 - b \int_\Omega v_n^2 dx - \eta \int_\Omega (v_n^-)^2 dx - \int_\Omega f_2(x, u_n, v_n, w_n) v_n dx. \\ o(1)\|w_n\| \quad &\geq \quad I_n'(u_n, v_n, w_n) \cdot (0, 0, w_n) \\ &= \quad \|w_n\|^2 - c \int_\Omega w_n^2 dx - \int_\Omega f_3(x, u_n, v_n, w_n) w_n dx. \end{aligned}$$

We know that $\|u\|^2 \geq \lambda_1 \|u\|_{L^2}^2$ for $u \in H$ and $\|u\|_{L^2}^2 \geq \int_\Omega (u^+)^2 dx$. Using (F2), we obtain

$$
\begin{aligned}
\|u_n\|^2 &+ \|v_n\|^2 + \|w_n\|^2 \\
&\leq \int_\Omega (au_n^2 + bv_n^2 + cw_n^2)dx + \delta \int_\Omega (u_n^+)^2 dx + \eta \int_\Omega (v_n^-)^2 dx \\
&\quad + \int_\Omega (u_n f_1 + v_n f_2 + w_n f_3)dx + o(1)(\|u_n\| + \|v_n\| + \|w_n\|) \\
&\leq \frac{a+\delta}{\lambda_1}\|u_n\|^2 + \frac{b+\eta}{\lambda_1}\|v_n\|^2 + \frac{c}{\lambda_1}\|w_n\|^2 \\
&\quad + o(1)(\|u_n\| + \|v_n\| + \|w_n\|) \\
&\quad + C_3 \int_\Omega (|u_n|^{r+1} + |v_n|^{r+1} + |w_n|^{r+1})dx + C_4. \qquad (4)
\end{aligned}
$$

(4) imply that if $a + \delta < \lambda_1$, $b + \eta < \lambda_1$ and $c < \lambda_1$ then

$$
\begin{aligned}
\|u_n\|^2 + \|v_n\|^2 + \|w_n\|^2 &\leq C_5 \int_\Omega (|u_n|^{r+1} + |v_n|^{r+1} + |w_n|^{r+1})dx \\
&\quad + o(1)C_6(\|u_n\| + \|v_n\| + \|w_n\|) + C_7. \qquad (5)
\end{aligned}
$$

Combining (3), (5) and using $\alpha = r + 1$, one infers that

$$
\|u_n\|^2 + \|v_n\|^2 + \|w_n\|^2 \leq o(1)C_8(\|u_n\| + \|v_n\| + \|w_n\|) + C_9.
$$

This yields $\{(u_n, v_n, w_n)\}$ is bounded. □

LEMMA 3.2. *Assume F satisfies (F1) and (F2) with $\alpha = r + 1$. If $a + \delta < \lambda_1$, $b + \eta < \lambda_1$ and $c < \lambda_1$, then the functional I satisfies the $(PS)_c^*$ condition with respect to E_n.*

Proof. By Lemma 3.1, any $(PS)_c^*$ sequence $\{(u_n, v_n, w_n)\}$ in E is bounded and hence $\{(u_n, v_n, w_n)\}$ has a weakly convergent subsequence. That is there exist a subsequence $\{(u_{n_j}, v_{n_j}, w_{n_j})\}$ and $(u, v, w) \in E$, with $u_{n_j} \rightharpoonup u$, $v_{n_j} \rightharpoonup v$ and $w_{n_j} \rightharpoonup w$. Since $\{u_{n_j}\}$, $\{v_{n_j}\}$ and $\{w_{n_j}\}$ are bounded, by Remark of the Rellich-Kondrachov compactness theorem [4], $u_{n_j} \to u$, $v_{n_j} \to v$ and $w_{n_j} \to w$ and thus I satisfies the $(PS)_c^*$ condition. □

4. Proof of main theorem

LEMMA 4.1. *Assume F satisfies (F3). If $c < \lambda_1$, then there exists $\rho_1 > 0$ such that*

$$\inf_{\partial B_{\rho_1}(H_3)} I > 0.$$

Proof. By (F3), for any $\varepsilon > 0$, there exists $\rho > 0$ such that

$$0 < \|w\| < \rho \Rightarrow |F(x,0,0,w)| < \varepsilon |w|^2.$$

Then $|\int_\Omega F(x,0,0,w)dx| < \int_\Omega |F(x,0,0,w)|dx < \int_\Omega \varepsilon |w|^2 dx < \frac{\varepsilon}{\lambda_1}\|w\|^2$ and hence

$$\begin{aligned} I(0,0,w) &= \frac{1}{2}\int_\Omega |\nabla w|^2 dx - \frac{c}{2}\int_\Omega w^2 dx - \int_\Omega F(x,0,0,w)dx \\ &> \frac{1}{2}\|w\|^2 - \frac{c}{2\lambda_1}\|w\|^2 - \frac{\varepsilon}{\lambda_1}\|w\|^2 \\ &= \frac{1}{2}(1 - \frac{c+2\varepsilon}{\lambda_1})\|w\|^2 > 0 \end{aligned}$$

which gives the result for sufficiently small ε. Therefore we can choose $0 < \rho_1 < \rho$ such that $I(0,0,w) > 0$ for any $\|w\| = \rho_1$. □

LEMMA 4.2. *Assume F satisfies (F1). If a, b, c, δ, and η are positive, then there exists an $R > 0$ such that for any $R_1 > R$*

$$\sup_{\partial Q_{R_1}(H_1 \oplus H_2, e_1^3)} I < 0.$$

Proof. In the following we denote different constants by C_1, C_2 etc. Remark implies that

$$\begin{aligned} I(u,v,\quad \beta_1 \quad e_1) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx + \frac{\lambda_1\beta_1^2}{2} - \frac{1}{2}\int_\Omega (au^2 + bv^2)dx \\ &\quad - \frac{c\beta_1^2}{2} - \frac{\delta}{2}\int_\Omega (u^+)^2 dx - \frac{\eta}{2}\int_\Omega (v^-)^2 dx - \int_\Omega F(x,u,v,\beta_1 e_1)dx \\ &\le \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 + \frac{\lambda_1\beta_1^2}{2} - \int_\Omega F(x,u,v,\beta_1 e_1)dx \\ &\le \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 + \frac{\lambda_1\beta_1^2}{2} - b_1\int_\Omega (|u|^\alpha + |v|^\alpha + |\beta_1 e_1|^\alpha)dx + C_1 \\ &\le \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 + \frac{\lambda_1\beta_1^2}{2} - C_2\|u\|^\alpha - C_2\|v\|^\alpha - C_3|\beta_1|^\alpha + C_4, \end{aligned}$$

for any $(u, v, 0) \in H_1 \oplus H_2$ and any constant β_1. Since $\alpha > 2$, $I(u, v, \beta_1 e_1) \to -\infty$ for $\|u\| \to \infty$ or $\|v\| \to \infty$ or $|\beta_1| \to \infty$. Therefore we can choose $0 < R_1 < \infty$ such that $I(u, v, \beta_1 e_1) < 0$ for any $\|(u, v, \beta_1 e_1)\|_E = R_1$. □

LEMMA 4.3. *Assume F satisfies (F3). If $b + \eta < \lambda_1$ and $c < \lambda_1$, then there exists $\rho_2 > 0$ such that*

$$\inf_{\partial B_{\rho_2}(H_2 \oplus H_3)} I > 0.$$

Proof. By (F3), for any $\varepsilon > 0$, there exists $\rho > 0$ such that

$$0 < \|v\|^2 + \|w\|^2 < \rho^2 \Rightarrow |F(x, 0, v, w)| < \varepsilon(|v|^2 + |w|^2).$$

Then $|\int_\Omega F(x, 0, v, w)dx| < \frac{\varepsilon}{\lambda_1}(\|v\|^2 + \|w\|^2)$ and hence

$$\begin{aligned} I(0, v, w) &= \frac{1}{2}\int_\Omega (|\nabla v|^2 + |\nabla w|^2)dx - \frac{b}{2}\int_\Omega v^2 dx - \frac{\eta}{2}\int_\Omega (v^-)^2 dx \\ &\quad - \frac{c}{2}\int_\Omega w^2 dx - \int_\Omega F(x, 0, v, w)dx \\ &> \frac{1}{2}(1 - \frac{b + \eta + 2\varepsilon}{\lambda_1})\|v\|^2 + \frac{1}{2}(1 - \frac{c + 2\varepsilon}{\lambda_1})\|w\|^2 > 0 \end{aligned}$$

which gives the result for sufficiently small ε. Therefore we can choose $0 < \rho_2 < \rho$ such that $I(0, v, w) > 0$ for any $\|v\|^2 + \|w\|^2 = \rho_2^2$. □

LEMMA 4.4. *Assume F satisfies (F1). If a, b, c, δ, and η are positive, then there exists an $R > 0$ such that for any $R_2 > R$*

$$\sup_{\partial Q_{R_2}(H_1, e_1^2)} I < 0.$$

Proof. In the following we denote different constants by C_1, C_2 etc. Remark implies that

$$\begin{aligned} I(u, \beta_2 e_1, 0) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\lambda_1 \beta_2^2}{2} - \frac{a}{2}\int_\Omega u^2 dx - \frac{b\beta_2^2}{2} \\ &\quad - \frac{\delta}{2}\int_\Omega (u^+)^2 dx - \int_\Omega F(x, u, \beta_2 e_1, 0)dx \\ &\le \frac{1}{2}\|u\|^2 + \frac{\lambda_1 \beta_2^2}{2} - \int_\Omega F(x, u, \beta_2 e_1, 0)dx \\ &\le \frac{1}{2}\|u\|^2 + \frac{\lambda_1 \beta_2^2}{2} - C_1\|u\|^\alpha - C_2|\beta_2|^\alpha + C_3, \end{aligned}$$

for any $u \in H$ and any constant β_2. Since $\alpha > 2$, $I(u, \beta_2 e_1, 0) \to -\infty$ for $\|u\| \to \infty$ or $|\beta_2| \to \infty$. Therefore we can choose $0 < R_2 < \infty$ such that $I(u, \beta_2 e_1, 0) < 0$ for any $\|(u, \beta_2 e_1, 0)\|_E = R_2$. $\square$

Proof of Theorem. By Lemma 4.1 and 4.2, there exists $0 < \rho_1 < R_1$ such that

$$\sup_{\partial Q_{R_1}(H_1 \oplus H_2, e_1^3)} I < 0 < \inf_{\partial B_{\rho_1}(H_3)} I.$$

And by Lemma 4.3 and 4.4, there exists $0 < \rho_2 < R_2 < R_1$ such that

$$\sup_{\partial Q_{R_2}(H_1, e_1^2)} I < 0 < \inf_{\partial B_{\rho_2}(H_2 \oplus H_3)} I.$$

By Theorem 1, $I(u, v, w)$ has at least three nonzero critical values c_1, c_2, c_3

$$\begin{aligned} \inf_{B_{\rho_2}(H_2 \oplus H_3)} I \le c_1 \le \sup_{\partial Q_{R_2}(H_1, e_1^2)} I < \inf_{\partial B_{\rho_2}(H_2 \oplus H_3)} I \le c_2 \le \sup_{Q_{R_2}(H_1, e_1^2)} I \\ \le \sup_{\partial Q_{R_1}(H_1 \oplus H_2, e_1^3)} I < \inf_{\partial B_{\rho_1}(H_3)} I \le c_3 \le \sup_{Q_{R_1}(H_1 \oplus H_2, e_1^3)} I. \end{aligned}$$

Therefore, (1) has at least three nontrivial solutions.

References

[1] A.M. Micheletti, C. Saccon, *Multiple solutions for an asymptotically linear problem in R^N*. Nonlinear Anal., **56** (2004), no. 1, 1–18.

[2] A. Szulkin, *Critical point theory of Ljusterni-Schnirelmann type and applications to partial differential equations.* Sem. Math. Sup., Vol. 107, pp. 35–96, Presses Univ. Montreal, Montreal, QC, 1989.

[3] D. Lupo, A.M. Micheletti, *Two applications of a three critical points theorem.* J. Differential Equations, **132** (1996), no. 2, 222–238.

[4] Evans, Lawrence C., *Partial differential equations.* American Mathematical Society, (1998).

[5] Jin Yinghua, *Nontrivial solutions of nonlinear elliptic equations and elliptic systems.* Inha U. 2004.

[6] Zou, Wenming, *Multiple solutions for asymptotically linear elliptic systems.* J. Math. Anal. Appl., **255** (2001), no. 1, 213–229.

Department of General Education
Namseoul University
Chonan 330-707, Korea
E-mail: hwnam@nsu.ac.kr