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ALMOST LINDELÖF FRAMES

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Abstract. Generalizing Lindelöf frames and almost compact frames, we introduce a concept of almost Lindelöf frames. Using a concept of δ-filters on frames, we characterize almost Lindelöf frames and then have their permanence properties. We also show that almost Lindelöf regular $D(\aleph_1)$ frames are exactly Lindelöf frames. Finally we construct an almost Lindelöfication of a frame L via the simple extension of L associated with the set of all δ-filters F on L with $\bigvee\{x^*|x \in F\} = e$.

1. Introduction and preliminaries

The concept of frames(=locales) was introduced by Ehresmann([3]) and Bénabou([1]) and Isbell has pointed out the importance of frames for a study of topological structures([5]). In 1981, Johnstone showed the Tychonoff theorem in the setting of frames without the axiom of choice([6]) and since then there were numerous authors who have produced remarkable results on frames([7]). We have introduced a concept of Lindelöf frames and Lindelöf biframes and obtained their Lindelöfications([8], [9]).

This paper is a sequel to the above papers. It is well known that almost compact spaces and its frame version are really important generalizations of compact spaces. Since dense elements can be defined in a frame, one can easily have a counterpart of dense subcover in a frame. The purpose of this paper is to introduce almost Lindelöf frames and study them. Introducing δ-filters on frames, we characterize almost Lindelöf frames and then using simple extension associated with a certain set of δ-filters on frames, we construct almost Lindelöfications of frames.

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First we collect basic definitions and results on frames. For general notions and facts concerning frames, we refer to Johnstone[7] and Khang[9].

Definition 1.1. (1) *A frame is a complete lattice L in which binary meet distributes over arbitrary join, that is,* $x \wedge \bigvee S = \bigvee\{x \wedge s \in S\}$ *for any* x *in L and any subset* S *of L.*

(2) *A frame homomorphism is a map* $h : L \to M$ *between frames L and M preserving all finitary meets and binary joins.*

We will denote the bottom element of a frame L by 0 or 0_L and the top element by e or e_L.

For any element a of a frame L, the map $a \wedge \_ : \mathrm{L} \to \mathrm{L}$ preserves arbitrary joins; hence it has a right adjoint, which will be denoted by $a \to \_ : \mathrm{L} \to \mathrm{L}$. In particular, $a \to 0$ exists for any a in L and we write $a \to 0 = a^*$, called the *pseudocomplement* of a.

Definition 1.2. (1) *An element* d *in a frame L is called dense if* $d^* = 0$.

(2) *A frame homomorphism* $h : L \to M$ *is called dense(codense, resp.) if* $h(x) = 0(h(x) = e$, *resp.) implies* $x = 0(e$, *resp.).*

We note that an element u in the frame $\Omega(\mathrm{X})$ of a topological space $(\mathrm{X}, \Omega(\mathrm{X}))$ is dense if and only if it is dense in the space.

Definition 1.3. (1) *Let L be a frame and* a, b *in L. We say that* a *is rather below* b *if there exists* c *in L such that* $a \wedge c = 0$ *and* $b \vee c = e$, *equivalently,* $a^* \vee b = e$. *In this case, we write* $a \prec b$.

(2) *A frame L is said to be regular if for any* a *in L,* $a = \bigvee\{b \in L | b \prec a\}$.

We note that $u \prec v$ in $\Omega(\mathrm{X})$ means $\overline{u} \subseteq v$, for a topological space $(\mathrm{X}, \Omega(\mathrm{X}))$ and it is clear that a topological space $(\mathrm{X}, \Omega(\mathrm{X}))$ is regular if and only if $\Omega(\mathrm{X})$ is a regular frame.

The following definition is a natural generalization of compact frames and Lindelöf spaces.

Definition 1.4. *A frame L is said to be a Lindelöf frame if for any subset* S *of L with* $\bigvee S = e$, *there is a countable subset* C *of* S *with* $\bigvee C = e$.

A 1-1 frame homomorphism is clearly codense and therefore the following is immediate :

PROPOSITION 1.5. *If $h : L \to M$ is a 1-1 frame homomorphism and M is a Lindelöf frame, then L is a Lindelöf frame.*

DEFINITION 1.6. ([2]) *A frame L is said to be a $D(\aleph_1)$ frame if for any a in L and any sequence $(b_n)_{n\in N}$ in L, $a \vee (\bigwedge_{n\in N} b_n) = \bigwedge_{n\in N} (a \vee b_n)$.*

PROPOSITION 1.7. *If $x_n \prec y$ for all n in N in a $D(\aleph_1)$ frame L, then $\bigvee_{n\in N} x_n \prec y$ in L.*

DEFINITION 1.8. *A nucleus k on a frame L is a map $k : L \to L$ such that for any a, b in L,*

(1) $a \leq k(a)$
(2) $k \circ k(a) = k(a)$
(3) $k(a \wedge b) = k(a) \wedge k(b)$

For a nucleus k on a frame L, $\mathrm{L}_k = \mathrm{Fix}(k) = \{x \in \mathrm{L} \mid k(x) = x\}$ is a frame, the corestriction $k_0 : \mathrm{L} \to \mathrm{Fix}(k)$ of k is an onto frame homomorphism and L_k is called a *sublocale* of L.

For any a in L, consider the nucleus $c_a : \mathrm{L} \to \mathrm{L}$ defined by $c_a(x) = a \vee x$. Then $\mathrm{Fix}(c_a) = \uparrow a$, which is called a *closed sublocale* of L.

Furthermore, the map $j : \mathrm{L} \to \mathrm{L}$ defined by $j(a) = a^{**}$ is also a nucleus and the sublocale $\mathrm{Fix}(j) = \{a \in \mathrm{L} \mid a^{**} = a\}$ is a smallest dense sublocale of L and the corresponding onto frame homomorphism will be denoted by $j_0 : \mathrm{L} \to \mathrm{L}_{**}$.

2. Almost Lindelöf frames

In this section, we introduce and study almost Lindelöf frames and almost Lindelöfications of frames.

DEFINITION 2.1. *A subset D of a poset L is said to be :*

(1) *countably down directed if every countable subset of D has a lower bound in D.*
(2) *a δ-filter if it is a countably down directed filter.*

We note that a filter on $\wp(X)$ is a δ-filter if and only if it is closed under countable intersections. More generally, a filter F in a complete lattice L is a δ-filter if and only if it is closed under countable meets. Thus the intersection of a non-empty family of δ-filters on a complete lattice L is again a δ-filter on L. Moreover, if F is a δ-filter on L, then for any sequence $(x_n)_{n\in N}$, $\bigwedge_{k\in N} x_k \neq 0$, i.e., F has the countable meet property.

In the following, we will assume that L is a complete lattice.

Remark. (1) *Let A be a subset of L. Then there is a δ-filter F with $A \subseteq F$ if and only if A has the countable meet property. Indeed, the condition is clearly necessary. For the converse, let $F = \{a \in L \mid$ there is a countable subset C of A with $\bigwedge C \leq a\}$, then F is clearly a δ-filter containing A, which will be called a δ-filter generated by A.*

(2) *Let $(F_\imath)_{\imath\in I}$ be a non-empty family of δ-filters on L. Then there is a δ-filter F on L with $F \supseteq \bigcup F_\imath$ if and only if for any countable subset J of I and $a_\imath \in F_\imath$ ($\imath \in J$), $\bigwedge_{\imath\in J} a_\imath \neq 0$. In particular, let F be a δ-filter on L and a in L. Then there is a δ-filter G on L with $G \supseteq F$ and $a \in G$ if and only if for any $d \in F$, $d \wedge a \neq 0$.*

Definition 2.2. *A frame L is said to be an almost Lindelöf frame if for any subset S of L with $\bigvee S = e$, there is a countable subset A of S such that $(\bigvee A)^* = 0$.*

Example 2.3. (1) *An almost compact frame is an almost Lindelöf frame.*

(2) *A Lindelöf frame is an almost Lindelöf frame.*

(3) *For a topological space $(X, \Omega(X))$, $\Omega(X)$ is an almost Lindelöf frame if and only if $(X, \Omega(X))$ is an almost Lindelöf space.*

(4) *The regular open set lattice $O_{reg}(R)$ on the real line is non-spatial since it is an atomless Boolean algebra([7]). Thus $O_{reg}(R)$ is a non-spatial almost Lindelöf frame.*

Theorem 2.4. *Let L be a frame. Then the following are equivalent :*

1) *L is an almost Lindelöf frame.*

2) *For any δ-filter F in L, $\bigvee\{x^*|x \in F\} \neq e$.*

Proof. 1) $\Rightarrow$ 2) Suppose not, then there is a δ-filter F in L such that $\bigvee\{x^*|x \in F\} = e$. By 1), there is a countable subset G of F such that $(\bigvee\{y^*|y \in G\})^* = 0$. Since $(\bigvee\{y^*|y \in G\})^* = \bigwedge\{y^{**}|y \in G\}$ and $y \leq y^{**}$ for any y in L, $\bigwedge G = 0$, which is a contradiction to the fact that F is a δ-filter.

2) $\Rightarrow$ 1) Suppose that there is a subset S of L such that $\bigvee S = e$ but for any countable subset A of S, $(\bigvee A)^* \neq 0$. Thus $\{x^*|x \in S\}$ has the countable meet property and hence generates a δ-filter, say U. Then
$U = \{y \in L \mid$ there is a countable subset A of S such that $\bigwedge\{x^*|x \in a\} \leq y\}$.
Using $x \leq x^{**}$ for any x in L, one has $e = \bigvee S \leq \bigvee\{x^{**} \mid x \in S\} \leq \bigvee\{u^* \mid u \in U\}$, because $\{x^* \mid x \in S\} \subseteq U$. Thus $\bigvee\{u^* \mid u \in U\} = e$, which is a contradiction to 2). □

Proposition 2.5. *Let $f : L \to M$ be a dense frame homomorphism. If M is an almost Lindelöf frame, then so is L*

Proof. Suppose not, then there is a δ-filter F in L such that $\bigvee\{x^*|x \in F\} = e$. Let $G = \{y \in \mathrm{M} \mid f(x) \leq y$ for some $x \in \mathrm{F}\}$. Then clearly G is an upper set. Take any sequence $(y_n)_{n\in N}$ in G, there is x_n in F such that $f(x_n) \leq y_n$. Thus $\bigwedge\limits_{n\in N} x_n$ is in F and $\bigwedge\limits_{n\in N} y_n \geq \bigwedge\limits_{n\in N} f(x_n) \geq f(\bigwedge\limits_{n\in N} x_n)$; hence $\bigwedge\limits_{n\in N} y_n$ is in G. Since f is dense, $0 \notin G$. Thus G is a δ-filter on M. Since $x \wedge x^* = 0$ in L implies that $0 = f(0) = f(x) \wedge f(x^*)$, $f(x^*) \leq f(x)^*$. Now, we have $e_M = f(e_L) = f(\bigvee\{x^*|x \in F\}) = \bigvee\{f(x^*) \mid x \in F\}$ $\leq \bigvee\{f(x)^* \mid x \in F\} \leq \bigvee\{y^* \mid y \in G\}$, for $\{f(x) \mid x \in F\} \subseteq G$. Thus $e_M = \bigvee\{y^* \mid y \in G\}$. But this contradicts to the fact that M is an almost Lindelöf frame. □

Corollary 2.6. *If L is a frame and $L_{**} = \{x \in L \mid x = x^{**}\}$ is an almost Lindelöf frame, then so is L.*

Proof. Since $j_0 : \mathrm{L} \to \mathrm{L}_{**}$ is a dense onto frame homomorphism, it is immediate from the above proposition. □

Proposition 2.7. *If L is an almost Lindelöf frame and a is in L, then $\uparrow a^* = L_{c(a^*)}$ is also an almost Lindelöf frame.*

Proof. Take any nonempty subset S of $L_{c(a^*)} = \uparrow a^*$ with $\bigvee_{\uparrow a^*} \mathrm{S} = e$. Since $\bigvee_{\uparrow a^*} \mathrm{S} = \bigvee_{\mathrm{L}} \mathrm{S}$, there is a countable subset T of S with $(\bigvee \mathrm{T})^* = 0$. If $\mathrm{T} = \phi$, then $e = 0$, that is, L is singleton ; hence we may assume that $\mathrm{T} \neq \phi$. For any y in $\uparrow a^*$ with $y \wedge (\bigvee \mathrm{T}) = a^*$, the bottom element of $\uparrow a^*$, $0 = a \wedge a^* = a \wedge y \wedge (\bigvee \mathrm{T})$ implies that $a \wedge y \leq (\bigvee \mathrm{T})^* = 0$ and hence

$a \wedge y = 0$, i.e. $y \leq a^*$. Thus $y = a^*$. Therefore the pseudocomplement of $\bigvee \mathrm{T}$ in $\uparrow a^*$ is a^*. So $L_{c(a^*)}$ is an almost Lindelöf frame. □

Example 2.8. *Let X be an uncountable set and p an element of X. Let $\Omega(X) = \{U \subseteq X \mid U = \phi$ or $p \in U\}$. Then $(X, \Omega(X))$ is not a Lindelöf space and hence $\Omega(X)$ is not a Lindelöf frame. But $(X, \Omega(X))$ is almost compact, for $\{p\}$ is an open dense subset of X. Thus $\Omega(X)$ is an almost Lindelöf frame.*

Proposition 2.9. *Let L be a regular $D(\aleph_1)$ frame. Then L is a Lindelöf frame if and only if L is an almost Lindelöf frame.*

Proof. The condition is clearly necessary. For the converse, take any subset S of L with $\bigvee \mathrm{S} = e$ and let $\mathrm{W}_s = \{x \in \mathrm{L} \mid x \prec s\}$ for any s in S. Since L is regular, $\bigvee \mathrm{W}_s = s$; $e = \bigvee \mathrm{S} = \bigvee\limits_{s \in S} (\bigvee \mathrm{W}_s) = \bigvee(\bigcup\limits_{s \in S} \mathrm{W}_s)$. Since L is an almost Lindelöf frame, there is a countable subset F of $\bigcup\limits_{s \in S} \mathrm{W}_s$ such that $(\bigvee \mathrm{F})^* = 0$. Thus for any y in F, there is s_y in S such that $y \prec s_y$; hence there is c_y in L such that $y \wedge c_y = 0$ and $s_y \vee c_y = e$. Since $(\bigvee \mathrm{F}) \wedge (\bigwedge\{c_y \mid y \in \mathrm{F}\}) = \bigvee\limits_{z \in F} (z \wedge (\bigwedge\{c_y \mid y \in \mathrm{F}\})) \leq \bigvee(z \wedge c_z) = 0$, $\bigwedge\{c_y \mid y \in \mathrm{F}\} \leq (\bigvee \mathrm{F})^* = 0$ so that $\bigwedge\{c_y \mid y \in \mathrm{F}\} = 0$. Since L is $D(\aleph_1)$,
$\bigvee\{s_y \mid y \in \mathrm{F}\} = \bigvee\{s_y \mid y \in \mathrm{F}\} \vee (\bigwedge\{c_y \mid y \in \mathrm{F}\}) = \bigwedge\limits_{z \in F} \{c_z \vee (\{s_y \mid y \in \mathrm{F}\})\} \geq \bigwedge\limits_{z \in F} \{c_z \vee s_z\} = e$.
Therefore $\bigvee\{s_y \mid y \in \mathrm{F}\} = e$ and $\{s_y \mid y \in \mathrm{F}\}$ is a countable subset of S. In all, L is a Lindelöf frame. □

In the following, we will construct almost Lindelöfication of a frame by a certain simple extension of a frame.

Notation 2.10. *Let L be a frame and X the set of all δ-filters F such that $\bigvee\{x^* \mid x \in F\} = e$. Then the subframe $s_X L = \{(x, \Sigma) \in L \times \wp(X) \mid$ for any F in $\Sigma, x \in F\}$ of $L \times \wp(X)$ is the simple extension of L associated with X. Furthermore, for any $x \in X$, $\Sigma_x = \{F \in \Sigma \mid x \in F\}$. And $s : s_X L \to L$ defined by the restriction of the first projection is open, dense and onto. (See [4] for the detail.)*

Using the above notation, one has the following :

Theorem 2.11. *Let L be a frame and X the set of all δ-filters F such that $\bigvee\{x^* \mid x \in F\} = e$. Then $s_X L$ is an almost Lindelöf frame.*

Proof. Suppose that there is a δ-filter G in $\mathrm{s}_X\mathrm{L}$ with $\bigvee\{v^* \mid v \in \mathrm{G}\} = (e, \mathrm{X})$. Since s is dense and onto, $s(\mathrm{G})$ is a filter in L. For any sequence $(x_n)_{n\in N}$ in $s(\mathrm{G})$, there is (x_n, Λ_n) in G for some $\Lambda_n \in \wp(\mathrm{X})$ for all $n \in N$. Since G is a δ-filter, there is (x, Λ) in G with $(x, \Lambda) \leq (x_n, \Lambda_n)$ for all $n \in N$. Thus $x \leq x_n$ for all $n \in N$ and $x \in s(\mathrm{G})$. Therefore $s(\mathrm{G})$ is a δ-filter. Since s is open, s preserves pseudocomplements. Thus $\bigvee\{u^* \mid u \in s(\mathrm{G})\} = e$; hence $s(\mathrm{G})$ is in X. For any $v = (x, \Lambda)$ in G,
$(e, \{s(\mathrm{G})\}) \wedge v^* = (e, \{s(\mathrm{G})\}) \wedge (x^*, \Sigma_{x^*}) = (x^*, \{s(\mathrm{G})\} \cap \Sigma_{x^*})$.
Since $x = s(v) \in s(\mathrm{G})$, $x^* \notin s(\mathrm{G})$; $(e, \{s(\mathrm{G})\}) \wedge v^* = (x^*, \phi)$. Hence $(e, \{s(\mathrm{G})\}) = (e, \{s(\mathrm{G})\}) \wedge (e, \mathrm{X}) = (e, \{s(\mathrm{G})\}) \wedge (\bigvee\{v^* \mid v \in \mathrm{G}\}) = \bigvee_{v\in\mathrm{G}} ((e, \{s(\mathrm{G})\}) \wedge v^*) = ((\bigvee\{x^* \mid (x, \Lambda) \in \mathrm{G}\}), \phi)$. This is a contradiction. □

The above almost Lindelöfication $s : \mathrm{s}_X\mathrm{L} \to \mathrm{L}$ will be denoted by $\delta : \delta\mathrm{L} \to \mathrm{L}$.

The proof of the following can be found in [4] and [10].

Proposition 2.12. *Let X be a set of filters in a frame L, then an element (x, Σ) in $s_X L$ is prime if and only if one of the following holds:*

1) *x is a prime element in L and $\Sigma = \Sigma_x$..*
2) *$x = e$ and $\Sigma = X - \{F\}$, for some $F \in X$.*

Using the above and the exactly same arguments as those in [4] and [10], we have the following :

Theorem 2.13. *A frame L is spatial if and only if δL is spatial.*

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