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LOCALLY DIVIDED DOMAINS OF THE FORM $\mathbf{D[X]_{N_v}}$

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Abstract. Let D be an integral domain, X be an indeterminate over D, and $N_v = \{f \in D[X] | (A_f)_v = D\}$. In this paper, we introduce the concept of t-locally divided domains, and we then prove that $D[X]_{N_v}$ is a locally divided domain if and only if D is a t-locally divided UMT-domain, if and only if $D[X]$ is a t-locally divided domain.

1. Introduction

Let D be an integral domain with quotient field K. As in [6], we say that a prime ideal P of D is *divided* if P is comparable to each principal ideal of D; equivalently, $P = PD_P$, while D is called a *divided domain* if each prime ideal of D is divided. It is easy to show that if D is divided, then Spec(D), the set of prime ideals of D, is linearly ordered under inclusion, and hence D is quasi-local. Following [6], we say that D is a *locally divided domain* if D_M is divided for each maximal ideal M of D. Examples of locally divided domains include Prüfer domains and integral domains of (Krull) dimension 1. A prime ideal P of D is said to be *strongly prime* if $xy \in P$ and $x, y \in K$ imply $x \in P$ or $y \in P$. Recall that D is a *pseudo-valuation domain* (PVD) if every prime ideal of D is strongly prime. Also, recall from [8] that D is called a *locally pseudo-valuation domain* (LPVD) if D_M is a pseudo-valuation domain for each maximal ideal M of D. It is well known that a strongly prime ideal is divided, and hence a PVD is a divided domain and an LPVD is locally divided. For more on (locally) divided domains and (locally) PVDs, see [1, 2, 6, 8, 11].

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Let X be an indeterminate over D, and let $D[X]$ be the polynomial ring over D. For any $f \in D[X]$, let A_f be the ideal of D generated by the coefficients of f. Let $N_v = \{f \in D[X] | (A_f)_v = D\}$; then $D[X]_{N_v}$ is an overring of $D[X]$. (Definitions related to the t-operation will be reviewed in the sequel.) In [4], the author introduced the notion of *t-locally pseudo-valuation domains* (t-LPVDs) to study when $D[X]_{N_v}$ is an LPVD. In particular, it was shown that $D[X]_{N_v}$ is an LPVD if and only if D is a t-LPVD and a UMT-domain [4, Corollary 3.8]. The purpose of this paper is to study when $D[X]_{N_v}$ is a locally divided domain. More precisely, we first introduce the concept of t-locally divided domains, and we then prove that $D[X]_{N_v}$ is a locally divided domain if and only if D is a t-locally divided UMT-domain, if and only if $D[X]$ is a t-locally divided domain.

We first review some definitions related to the t-operation. Let $\mathbf{F}(D)$ (resp., $\mathbf{f}(D)$) be the set of nonzero fractional ideals (resp., finitely generated fractional ideals) of D; so $\mathbf{f}(D) \subseteq \mathbf{F}(D)$. For any $I \in \mathbf{F}(D)$, let $I^{-1} = \{x \in K | xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, $I_t = \cup\{J_v | J \subseteq I \text{ and } J \in \mathbf{f}(D)\}$, and $I_w = \{x \in K | xJ \subseteq I \text{ for some } J \in \mathbf{f}(D) \text{ with } J^{-1} = D\}$. Let $* = t$ or w. An $I \in \mathbf{F}(D)$ is called a *$*$-ideal* if $I_* = I$; while I is said to be *$*$-invertible* if $(II^{-1})_* = D$. Let $*$-Max(D) denote the set of $*$-ideals of D maximal among proper integral $*$-ideals of D. Each (necessarily prime) ideal in $*$-Max(D) is called a *maximal $*$-ideal*. It is well known that $*$-Max$(D) \neq \emptyset$ if D is not a field; each (integral) $*$-ideal is contained in a maximal $*$-ideal; and t-Max$(D) = w$-Max(D). Recall that D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal is t-invertible, while D is a UMT-domain if each upper to zero in $D[X]$ is a maximal t-ideal. (An upper to zero Q in $D[X]$ is a nonzero prime ideal of $D[X]$ such that $Q \cap D = (0)$.) It is well known that D is a PvMD if and only if $D[X]_{N_v}$ is a Prüfer domain [13, Theorem 3.7], if and only if D is an integrally closed UMT-domain [12, Proposition 3.2]. For any undefined notation and definition, see [10].

2. On t-locally divided domains

Throughout D is an integral domain with quotient field K (we assume $D \neq K$), X is an indeterminate over D, and $D[X]$ is the polynomial ring over D. Let $S = \{f \in D[X] | A_f = D\}$, $N_v = \{f \in D[X] | (A_f)_v = D\}$, and $D(X) = D[X]_S$ the Nagata ring of D; so if D is quasi-local with maximal ideal P, then $D(X) = D[X]_{P[X]}$.

LEMMA 1. *If D is a locally divided domain, then each nonzero prime ideal of D is a t-ideal.*

Proof. If P is a nonzero prime ideal of D, then D_P is divided, and hence PD_P is a t-ideal [13, Theorem 3.19]. Thus $P = PD_P \cap D$ is a t-ideal [13, Lemma 3.17]. □

LEMMA 2. *If $D[X]_{N_v}$ is divided, then D is quasi-local whose maximal ideal is a t-ideal. Hence $D[X]_{N_v} = D(X)$.*

Proof. Recall that a divided domain is quasi-local [6, Proposition 2.1]. Hence $D[X]_{N_v}$ is quasi-local, and since $\mathrm{Max}(D[X]_{N_v}) = \{P[X]_{N_v} | P \in t\text{-}\mathrm{Max}(D)\}$ [13, Proposition 2.1], D has a unique maximal t-ideal. Next, let P be the maximal t-ideal of D. Let $a \in D$ be a nonzero nonunit. Then aD is a proper t-ideal of D, and since each t-ideal is contained in a maximal t-ideal, we have $a \in aD \subseteq P$. Thus D is quasi-local with maximal ideal P and $D[X]_{N_v} = D[X]_{P[X]} = D(X)$. □

LEMMA 3. *The following statements are equivalent for an integral domain D.*

1. *$D[X]_{N_v}$ is a divided domain.*
2. *$D(X)$ is a divided domain.*
3. *D is a divided UMT-domain.*

Proof. (1) $\Rightarrow$ (2) Lemma 2.

(2) $\Rightarrow$ (1) and (3) Let P be a nonzero prime ideal of D. Then $P(X) = PD(X)$ is a prime ideal of $D(X)$, and hence $P(X)$ is divided. Since $P(X) = P(X)_{P(X)} = P[X]_{P[X]}$, we have $P = P(X) \cap K = P[X]_{P[X]} \cap K = PD_P$; so P is divided. Thus D is divided and D is quasi-local. Let M be the maximal ideal of D; then M is a t-ideal by Lemma 1. In particular, $D[X]_{N_v} = D(X)$.

Next, assume that D is not a UMT-domain, and let $f \in M[X]$ such that $Q := fK[X] \cap D[X]$ is a nonzero prime ideal and Q is not a maximal t-ideal. Then $(\sum_{y \in Q} A_y)_t \subsetneq D$ [12, Theorem 1.4], and since M is a t-ideal, we have $Q \subsetneq M[X]$. Hence $Q_S \subsetneq M(X)$ and $Q_Q = (Q_S)_{Q_S} = Q_S$ by (2). Let a be a nonzero coefficient of f. Then $a^2 \in D[X] \setminus Q$ and $f \in Q$, and hence $\frac{f}{a^2} \in Q_Q = Q_S$. So there are some $g \in Q$ and $h \in S$ such that $\frac{f}{a^2} = \frac{g}{h}$ or $fh = a^2 g$. Let m be a positive integer such that $A_h^{m+1}A_f = A_h^m A_{fh}$ [10, Theorem 28.1]. Since $A_h = D$, we have $A_f = A_{fh}$, and hence $A_f = a^2A_g \subseteq a^2D$. So $a \in a^2D$, and hence a is a unit of D. Thus $f \notin M[X]$, a contradiction. Therefore, D is a UMT-domain.

(3) $\Rightarrow$ (2) First, note that D is quasi-local whose maximal ideal is a t-ideal by Lemma 1. So $D(X) = D[X]_{N_v}$, and hence each prime ideal of $D(X)$ is extended from D [12, Theorem 3.1], i.e., $\text{Spec}(D(X)) = \{P(X) | P \in \text{Spec}(D)\}$. Let P be a nonzero prime ideal of D; then $P(X)$ is a prime ideal of $D(X)$ and $P(X)_{P(X)} = P[X]_{P[X]}$. Hence, to prove that $P(X)$ is divided, it suffices to show that $P[X]_{P[X]} \subseteq P(X)$.

Let $f \in P[X]$ and $g \in D[X] \setminus P[X]$. Since D is a UMT-domain, there is a polynomial $h \in K[X]$ such that $(A_{gh})_v = D$ [9, Lemma 3.4]. Let m be a positive integer such that $A_g^{m+1}A_h = A_g^m A_{gh} \subseteq D$ [10, Theorem 28.1]. Since $A_g \nsubseteq P$, we have $A_g^{m+1} \nsubseteq P$. Choose $a \in A_g^{m+1} \setminus P$; then $ah \subseteq D[X]$. Since P is divided, we have $A_f \subseteq P \subsetneq aD$, and hence $A_{fh} \subseteq A_f A_h \subseteq aA_h = A_{ah} \subseteq D$; thus $fh \in D[X]$. So $\frac{f}{g} = \frac{fh}{gh} \in D(X)$, and since $P[X]_{P[X]} \cap D(X) = P(X)$, we have $\frac{f}{g} \in P(X)$. □

Recall that D is a *locally divided domain* if D_M is divided for each maximal ideal M of D. Hence it is natural to say that D is a *t-locally divided domain* if D_P is divided for each maximal t-ideal P of D. Recall that D is a PvMD if and only if D_P is a valuation domain for each maximal t-ideal P of D [13, Theorem 3.2]. Thus a PvMD is a t-locally divided domain.

Lemma 4. *The following statements are equivalent for an integral domain D.*

1. *D is a locally divided domain.*
2. *D is t-locally divided and each maximal ideal of D is a t-ideal.*

Proof. Assume that D is a locally divided domain, and let M be a maximal ideal of D. Then M is a t-ideal by Lemma 1. Thus each maximal ideal of D is a t-ideal and D is t-locally divided. The converse is clear. □

We next give the main result of this paper.

Theorem 5. *The following statements are equivalent for an integral domain D.*

1. *D is a t-locally divided and a UMT-domain.*

2. *$D[X]$ is a t-locally divided domain.*
3. *$D[X]_{N_v}$ is a locally divided domain.*
4. *$D_P(X)$ is a divided domain for each maximal t-ideal P of D.*
5. *D_P is a divided UMT-domain for each maximal t-ideal P of D.*

Proof. (1) $\Rightarrow$ (2) Let Q be a maximal t-ideal of $D[X]$. If $Q \cap D = (0)$, then $D[X]_Q$ is a local PID, and hence a divided domain. Next, assume that $Q \cap D \neq (0)$, and let $Q \cap D = P$. Then $Q = P[X]$ [12, Proposition 1.1] and P is a maximal t-ideal of D (cf. [13, Corollary 2.3]). Hence D_P is divided by (1), and since D_P is a UMT-domain [9, Proposition 1.2] and $D[X]_Q = D[X]_{P[X]} = D_P[X]_{PD_P[X]} = D_P(X)$, it follows from Lemma 3 that $D[X]_Q$ is divided.

(2) $\Rightarrow$ (3) Let Q be a maximal ideal of $D[X]_{N_v}$. Then $Q = P[X]_{N_v}$ for some maximal t-ideal P of D [13, Proposition 2.1]. Note that $P[X]$ is a maximal t-ideal of $D[X]$ (cf. [12, Proposition 1.1] and [13, Corollary 2.3]); hence $D[X]_{P[X]}$ is divided by (2). Since $(D[X]_{N_v})_Q = D[X]_{P[X]}$, it follows that $(D[X]_{N_v})_Q$ is divided.

(3) $\Rightarrow$ (1) It suffices to show that D_P is a divided UMT-domain for each maximal t-ideal P of D (cf. [4, Lemma 2.2]). Let P be a maximal t-ideal of D. Then $P[X]_{N_v}$ is a maximal ideal of $D[X]_{N_v}$ [13, Proposition 2.1] and $D[X]_{P[X]} = (D[X]_{N_v})_{P[X]_{N_v}}$. Hence $D[X]_{P[X]}$ is divided by (3). Note that $D[X]_{P[X]} = (D_P[X])_{PD_P[X]} = D_P(X)$. Thus D_P is a divided UMT-domain by Lemma 3.

(1) $\Leftrightarrow$ (5) This follows from [4, Lemma 2.2] and Lemma 1.

(4) $\Leftrightarrow$ (5) Lemma 3. □

Corollary 6. *The following statements are equivalent for an integral domain D.*

1. *D is a locally divided domain and a UMT-domain.*
2. *D is a locally divided domain and D has Prüfer integral closure.*
3. *$D(X)$ is a locally divided domain.*

Proof. (1) $\Leftrightarrow$ (2) Recall that if each maximal ideal of D is a t-ideal, then D is a UMT-domain if and only if the integral closure of D is a Prüfer domain (cf. [9, Theorem 1.5]). Thus the result follows directly from Lemma 1. (1) $\Leftrightarrow$ (3) This is an immediate consequence of Lemma 4 and Theorem 5. □

An integral domain D is called a *strong Mori domain* (SM-domain) if D satisfies the ascending chain condition on integral w-ideals of D; equivalently, each w-ideal of D is of finite type. Clearly Noetherian domains are SM-domains. It is known that an SM-domain D is a UMT-domain if and only if each prime t-ideal of D is a maximal t-ideal [5, Corollary 3.2] and that D is an SM-domain if and only if $D[X]_{N_v}$ is a Noetherian domain [3, Theorem 2.2]. Let D be a locally divided Noetherian domain. If P is a maximal ideal of D, then D_P is a divided Noetherian domain, and hence D_P must be of (Krull) dimension 1. Thus a Noetharian domain D is locally divided if and only if the (Krull) dimension of D is 1. We mean by t-dim$(D) = 1$ that each prime t-ideal of D is a maximal t-ideal.

Corollary 7. *The following statements are equivalent for an SM-domain D.*

1. *D is a t-locally divided domain.*
2. *$D[X]$ is a t-locally divided domain.*
3. *t-dim$(D) = 1$.*
4. *$D[X]_{N_v}$ is a locally divided domain.*

Proof. (1) $\Leftrightarrow$ (3) This follows because an SM domain D is t-locally Noetherian, i.e., D_P is Noetherian for each maximal t-ideal P of D [3, Theorem 2.2].

(1) $\Rightarrow$ (2) Let P be a maximal t-ideal of D. Then D_P is of (Krull) dimension 1 by the equivalence of (1) and (3). Hence D is a UMT-domain [5, Corollary 3.2], and thus $D[X]$ is t-locally divided by Theorem 5.

(2) $\Leftrightarrow$ (4) $\Rightarrow$ (1) Theorem 5. □

We end this paper with two examples of t-locally divided domains that are not locally divided domains.

Example 8. (1) Let D be a Noetherian domain of (Krull) dimension 1. Then $D[X]$ is a t-locally divided domain by Corollary 7. However, if M is a maximal ideal of D, then $M_0 := M[X] + XD[X]$ is a maximal ideal of $D[X]$ such that $D[X]_{M_0}$ is not a divided domain. Thus $D[X]$ is not a locally divided domain.

(2) Let D be a PvMD. Then D is a t-locally divided UMT-domain, and hence $D[X]$ is a t-locally divided domain by Theorem 5. However, if P is a maximal t-ideal of D, then $Q := P[X] + XD[X]$ is a prime ideal of $D[X]$ but $D[X]_Q$ is not a divided domain. Thus $D[X]$ is not a locally divided domain.

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