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FERMAT-TYPE EQUATIONS FOR MÖBIUS TRANSFORMATIONS

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Abstract. A Fermat-type equation deals with representing a nonzero constant as a sum of kth powers of nonconstant functions. Suppose that $k \geq 2$. Consider $\sum_{i=1}^{p} f_i(z)^k = 1$. Let p be the smallest number of functions that give the above identity. We consider the Fermat-type equation for Möbius transformations and obtain $k \leq p \leq k+1$.

1. Introduction

A Fermat-type equation is to represent a nonzero constant as a sum of kth powers of nonconstant functions. We allow complex coefficients in these problems. Let k and n be natural numbers. Consider the equation of the form

$$\sum_{i=1}^{n} f_i(z)^k = C,$$

where C is a nonzero constant. Suppose that

$$\sum_{i=1}^{n} f_i(z)^k = 1. \tag{1.1}$$

Then, for any choice of the branch of $C^{1/k}$, we get

$$\sum_{i=1}^{n} \left(C^{\frac{1}{k}} f_i(z)\right)^k = C.$$

Thus any nonzero constant can be represented by the sum of n kth powers of nonconstant functions. Equations of the form (1.1) are called Fermat-type equations.

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DEFINITION 1.1. Suppose that $k \geq 2$ and that $n \geq 2$. Suppose that S is a set of functions. Let $f_1, f_2, \ldots, f_n$ be nonconstant functions in S satisfying

$$\sum_{i=1}^{n} f_i(z)^k = 1. \tag{1.2}$$

$F_S(k)$ denotes the smallest number n satisfying the equation (1.2).

We denote the sets of linear polynomials, polynomials, entire functions, rational functions, and meromorphic functions by L, P, E, R and M respectively. Newman and Slater showed that any nonzero constant can be represented by a sum of $(k+1)$ kth powers of nonconstant polynomials [9]. Therefore Fermat-type equations for P, E, R and M are solvable.

THEOREM 1.1 ([8]). *We have the following result for the equation* (1.2).

$$F_P(k) \leq \left[(4k+1)^{1/2}\right], \tag{1.3}$$

where $[x] = \max\{m \in \mathbb{Z} \mid m \leq x\}$.

Hence $\left[\sqrt{4k+1}\right]$ is an upper bound for $F_P(k)$, $F_E(k)$, $F_R(k)$ and $F_M(k)$.

THEOREM 1.2. *We have the following results for the equation* (1.2).

$$F_P(k) > \frac{1}{2} + \sqrt{k + \frac{1}{4}}. \tag{1.4}$$

$$F_E(k) \geq \frac{1}{2} + \sqrt{k + \frac{1}{4}}. \tag{1.5}$$

$$F_R(k) > \sqrt{k+1}. \tag{1.6}$$

$$F_M(k) \geq \sqrt{k+1}. \tag{1.7}$$

Theorem 1.2 is a collection of results to be found in [2], [3], [5], [9] and [10].

THEOREM 1.3 ([7]). *We have the following result for the equation* (1.2).

$$F_L(k) = k + 1. \tag{1.8}$$

More details and results can be found in the survey papers; see [4] and [6].

Definition 1.2. A *Möbius transformation*, also called a *linear fractional transformation* or a *bilinear transformation*, is a map

$$f(z) = \frac{az+b}{cz+d}, \ (ad - bc \neq 0). \tag{1.9}$$

We denote the set of Möbius transformations by T.

2. Fermat-type equations for Möbius transformations

Now we prove our theorems.

Lemma 2.1. *Suppose that $k \geq 2$ and that $n \geq 2$. Let $f_1, f_2, \ldots, f_n$ be nonconstant linear polynomials satisfying*

$$\sum_{i=1}^{n} f_i(z)^k = z^k. \tag{2.1}$$

Suppose that q is the smallest number n satisfying the equation (2.1)*. Then, $q \geq k$.*

Proof. Let q be the smallest number n satisfying the equation (2.1). Then we can write

$$\sum_{i=1}^{q}(a_i z + b_i)^k = z^k. \tag{2.2}$$

Now, we suppose that $q < k$ and will obtain a contradiction. According to the minimality of q, all the $(a_i z + b_i)^k$ with $1 \leq i \leq q$ are linearly independent. Hence we can have $b_i = 0$ for at most one i. Then $(b_i + a_i z)^k = {b_i}^k \left(1 + \frac{a_i}{b_i} z\right)^k$ if $b_i \neq 0$. Suppose that ${b_i}^k = B_i$ and that $\frac{a_i}{b_i} = A_i$ for each i.

Suppose that $b_q = 0$ and $b_i \neq 0$ for $1 \leq i \leq q-1$. Then

$$\begin{aligned}
\sum_{i=1}^{q}(a_i z + b_i)^k &= {a_q}^k z^k + \sum_{i=1}^{q-1} B_i (1 + A_i z)^k \\
&= {a_q}^k z^k + \sum_{i=1}^{q-1} B_i \left(\sum_{r=0}^{k} \binom{k}{r} {A_i}^r z^r \right) \\
&= {a_q}^k z^k + \sum_{r=0}^{k} \binom{k}{r} z^r \left(\sum_{i=1}^{q-1} B_i {A_i}^r \right).
\end{aligned}$$

Since the right hand side of the equation (2.2) is equal to z^k, we get, in particular, the system of equations

$$\sum_{i=1}^{q-1} {A_i}^r B_i = 0 \qquad \text{for } 0 \le r \le k-1. \tag{2.3}$$

Because $q < k$, we use $q-1$ equations for $0 \le r \le q-2$. Now consider B_i for $1 \le i \le q-1$ as unknowns. Then the coefficients form a square matrix M_1 whose determinant is given by

$$|M_1| = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ A_1 & A_2 & \cdots & A_{q-1} \\ {A_1}^2 & {A_2}^2 & \cdots & {A_{q-1}}^2 \\ \vdots & \vdots & \ddots & \vdots \\ {A_1}^{q-2} & {A_2}^{q-2} & \cdots & {A_{q-1}}^{q-2} \end{vmatrix}.$$

Since the determinant of M_1 is the van der Monde determinant [1], we get

$$|M_1| = \prod_{i<j} (A_j - A_i).$$

Since all the $(a_i z + b_i)^k$ with $1 \le i \le q$ are linearly independent, we have $A_i \ne A_j$ for $i \ne j$ and we get $|M_1| \ne 0$. Hence the system (2.3) of homogeneous linear equations has only the trivial solution and so ${b_i}^k = B_i = 0$ for all i with $1 \le i \le q-1$. Thus $b_i = 0$ for all i with $1 \le i \le q-1$. This is a contradiction.

Suppose that $b_i \ne 0$ for each i. Then

$$\sum_{i=1}^{q} (a_i z + b_i)^k = \sum_{r=0}^{k} \binom{k}{r} z^r \left(\sum_{i=1}^{q} B_i {A_i}^r \right).$$

Because the right hand side of the equation (2.2) is equal to z^k, we get

$$\sum_{i=1}^{q} {A_i}^r B_i = 0 \qquad \text{for } 0 \le r \le k-1. \tag{2.4}$$

By using q equations for $0 \le r \le q-1$, we have a coefficient matrix M_2 whose determinant is given by

$$|M_2| = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ A_1 & A_2 & \cdots & A_q \\ {A_1}^2 & {A_2}^2 & \cdots & {A_q}^2 \\ \vdots & \vdots & \ddots & \vdots \\ {A_1}^{q-1} & {A_2}^{q-1} & \cdots & {A_q}^{q-1} \end{vmatrix}.$$

Since $|M_2| \neq 0$, the system (2.4) has only the trivial solution and so ${b_i}^k = B_i = 0$ for all i. Thus $b_i = 0$ for all i. This is a contradiction. Therefore we get $q \ge k$. □

EXAMPLE 2.1. We define $\omega = e^{2\pi i/(k+1)}$. Then

$$\sum_{j=1}^{k+1} \left(\frac{z + \omega^j}{(k+1)^{1/k}} \right)^k = z^k. \tag{2.5}$$

Thus, the equation (2.1) is solvable.

THEOREM 2.2. *Suppose that* $k \ge 2$ *and that* $n \ge 2$. *Let* $f_1, f_2, \ldots, f_n$ *be nonconstant Möbius transformations satisfying*

$$\sum_{i=1}^{n} f_i(z)^k = 1. \tag{2.6}$$

Suppose that at least one of the f_i *is not a linear polynomial and that* p *is the smallest number* n *satisfying the equation* (2.6). *Then,* $p \ge k$.

We do not need to consider the case that all functions f_i are linear polynomials because of Theorem 1.3.

Proof. Let p be the smallest number n satisfying the equation (2.6). Suppose that

$$\sum_{i=1}^{p} f_i(z)^k = 1, \tag{2.7}$$

where each f_i is a Möbius transformation. Since at least one of the f_i is not a linear polynomial, without loss of generality, we can suppose that $f_1, \ldots, f_s$ are linear polynomials (if $s = 0$, then there are no linear polynomials) while the remaining f_i have finite poles.

Any f_i with a finite pole z_0 can be written as $(az+b)/(z-z_0)$. Divide the functions f_i with finite poles into groups $G_1, \ldots, G_t$ so that those functions in a group G_j have the same finite pole z_j. Suppose that the group G_j consists of $f_{j_1}, \ldots, f_{j_m}$ with the pole z_j. Since all the other functions appearing in the equation (2.7) have no pole at z_j,

$$\sum_{i=j_1}^{j_m} f_i(z)^k \tag{2.8}$$

must have no pole at z_j. Since no function f_i in G_j is a constant function, there are at least two functions, that is, $j_m - j_1 \geq 1$.

For $j_1 \leq i \leq j_m$, we can write

$$f_i(z) = (a_i z + b_i)/(z - z_j).$$

Then we get

$$\sum_{i=j_1}^{j_m} f_i(z)^k = \frac{1}{(z - z_j)^k} \sum_{i=j_1}^{j_m} (a_i z + b_i)^k.$$

It follows that

$$\sum_{i=j_1}^{j_m} (a_i z + b_i)^k,$$

which is a polynomial of degree at most k, must have a zero of order at least k at z_j. Hence for some nonzero constant C_j, we must have

$$\sum_{i=j_1}^{j_m} (a_i z + b_i)^k = C_j (z - z_j)^k. \tag{2.9}$$

Thus we get

$$\sum_{i=j_1}^{j_m} f_i(z)^k = \sum_{i=j_1}^{j_m} \frac{(a_i z + b_i)^k}{(z - z_j)^k} = C_j$$

and so any such group adds up to a constant.

Since we may replace $z - z_j$ by z in the equation (2.9), for any choice of the branch of ${C_j}^{1/k}$, we get

$$\sum_{i=j_1}^{j_m} \left(\frac{a_i (z + z_j) + b_i}{{C_j}^{1/k}} \right)^k = z^k. \tag{2.10}$$

Hence, by Lemma 2.1, we get $j_m \geq k$ for each group G_j. Therefore we obtain $p = s + \sum_{j=1}^{t} j_m \geq k$. $\square$

THEOREM 2.3. *We have the following result for the equation* (1.2).

$$k \leq F_T(k) \leq k+1.$$

Proof. We get this result by Theorem 1.3 and Theorem 2.2. □

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