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SOME GEOMETRIC RESULTS ON A PARTICULAR SOLUTION OF EINSTEIN'S EQUATION

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Abstract. In the unified field theory(UFT), many works on the solutions of Einstein's equation have been published. The main goal in the present paper is to obtain some geometric results on a particular solution of Einstein's equation under some condition in even-dimensional UFT X_n.

1. Introduction

Einstein ([1], 1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Characterizing Einstein's unified field theory as a set of geometrical postulates in a 4-dimensional generalized Riemannian space X_4 (i.e., space-time), Hlavatý ([9], 1957) gave the mathematical foundation of the 4-dimensional unified field theory(UFT X_4) for the first time. Generalizing X_4 to the n-dimensional generalized Riemannian manifold X_n, n-dimensional generalization of this theory, the so-called *Einstein's n-dimensional unified field theory*(UFT X_n), had been obtained by Mishra ([8], 1958). Since then many consequences of this theory has been obtained by a number of mathematicians. The main goal in the present paper is to obtain some geometric results on a particular solution of Einstein's equation under some condition in even-dimensional UFT X_n. The obtained results and discussions in the present paper will be useful for the even-dimensional considerations of the unified field theory.

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2. Preliminary

This section is a brief collection of basic concepts, notations, and results, which are needed in our further considerations in the present paper.

Let X_n be an n-dimensional generalized Riemannian manifold covered by a system of real coordinate neighborhoods $\{U; x^\nu\}$, where, here and in the sequel, Greek indices run over the range $\{1, 2, \cdots, n\}$ and follow the summation convention. In the Einstein's usual n-dimensional unified field theory(UFT X_n), the algebraic structure on X_n is imposed by a basic real non-symmetric tensor $g_{\lambda\mu}$, the so-called *unified field tensor*, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.1}$$

where we assume that

$$G = det(g_{\lambda\mu}) \neq 0, \quad H = det(h_{\lambda\mu}) \neq 0. \tag{2.2}$$

Since $det(h_{\lambda\mu}) \neq 0$, we may define a unique tensor $h^{\lambda\nu} (= h^{\nu\lambda})$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.3}$$

We use the tensors $h^{\lambda\nu}$ and $h_{\lambda\mu}$ as tensors for raising and/or lowering indices for all tensors defined in UFT X_n in the usual manner. Then we may define new tensors by

$$k^\alpha{}_\mu = k_{\lambda\mu} h^{\lambda\alpha}, \quad k_\lambda{}^\alpha = k_{\lambda\mu} h^{\mu\alpha}. \tag{2.4}$$

In UFT X_n, the differential geometric structure is imposed by the tensor $g_{\lambda\mu}$ by means of a connection $\Gamma^\nu_{\lambda\mu}$ defined by the Einstein's equation:

$$\partial_\omega g_{\lambda\mu} - g_{\alpha\mu}\Gamma^\alpha_{\lambda\omega} - g_{\lambda\alpha}\Gamma^\alpha_{\omega\mu} = 0 \ \ (\partial_\nu = \frac{\partial}{\partial x^\nu}), \tag{2.5a}$$

or equivalently

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}, \tag{2.5b}$$

where D_ω denotes the symbolic vector of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$, and $S_{\lambda\mu}{}^\nu$ is the torsion tensor of $\Gamma^\nu_{\lambda\mu}$.

In UFT X_n, the following quantities are frequently used, where $p = 1, 2, 3, ...$:

$$(2.6)\quad \begin{aligned} &(a)\ \ g = \frac{G}{H}, \quad k = \frac{T}{H}, \\ &(b)\ \ K_0 = 1, \qquad K_p = k_{[\alpha_1}{}^{\alpha_1}\, k_{\alpha_2}{}^{\alpha_2} \ldots k_{\alpha_p]}{}^{\alpha_p}, \\ &(c)\ \ {}^{(0)}k_\lambda{}^\nu = \delta_\lambda^\nu, \quad {}^{(p)}k_\lambda{}^\nu = k_\lambda{}^{\alpha\,(p-1)}k_\alpha{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha\, k_\alpha{}^\nu, \\ &(d)\ \ \phi = {}^{(2)}k_\alpha{}^\alpha. \end{aligned}$$

It should be remarked that the tensor ${}^{(p)}k_{\lambda\nu}$ is symmetric if p is even, and skew-symmetric if p is odd.

Remark 2.1. From now on, we shall assume that

$$(2.7)\qquad T = det(k_{\lambda\mu}) \neq 0.$$

Hence there exists a unique skew-symmetric tensor $\overline{k}^{\lambda\mu}$ in X_n satisfying

$$(2.8)\qquad k_{\lambda\mu}\,\overline{k}^{\lambda\nu} = \delta_\mu^\nu.$$

Since $k_{\lambda\mu}$ is skew-symmetric, and $T \neq 0$, the dimension of X_n is even. That is, n is *even*. Hence *all our further considerations in the present paper are dealt in even-dimensional UFT X_n.*

Our investigation is based on the skew-symmetric tensor

$$(2.9)\qquad P_{\lambda\mu} = (1-\phi)k_{\lambda\mu} + {}^{(3)}k_{\lambda\mu},$$

where ϕ is given by (2.6)(d). And the following quantities are used in our further considerations. For $s = 2, 4, ..., n+2$,

$$(2.10)\qquad \Omega_0 = 0, \qquad \Omega_s = (\phi-1)\Omega_{s-2} + K_{s-2}.$$

A direct calculation shows that

$$(2.11)\quad \begin{aligned} \Omega_{n+2} &= (\phi-1)^{\frac{n}{2}}K_0 + (\phi-1)^{\frac{n-2}{2}}K_2 + (\phi-1)^{\frac{n-4}{2}}K_4 + \\ &\quad \ldots + (\phi-1)K_{n-2} + K_n \\ &= \sum_{p=0}^{n}\{\sqrt{\phi-1}\}^{n-p}K_p \end{aligned}$$

The following theorems were proved by Lee[3, 2009]:

Theorem 2.2. *The determinant of the tensor $P_{\lambda\mu}$, given by (2.9), never vanishes, i.e.,*

$$(2.12)\qquad det(P_{\lambda\mu}) \neq 0,$$

if and only if

$$\Omega_{n+2} \neq 0. \tag{2.13}$$

REMARK 2.3. In our further considerations in the present paper, we assume that $\Omega_{n+2} \neq 0$, that is, $det(P_{\lambda\mu}) \neq 0$. Therefore there exists a unique skew-symmetric tensor $Q^{\lambda\nu}$ satisfying

$$P_{\lambda\mu}\, Q^{\lambda\nu} = \delta^{\nu}_{\mu}. \tag{2.14}$$

THEOREM 2.4. *The representation of the tensor $Q^{\lambda\mu}$, given by (2.14), may be given by*

$$Q^{\lambda\mu} = \frac{1}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2}\ {}^{(n-s-3)}k^{\lambda\mu}, \tag{2.15}$$

Here and in what follows, the index s is assumed to take the values 0, 2, 4, ... , n in the specified range, and

$$^{(-1)}k^{\lambda\mu} = -\overline{k}^{\lambda\mu} = -\frac{1}{k} \sum_{s=0}^{n-2} K_s\ {}^{(n-s-1)}k^{\lambda\mu}. \tag{2.16}$$

THEOREM 2.5. *A necessary and sufficient condition for the Einstein's equation (2.5) to admit exactly one particular solution $\Gamma^{\nu}_{\lambda\mu}$ of the form*

$$S_{\lambda\mu}{}^{\nu} = k_{\lambda\mu} Y^{\nu}, \tag{2.17}$$

for some nonzero vector Y^{ν}, is that the basic tensor $g_{\lambda\mu}$ satisfies the following condition:

$$\nabla_{\nu}\, k_{\lambda\mu} = -2(k_{\nu[\lambda}\, h_{\mu]\alpha} - {}^{(2)}k_{\nu[\lambda}\, k_{\mu]\alpha}) Q^{\gamma\alpha}\, \nabla_{\beta}\, k_{\gamma}{}^{\beta}, \tag{2.18}$$

where $Q^{\lambda\mu}$ is given by (2.15), and ∇_{ω} is the symbolic vector of the covariant derivative with respect to the Christoffel symbols $\{{}_{\lambda}{}^{\nu}{}_{\mu}\}$ defined by $h_{\lambda\mu}$. If this condition is satisfied, then the vector Y^{ν} which defines the particular solution is given by

$$Y^{\alpha} = Q^{\lambda\alpha}\, \nabla_{\beta} k_{\lambda}{}^{\beta}, \tag{2.19}$$

and hence the complete representation of the particular solution in terms of the basic tensor $g_{\lambda\mu}$ may be given by

(2.20)

$$\Gamma^{\nu}_{\lambda\mu} = \{{}_{\lambda}{}^{\nu}{}_{\mu}\} - \frac{1}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2} (2k_{(\lambda}{}^{\nu} k_{\mu)\alpha} - k_{\lambda\mu}\delta^{\nu}_{\alpha})\ {}^{(n-s-3)}k^{\gamma\alpha}\, \nabla_{\beta} k_{\gamma}{}^{\beta}.$$

3. Some geometric results

REMARK 3.1. Our further considerations in the present paper, we assume that the condition (2.18) is always satisfied by the basic unified field tensor $g_{\lambda\mu}$.

THEOREM 3.2. *When a connection $\Gamma^{\nu}_{\lambda\mu}$ of the form (2.17) is a solution of the Einstein's equation (2.5), its torsion vector $S_\lambda = S_{\lambda\alpha}{}^{\alpha}$ is given by*

$$S_\lambda = -\frac{1}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2} \ {}^{(n-s-2)}k_\lambda{}^\gamma \nabla_\beta k_\gamma{}^\beta, \tag{3.1}$$

Proof. From (2.20), we obtain

$$S_{\lambda\mu}{}^{\nu} = \frac{1}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2} \ k_{\lambda\mu} \ {}^{(n-s-3)}k^{\gamma\nu} \nabla_\beta k_\gamma{}^\beta, \tag{3.2}$$

Contracting for μ and ν in (3.2), and making use of (2.6)(c), we obtain (3.1) □

THEOREM 3.3. *The Nijenhuis tensor $N_{\lambda\mu}{}^{\nu}$,*

$$N_{\lambda\mu}{}^{\nu} = 2(\partial_\alpha k_{[\lambda}{}^{\nu}) k_{\mu]}{}^{\alpha} - 2k_\alpha{}^\nu (\partial_{[\mu} k_{\lambda]}{}^{\alpha}), \tag{3.3}$$

is given by

$$N_{\lambda\mu}{}^{\nu} = -\frac{2(k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu})}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2} \ {}^{(n-s-2)}k^{\nu\gamma} \nabla_\beta k_\gamma{}^\beta, \tag{3.4}$$

Proof. The symbol ∂ in (3.3) may be replaced by ∇, that is,

$$N_{\lambda\mu}{}^{\nu} = 2(\nabla_\alpha k_{[\lambda}{}^{\nu}) k_{\mu]}{}^{\alpha} - 2k_\alpha{}^\nu (\nabla_{[\mu} k_{\lambda]}{}^{\alpha}). \tag{3.5}$$

In this case, substituting the condition (2.18) into (3.5), the Nijenhuis tensor $N_{\lambda\mu}{}^{\nu}$ may be given by

$$N_{\lambda\mu}{}^{\nu} = 2(k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu}) k^{\nu}{}_{\alpha} Q^{\gamma\alpha} \nabla_\beta k_\gamma{}^\beta, \tag{3.6}$$

by a straightforward computation. Substituting (2.15) into (3.6), we obtain (3.4). □

THEOREM 3.4. *The covariant derivatives of the determinants G and H, given by (2.2), with respect to (2.20) may be given by*

$$D_\omega G = -\frac{2G}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2} \ {}^{(n-s-2)}k_\omega{}^\gamma \nabla_\beta k_\gamma{}^\beta, \tag{3.7}$$

$$D_\omega H = -\frac{2H}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2} \; ({}^{(n-s-2)}k_\omega{}^\gamma + {}^{(n-s-1)}k_\omega{}^\gamma)\nabla_\beta k_\gamma{}^\beta. \tag{3.8}$$

Proof. According to (2.2), there is a unique tensor

$$^*g^{\lambda\nu} = \frac{\partial\, ln\, G}{\partial g_{\lambda\nu}}, \tag{3.9}$$

satisfying the condition

$$g_{\lambda\mu}\, {}^*g^{\lambda\nu} = g_{\mu\lambda}\, {}^*g^{\nu\lambda} = \delta^\nu_\mu. \tag{3.10}$$

Multiplying ${}^*g^{\lambda\nu}$ to both sides of (2.5)(b), and making use of (3.10), we obtain

$$^*g^{\lambda\nu} D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\nu, \tag{3.11}$$

Contracting for μ and ν in (3.11), and making use of (3.9), we obtain

$$^*g^{\lambda\nu} D_\omega g_{\lambda\nu} = G^{-1} D_\omega G = 2\, S_\omega,$$

which implies that

$$D_\omega G = 2G\, S_\omega. \tag{3.12}$$

Substituting (3.1) into (3.12), we obtain (3.7). Next, making use of (2.1) and (2.5)(b), we obtain

$$D_\omega h_{\lambda\mu} = D_\omega g_{(\lambda\mu)} = 2S_{\omega(\mu}{}^\alpha\, g_{\lambda)\alpha}. \tag{3.13}$$

Multiplying $h^{\lambda\mu}$ to both sides of (3.13), and making use of (2.4), we obtain

$$\begin{aligned} h^{\lambda\mu} D_\omega h_{\lambda\mu} = H^{-1} D_\omega H &= 2S_{\omega\mu\lambda}\, h^{\mu\lambda} + 2S_{\omega(\mu}{}^\alpha k_{\lambda)\alpha}\, h^{\mu\lambda} \\ &= 2S_\omega + 2S_{\omega\lambda}{}^\nu\, k^\lambda{}_\nu, \end{aligned}$$

which implies that

$$D_\omega H = 2H(S_\omega - S_{\omega\lambda}{}^\nu\, k_\nu{}^\lambda). \tag{3.14}$$

Substituting (3.1) and (3.2) into (3.14), and making use of (2.6)(c), we obtain (3.8). □

THEOREM 3.5. *The partial derivative of ϕ, given by (2.6)(d), is given by*

$$\partial_\omega \phi = -\frac{4}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2} \; ({}^{(n-s-1)}k_\omega{}^\gamma + {}^{(n-s+1)}k_\omega{}^\gamma)\nabla_\beta k_\gamma{}^\beta, \tag{3.15}$$

Proof. Making use of the definition of the covariant derivative with respect to $\{{}_\lambda{}^\nu{}_\mu\}$, and $\nabla_\omega h_{\lambda\mu} = 0$, we obtain

$$\begin{aligned}\partial_\omega \phi &= \partial_\omega ({}^{(2)}k_\beta{}^\beta) = -\partial_\omega (k_{\alpha\beta} k^{\alpha\beta}) \\ &= -\nabla_\omega (k_{\alpha\beta} k^{\alpha\beta}) = -2k^{\alpha\beta}\,\nabla_\omega k_{\alpha\beta}.\end{aligned} \tag{3.16}$$

Substituting (2.18) into (3.16), and making use of (2.6)(c), we obtain (3.15). □

THEOREM 3.6. *The partial derivative of g, given by (2.6)(a), is given by*

$$\partial_\omega g = \frac{2g}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2}\ {}^{(n-s-1)}k_\omega{}^\gamma\, \nabla_\beta k_\gamma{}^\beta, \tag{3.17}$$

Proof. In (2.20), let

$$\Gamma^\nu_{\lambda\mu} = \{{}_\lambda{}^\nu{}_\mu\} + U^\nu{}_{\lambda\mu} + S_{\lambda\mu}{}^\nu, \tag{3.18}$$

then

$$U^\nu{}_{\lambda\mu} = -\frac{2}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2}\, k_{(\lambda}{}^\nu k_{\mu)\alpha}\ {}^{(n-s-3)}k^{\gamma\alpha}\, \nabla_\beta k_\gamma{}^\beta. \tag{3.19}$$

Since G is a density of weigh 2, making use of (3.12) and (3.18), we obtain

$$\begin{aligned}2G\, S_\omega &= D_\omega G = G\{\partial_\omega (ln\, G) - 2\Gamma^\alpha_{\alpha\omega}\} \\ &= G\{\partial_\omega (ln\, G) - \partial_\omega (ln\, H) + 2S_\omega - 2U_\omega\},\end{aligned} \tag{3.20}$$

where

$$U^\alpha{}_{\alpha\omega} = U_\omega. \tag{3.21}$$

Making use of (2.6)(a), the equation (3.20) is equivalent to

$$2U_\omega = \partial_\omega (ln\, g) = \frac{1}{g}\partial_\omega g. \tag{3.22}$$

On the other hand, making use of (2.6)(c), (3.19) and (3.21), we obtain

$$U_\omega = \frac{1}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2}\ {}^{(n-s-1)}k_\omega{}^\gamma\, \nabla_\beta k_\gamma{}^\beta. \tag{3.23}$$

Substituting (3,23) into (3.22), we obtain (3.17). □

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