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GENERALIZED CHRISTOFFEL FUNCTIONS

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Abstract. Let $W(x) = \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot \exp(-|x|^{\alpha})$. Associated with the weight W, upper and lower bounds of the generalized Christoffel functions for generalized nonnegative polynomials are obtained.

1. Introduction

A generalized nonnegative algebraic polynomial is a function of the type

$$f(z) = |\omega| \prod_{j=1}^{m} |z - z_j|^{r_j} \quad (0 \neq \omega \in \mathbb{C})$$

with $r_j \in \mathbb{R}^+$, $z_j \in \mathbb{C}$, and the number

$$n \stackrel{\text{def}}{=} \sum_{j=1}^{m} r_j$$

is called the generalized degree of f.

We denote by GANP_n the set of all generalized nonnegative algebraic polynomials of degree at most $n \in \mathbb{R}^+$ and we denote by $\mathbb{P}_n$ the set of all polynomials of degree at most $n = 0, 1, 2, \cdots$.

Using

$$|z - z_j|^{r_j} = ((z - z_j)(z - \bar{z}_j))^{r_j/2}, \quad z \in \mathbb{R},$$

we can easily check that when $f \in \mathrm{GANP}_n$ is restricted to the real line, then it can be written as

$$f = \prod_{j=1}^{m} P_j^{r_j/2}, \quad 0 \leq P_j \in \mathbb{P}_2, \quad r_j \in \mathbb{R}^+, \quad \sum_{j=1}^{m} r_j \leq n,$$

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which is the product of nonnegative polynomials raised to positive real powers. Many properties of generalized nonnegative polynomials were investigated in a series of papers ([1,2,3,4]).

Let $w(x)$ be a function, positive in $(-\infty,\infty)$, for which all moments $\int_{-\infty}^{\infty} x^j w(x)dx$, $j=0,1,2,\cdots$, are finite. Let $p_n(w^2;x)$, $n=0,1,2,\cdots$, be the sequence of orthonormal polynomials for $w^2(x)$, that is,

$$\begin{aligned}\int_{-\infty}^{\infty} p_n(w^2;x)p_m(w^2;x)w^2(x)dx &= 1, \quad m=n,\\ &= 0, \quad m\neq n.\end{aligned}$$

The classical Christoffel functions are defined by

$$\begin{aligned}\lambda_n(w;x) &= \min_{P\in\mathbb{P}_{n-1}} \int_{-\infty}^{\infty} \frac{(P(t)w(t))^2dt}{(P(x))^2}\\ &= \left(\sum_{k=0}^{n-1}(p_k(w^2;x))^2\right)^{-1},\end{aligned}$$

for $n=1,2,3,\cdots$.

Next we define generalized Christoffel functions. Let $0<p<\infty$. Then the generalized Christoffel functions for ordinary polynomials are defined by

$$\lambda_{n,p}(w;x) = \min_{P\in\mathbb{P}_{n-1}} \int_{-\infty}^{\infty} \frac{|P(t)w(t)|^p}{|P(x)|^p}dt, \quad n\in\mathbb{N}.$$

The generalized Christoffel functions for generalized nonnegative polynomials are defined by

$$\omega_{n,p}(w;x) = \inf_{f\in\mathrm{GANP}_n} \int_{-\infty}^{\infty} \frac{(f(t)w(t))^p}{f^p(x)}dt, \quad n\in\mathbb{R}^+.$$

The upper and lower bounds of the classical Christoffel functions for various weights were investigated in [5], [13]and [11] and for the generalized Christoffel functions for ordinary polynomials, their bounds were obtained in [12]. When w is supported on $[-1,1]$, upper and lower bounds of the generalized Christoffel functions $\omega_{n,p}$ for generalized nonnegative polynomials were obtained in [4], and for the Freud weights $W_\alpha(x)=\exp(-|x|^\alpha)$, $\alpha>1$, upper and lower bounds of $\omega_{n,p}$ were given in [6].

In this paper we obtain upper and lower bounds of the generalized Christoffel functions $\omega_{n,p}(W;x)$ for generalized nonnegative polynomials where $W(x)=\prod_{k=1}^{m}|x-x_k|^{\gamma_k}\cdot\exp(-|x|^\alpha)$.

Associated with the Freud weight $W_\alpha(x)=\exp(-|x|^\alpha)$, $\alpha>0$, there are Mhaskar-Rahmanov-Saff numbers $a_n=a_n(\alpha)$, which is the positive solution of the equation

$$n=\frac{2}{\pi}\int_0^1 a_n t Q'(a_n t)(1-t^2)^{-\frac{1}{2}}dt, \quad n\in\mathbb{R}^+,$$

where $Q(x)=|x|^\alpha$, $\alpha>0$. Explicitly,

$$a_n=a_n(\alpha)=\left(\frac{n}{\lambda_\alpha}\right)^{1/\alpha}, \quad n\in\mathbb{R}^+,$$

where

$$\lambda_\alpha=\frac{2^{2-\alpha}\Gamma(\alpha)}{\{\Gamma(\alpha/2)\}^2}.$$

Its importance lies partly in the identity [9]

$$\|PW_\alpha\|_{L^\infty(\mathbb{R})}=\|PW_\alpha\|_{L^\infty([-a_n,a_n])}, \quad P\in\mathbb{P}_n.$$

Now we state our results. For upper bounds of $\omega_{n,p}(W;x)$, we have the following.

THEOREM 1.1. *Let $0<p<\infty$. Let*

$$\begin{aligned} W(x) &= \prod_{k=1}^{m}|x-x_k|^{\gamma_k}\cdot\exp(-|x|^\alpha) \\ &= \prod_{k=1}^{m}|x-x_k|^{\gamma_k}\cdot W_\alpha(x), \end{aligned}$$

where $\alpha>1$, $x_k,\gamma_k\in\mathbb{R}$ and $p\gamma_k>-1$, for $k=1,\cdots,m$. Let

$$W_n(x)=\prod_{k=1}^{m}\left(|x-x_k|+\frac{a_n}{n}\right)^{\gamma_k}\cdot\exp(-|x|^\alpha), \quad n\in\mathbb{R}^+.$$

Let $M=2\sum_{\gamma_k<0}(-\gamma_k)$. Then there exist positive constants C_1 and δ such that

$$\omega_{n,p}(W;x)\le C_1\frac{a_n}{n}W_n^p(x), \quad |x|\le\delta a_n, \quad M\le n\in\mathbb{R}^+.$$

For lower bounds of $\omega_{n,p}(W;x)$, we have the following.

Theorem 1.2. *Let $\epsilon > 0$ and $0 < p < \infty$. Let*

$$\begin{aligned} W(x) &= \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot \exp(-|x|^\alpha) \\ &= \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot W_\alpha(x), \end{aligned}$$

where $\alpha > 1$, $x_k, \gamma_k \in \mathbb{R}$ and $p\gamma_k > -1$, for $k = 1, \cdots, m$. Let

$$W_n(x) = \prod_{k=1}^{m} \left(|x - x_k| + \frac{a_n}{n} \right)^{\gamma_k} \cdot \exp(-|x|^\alpha), \quad n \in \mathbb{R}^+.$$

Then there exist positive constants C_2 such that

$$\omega_{n,p}(W; x) \geq C_2 \frac{a_n}{n} W_n^p(x), \quad x \in \mathbb{R}, \quad \epsilon \leq n \in \mathbb{R}^+.$$

Throughout this paper we write $g_n(x) \sim h_n(x)$ if for every n and for every x in consideration

$$0 < c_1 \leq \frac{g_n(x)}{h_n(x)} \leq c_2 < \infty,$$

and $g(x) \sim h(x)$, $n \sim N$ have similar meanings.

2. Proof of Theorems

In order to prove Theorems, first we need infinite finite range inequalities for generalized polynomials with the Freud weight $W_\alpha(x) = \exp(-|x|^\alpha)$. We restate Theorem 2.2 in [6. p. 124].

Lemma 2.1. *Let $\epsilon > 0$ and $d > 0$. Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$. Let*

$$s_n = \min \left\{ \frac{d a_n}{n}, a_n \right\}, \quad n \in \mathbb{R}^+.$$

If $0 < p < \infty$, then there exist positive constants B^* and C_1 such that for all measurable sets $\Delta_n \subset [-B^* a_n, B^* a_n]$ with $m(\Delta_n) \leq s_n/2$,

$$\int_{-\infty}^{\infty} f^p(x) W_\alpha^p(x) dx \leq C_1 \int_{\substack{|x| \leq B^* a_n \\ x \notin \Delta_n}} f^p(x) W_\alpha^p(x) dx, \tag{2.1}$$

for all $f \in \mathrm{GANP}_n$, $\epsilon \leq n \in \mathbb{R}^+$.

Proof. See the proof of Theorem 2.2 in [6. p. 124]. □

For the estimates of $\omega_{n,p}(W_\alpha; x)$, we need the following lemma, which is the restatement of Theorem 2.3 in [6, p. 125].

LEMMA 2.2. *Let $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 1$. Let $0 < p < \infty$. Then*

$$\omega_{n,p}(W_\alpha; x) \geq C\frac{a_n}{n}W_\alpha^p(x), \quad x \in \mathbb{R}, \quad n \in \mathbb{R}^+,$$

and

$$\omega_{n,p}(W_\alpha; x) \leq \lambda_{[n]+1,p}(W_\alpha; x), \quad x \in \mathbb{R}, \quad n \in \mathbb{R}^+,$$

where $[n]$ denotes the integer part of n.

Proof. See the proof of Theorem 2.3 in [6, p. 125]. □

REMARK. *It is well known (see, for example, [8]) that if $\alpha > 1$, then there exist positive constants C_1 and C_2 depending on p and α, such that*

$$\lambda_{[n]+1,p}(W_\alpha; x) \leq C_1\frac{a_n}{n}W_\alpha^p(x), \quad |x| \leq C_2 a_n.$$

Consequently

$$\omega_{n,p}(W_\alpha; x) \sim \frac{a_n}{n}W_\alpha^p(x), \quad |x| \leq C_2 a_n.$$

Now we prove our results.

Proof of Theorem 1.1.

Proof. Let $0 < p < \infty$. Let

$$W(x) = \prod_{k=1}^{m} |x - x_k|^{\gamma_k} \cdot \exp(-|x|^\alpha),$$

and

$$W_n(x) = \prod_{k=1}^{m} \left(|x - x_k| + \frac{a_n}{n}\right)^{\gamma_k} \cdot \exp(-|x|^\alpha),$$

where $n \in \mathbb{R}^+$, $\alpha > 1$, $x_k, \gamma_k \in \mathbb{R}$ and $p\gamma_k > -1$, for $k = 1, \cdots, m$. Let

$$v_k(x) = |x - x_k|^{\gamma_k}, \quad 1 \leq k \leq m,$$

and

$$W_\alpha(x) = \exp(-|x|^\alpha).$$

Then

$$W(x) = \prod_{k=1}^{m} v_k(x) \cdot W_\alpha(x).$$

Assume that

$$\gamma_k < 0, \quad 1 \leq k \leq i,$$

and

$$\gamma_k \geq 0, \quad i < k \leq m.$$

Let

$$M = 2\sum_{k=1}^{i}(-\gamma_k).$$

By Theorem 1.1 in [7], there exist constants $C > 0$, $B > 0$ and $d > 0$ such that

$$\int_{-\infty}^{\infty} f^p(t)W^p(t)dt \leq C\int_{I_n\setminus J_n} f^p(t)W^p(t)dt, \tag{2.2}$$

for $f \in \mathrm{GANP}_n$, where

$$I_n = [-Ba_n, Ba_n]$$

and

$$J_n = \bigcup_{k=1}^{m}\left(x_k - \frac{da_n}{n}, x_k + \frac{da_n}{n}\right).$$

Now denote by $P_j(\alpha, \beta, x)$, $(\alpha > -1, \beta > -1)$, $j = 0, 1, 2, \cdots$, the orthonormalized Jacobi polynomials and let

$$K_\ell(\alpha, \beta, x) = \sum_{j=0}^{\ell-1} P_j^2(\alpha, \beta, x).$$

Let

$$Q_{\ell,k}(x) = \frac{1}{\ell}K_\ell\left(-\frac{1}{2}, \frac{\gamma_k - 1}{2}, 2x^2 - 1\right), \quad \ell \in \mathbb{N}, \quad i < k \leq m.$$

It is well known (see [10, Lemma 2, p. 241] and [12, p.108]) that

$$Q_{\ell,k}(x) \sim \left(|x| + \frac{1}{\ell}\right)^{-\gamma_k}, \quad |x| \leq 1, \quad \ell \in \mathbb{N}, \quad i < k \leq m.$$

Using $Q_{\ell,k}$, we can construct polynomials $R_{n,k}$, $i < k \leq m$, which has degree at most $n/4(m-i)$ and

$$R_{n,k}(t) \sim \left(|t - x_k| + \frac{a_n}{n}\right)^{-\gamma_k}, \quad t \in I_n$$

and

$$R_{n,k}(t) \sim |t - x_k|^{-\gamma_k} = v_k^{-1}(t), \quad t \in I_n \setminus J_n.$$

Now let $n \geq M$. Then

$$\begin{aligned}
\omega_{n,p} \quad (W;x) &= \inf_{f\in \mathrm{GANP}_n} \int_{-\infty}^{\infty} \frac{f^p(t)W^p(t)}{f^p(x)}dt \\
&\leq c_1 \inf_{f\in \mathrm{GANP}_n} \int_{I_n\setminus J_n} \frac{f^p(t)W^p(t)}{f^p(x)}dt \\
&\leq c_1 \inf_{f\in \mathrm{GANP}_{n/4}} \int_{I_n\setminus J_n} \frac{f^p(t)v_1^{-p}(t)\cdots v_i^{-p}(t)R^p_{n,i+1}(t)\cdots R^p_{n,m}(t)W^p(t)}{f^p(x)v_1^{-p}(x)\cdots v_i^{-p}(x)R^p_{n,i+1}(x)\cdots R^p_{n,m}(x)}dt \\
&\leq c_1 v_1^p(x)\cdots v_i^p(x)R^{-p}_{n,i+1}(x)\cdots R^{-p}_{n,m}(x) \\
&\quad \times \inf_{f\in \mathrm{GANP}_{n/4}} \int_{I_n\setminus J_n} \frac{f^p(t)W^p_\alpha(t)}{f^p(x)}dt.
\end{aligned}$$

Hence, by Lemma 2.2, there exists some $\delta > 0$ such that if $|x| \leq \delta a_n$ and $x \notin J_n$,

$$\omega_{n,p}(W;x) \leq c_2 \frac{a_n}{n} W_n^p(x). \tag{2.3}$$

If $x \in J_n$, then using the above method and

$$v_k^p(t) \leq \left(\frac{da_n}{n}\right)^{p\gamma_k}, \quad t \in I_n \setminus J_n, \quad 1 \leq k \leq i,$$

we obtain

$$\begin{aligned}
\omega_{n,p}(W;x) &= \inf_{f\in \mathrm{GANP}_n} \int_{-\infty}^{\infty} \frac{f^p(t)W^p(t)}{f^p(x)}dt \\
&\leq c_1 \inf_{f\in \mathrm{GANP}_n} \int_{I_n\setminus J_n} \frac{f^p(t)W^p(t)}{f^p(x)}dt \\
&\leq c_1 \prod_{k=1}^{i} \left(\frac{da_n}{n}\right)^{p\gamma_k} \inf_{f\in \mathrm{GANP}_n} \int_{I_n\setminus J_n} \frac{f^p(t)v^p_{i+1}(t)\cdots v^p_m(t)W^p_\alpha(t)}{f^p(x)}dt \\
&\leq c_3 \frac{a_n}{n} W_n^p(x), \quad |x| \leq \delta a_n.
\end{aligned}$$

From (2.3) and the above inequality, we have

$$\omega_{n,p}(W;x) \leq c_4 \frac{a_n}{n} W_n^p(x),$$

for $|x| \leq \delta a_n$ and $n \geq M$, hence, Theorem 1.1 is proved. □

Proof of Theorem 1.2.

Proof. Let $\epsilon > 0$ and $0 < p < \infty$. For simplicity we consider the weight

$$W(x) = |x - x_1|^{\gamma_1} |x - x_2|^{\gamma_2} W_\alpha(x),$$

where

$$W_\alpha(x) = \exp(-|x|^\alpha)$$

and

$$\gamma_1 < 1 \text{ and } \gamma_2 \geq 1.$$

General case follows by the same method. Now let

$$\beta(n) = 5n + \gamma_2.$$

Let B^* be the constant which satisfies (2.1). Choose $B > 0$ big enough so that

$$B^* a_{\beta(n)} \leq B a_n, \quad \text{for } n \geq \epsilon, \tag{2.4}$$

and

$$|x_k| < B a_n, \quad \text{for } k = 1, 2, \text{ and } n \geq \epsilon. \tag{2.5}$$

Let

$$I_n = [-Ba_n, Ba_n],$$

and let

$$A_{n,k} = \left(x_k - \frac{da_n}{n}, x_k + \frac{da_n}{n} \right), \quad k = 1, 2,$$

and

$$J_n = \cup_{k=1}^2 A_{n,k}.$$

Here, we can choose $d \in (0, 1)$ small enough so that $A_{n,k}$'s are disjoint and $J_n \subset I_n$ and

$$m(J_n) = \frac{4da_n}{n} \leq a_n. \tag{2.6}$$

Similarly as done in the proof of Theorem 1.1, we can construct the polynomial $R_{n,1}$ such that $R_{n,1}$ has degree at most $4n$ and

$$R_{n,1}(x) \sim \left(|x - x_1| + \frac{a_n}{n} \right)^{\gamma_1}, \quad x \in I_n, \tag{2.7}$$

and

$$R_{n,1}(x) \sim |x - x_1|^{\gamma_1}, \quad x \in I_n \setminus A_{n,1}. \tag{2.8}$$

Note that

$$c_1(p)(|a|+|b|)^p \le (|a|^p+|b|^p) \le c_2(p)(|a|+|b|)^p, \quad (0<p<\infty). \tag{2.9}$$

If $x \notin A_{n,2}$, then

$$|x-x_2| \ge \frac{d}{2}\left(\left|x-x_2+\frac{a_n}{n}\right|+\left|x-x_2-\frac{a_n}{n}\right|\right),$$

hence by (2.9),

$$|x-x_2|^{p\gamma_2} \ge c_3\left(\left|x-x_2+\frac{a_n}{n}\right|^{p\gamma_2}+\left|x-x_2-\frac{a_n}{n}\right|^{p\gamma_2}\right), \tag{2.10}$$

for $x \notin A_{n,2}$.
Then by (2.8) and (2.10),

$$\begin{aligned}
\omega_{n,p} \quad (W;x) &= \inf_{f\in \mathrm{GANP}_n}\int_{-\infty}^{\infty}\frac{f^p(t)W^p(t)}{f^p(x)}dt\\
&\ge \inf_{f\in \mathrm{GANP}_n}\int_{I_n\setminus J_n}\frac{f^p(t)W^p(t)}{f^p(x)}dt\\
&\ge c_4\inf_{f\in \mathrm{GANP}_n}\int_{I_n\setminus J_n}\frac{f^p(t)R^p_{n,1}(t)|t-x_2|^{p\gamma_2}W^p_\alpha(t)}{f^p(x)}dt\\
&\ge c_5\inf_{f\in \mathrm{GANP}_n}\left(\int_{I_n\setminus J_n}\frac{f^p(t)R^p_{n,1}(t)\left|t-x_2+\frac{a_n}{n}\right|^{p\gamma_2}W^p_\alpha(t)}{f^p(x)}dt\right.\\
&\qquad\left.+\int_{I_n\setminus J_n}\frac{f^p(t)R^p_{n,1}(t)\left|t-x_2-\frac{a_n}{n}\right|^{p\gamma_2}W^p_\alpha(t)}{f^p(x)}dt\right)\\
&\ge c_5\left(\inf_{f\in \mathrm{GANP}_n}\int_{I_n\setminus J_n}\frac{f^p(t)R^p_{n,1}(t)\left|t-x_2+\frac{a_n}{n}\right|^{p\gamma_2}W^p_\alpha(t)}{f^p(x)}dt\right.\\
&\qquad\left.+\inf_{f\in \mathrm{GANP}_n}\int_{I_n\setminus J_n}\frac{f^p(t)R^p_{n,1}(t)\left|t-x_2-\frac{a_n}{n}\right|^{p\gamma_2}W^p_\alpha(t)}{f^p(x)}dt\right).
\end{aligned} \tag{2.11}$$

Since

$$f(t)R_{n,1}(t)\left|t-x_2+\frac{a_n}{n}\right|^{\gamma_2}$$

has degree at most $\beta(n) = \mathcal{O}(n)$, by Lemma 2.1 and (2.4) and (2.6), we have for $x \in I_n$, $n \geq \epsilon$,

$$
\begin{aligned}
&\inf_{f\in \mathrm{GANP}_n} \int_{I_n\setminus J_n} \frac{f^p(t)R^p_{n,1}(t)\left|t - x_2 + \frac{a_n}{n}\right|^{p\gamma_2} W^p_\alpha(t)}{f^p(x)}dt \\
&\geq c_6 \inf_{f\in \mathrm{GANP}_n} \int_{-\infty}^{\infty} \frac{f^p(t)R^p_{n,1}(t)\left|t - x_2 + \frac{a_n}{n}\right|^{p\gamma_2} W^p_\alpha(t)}{f^p(x)}dt \\
&= c_6 R^p_{n,1}(x)\left|x - x_2 + \frac{a_n}{n}\right|^{p\gamma_2} \\
&\quad\times \inf_{f\in \mathrm{GANP}_n} \int_{-\infty}^{\infty} \frac{f^p(t)R^p_{n,1}(t)\left|t - x_2 + \frac{a_n}{n}\right|^{p\gamma_2} W^p_\alpha(t)}{f^p(x)R^p_{n,1}(x)\left|x - x_2 + \frac{a_n}{n}\right|^{p\gamma_2}}dt \\
&\geq c_6 R^p_{n,1}(x)\left|x - x_2 + \frac{a_n}{n}\right|^{p\gamma_2} \inf_{f\in \mathrm{GANP}_{\beta(n)}} \int_{-\infty}^{\infty} \frac{f^p(t)W^p_\alpha(t)}{f^p(x)}dt,
\end{aligned}
$$

hence, by Lemma 2.2 and (2.7),

$$
\begin{aligned}
&\inf_{f\in \mathrm{GANP}_n} \int_{I_n\setminus J_n} \frac{f^p(t)R^p_{n,1}(t)\left|t - x_2 + \frac{a_n}{n}\right|^{p\gamma_2} W^p_\alpha(t)}{f^p(x)}dt \\
&\geq c_7 \frac{a_n}{n} R^p_{n,1}(x)\left|x - x_2 + \frac{a_n}{n}\right|^{p\gamma_2} W^p_\alpha(x) \\
&\geq c_8 \frac{a_n}{n}\left(|x - x_1| + \frac{a_n}{n}\right)^{p\gamma_1}\left|x - x_2 + \frac{a_n}{n}\right|^{p\gamma_2} W^p_\alpha(x), \qquad (2.12)
\end{aligned}
$$

for $x \in I_n$, $n \geq \epsilon$.
Similarly, we obtain for $x \in I_n$, $n \geq \epsilon$,

$$
\begin{aligned}
&\inf_{f\in \mathrm{GANP}_n} \int_{I_n\setminus J_n} \frac{f^p(t)R^p_{n,1}(t)\left|t - x_2 - \frac{a_n}{n}\right|^{p\gamma_2} W^p_\alpha(t)}{f^p(x)}dt \\
&\geq c_9 \frac{a_n}{n}\left(|x - x_1| + \frac{a_n}{n}\right)^{p\gamma_1}\left|x - x_2 - \frac{a_n}{n}\right|^{p\gamma_2} W^p_\alpha(x). \qquad (2.13)
\end{aligned}
$$

Then by (2.11), (2.12), (2.13) and (2.9), we have for $x \in I_n$, $n \geq \epsilon$,

$$\begin{aligned}
&\omega_{n,p}(W;x)\\
&\geq c_{10}\frac{a_n}{n}\left(|x-x_1|+\frac{a_n}{n}\right)^{p\gamma_1}\\
&\qquad\qquad\left(\left|x-x_2+\frac{a_n}{n}\right|^{p\gamma_2}+\left|x-x_2-\frac{a_n}{n}\right|^{p\gamma_2}\right)W_\alpha^p(x)\\
&\geq c_{11}\frac{a_n}{n}\left(|x-x_1|+\frac{a_n}{n}\right)^{p\gamma_1}\\
&\qquad\qquad\left(\left|x-x_2+\frac{a_n}{n}\right|+\left|x-x_2-\frac{a_n}{n}\right|\right)^{p\gamma_2}W_\alpha^p(x).
\end{aligned}$$

Since

$$\left|x-x_2+\frac{a_n}{n}\right|+\left|x-x_2-\frac{a_n}{n}\right|\geq\left(|x-x_2|+\frac{a_n}{n}\right),$$

we obtain

$$\begin{aligned}
&\omega_{n,p}(W;x)\\
&\quad\geq c_{11}\frac{a_n}{n}\left(|x-x_1|+\frac{a_n}{n}\right)^{p\gamma_1}\left(|x-x_2|+\frac{a_n}{n}\right)^{p\gamma_2}W_\alpha^p(x)\\
&\quad= c_{11}\frac{a_n}{n}W_n^p(x),
\end{aligned}$$

for $x \in I_n$, $n \geq \epsilon$.
Then by Theorem 1.3 in [7] and the above inequality, we have

$$\begin{aligned}
||fW_n||_{L^\infty(\mathbb{R})} &\leq c_{12}||fW_n||_{L^\infty(I_n)}\\
&\leq c_{13}\left(\frac{n}{a_n}\right)^{\frac{1}{p}}||fW||_{L^p(\mathbb{R})},
\end{aligned}$$

thus,

$$\omega_{n,p}(W;x)\geq c_{14}\frac{a_n}{n}W_n^p(x),\quad x\in\mathbb{R},\quad n\geq\epsilon,$$

which proves Theorem 1.2. □

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