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REGULARLY QUASI-ORDERED SPACES AND NORMALLY QUASI-ORDERED SPACES

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Abstract. Generalizing normally quasi-ordered spaces, we introduce a concept of regularly quasi-ordered spaces and study their categorical properties. We obtain well behaved reflective subcategories of the category **Rqos** of regularly quasi-ordered spaces and continuous isotones, namely the full subcategory of **Rqos** determined by T_0-objects among others, and this result can be extended to that in the category **Nqos** of normally quasi-ordered spaces and continuous isotones.

1. Introduction and main results

In 1940's, Nachbin extended the measure theory and function theory on topological spaces to those on topological partially ordered spaces([7]). After that, there have been many attempts to study topological partially ordered spaces by various authors([3], [4], [6], [8], [9], [11]).

It is well known that the category **Pord** of partially ordered sets and isotones is mono-topological but the category **Qord** of quasi-ordered sets and isotones is topological and therefore **Qord** is more convenient than **Pord**. So we are interested in topological quasi-ordered spaces determined by quasi-order relations instead of partial order relations.

A *topological quasi-ordered space* $(X, \tau, \leq)$ is a set X endowed with both a topology τ and a quasi-order $\leq$. A topological space (X, τ) may be considered as a topological quasi-ordered space when it is realized that X is a quasi-ordered set endowed with the discrete order and a quasi-ordered set $(X, \leq)$ may be considered as a topological quasi-ordered space with discrete or indiscrete topology. Therefore the study of topological quasi-ordered spaces not only includes the study of general

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topology and quasi-ordered sets, but also reveals many generalizations of results in functional analysis for those in function spaces and measure theory.

In this paper, we study some properties of regularly quasi-ordered spaces and normally quasi-ordered spaces, and also categorical properties of the category **Rqos** of regularly quasi-ordered spaces and the category **Nqos** of normally quasi-ordered spaces. Moreover we obtain well behaved subcategories of them. In particular, we show that **Rqos** and **Nqos** are order compatible hereditary. Using some results about normality of quasi-order in Nachbin([7]), we suggest a generalization of Urysohn's Theorem on topological spaces to topological quasi-ordered spaces. We also show that the categories $\mathbf{T_0Rqos}$, $\mathbf{T_1Rqos}$, $\mathbf{T_2Rqos}$ and **Rpos** are epireflective subcategories, initially and finally dense in **Rqos**, $\mathbf{R_0Rqos}$, $\mathbf{R_1Rqos}$ and $\mathbf{T_0^{-1}Rqos}$, respectively. Similarly, by replacing **Rqos** with **Nqos**, we show that $\mathbf{T_0Nqos}$, $\mathbf{T_1Nqos}$, $\mathbf{T_2Nqos}$ and **Npos** are epireflective subcategories, initially and finally dense in **Nqos**, $\mathbf{R_0Nqos}$, $\mathbf{R_1Nqos}$ and $\mathbf{T_0^{-1}Nqos}$, respectively.

For the terminology not introduced in the paper, we refer to Adamek, Herrlich and Strecker [1] for the category theory and Bourbaki [2] for topology and Davey and Priestley [5] for the order theory. Also we assume throughout this paper that a subcategory of a category is full and isomorphism closed.

2. Regularly quasi-ordered spaces

A *continuous quasi-ordered space* $(X, \tau, \leq)$ is a topological quasi-ordered space with a continuous order $\leq$, i.e., for any $x \not\leq y$ in X, there are neighborhoods U of x and V of y such that $u \not\leq v$ for all $u \in U$ and $v \in V$([7]).

It is well known ([7, 9]) that a continuous partially ordered space is a T_2-space but every topological space is a continuous quasi-ordered space when it is endowed with the indiscrete order. Furthermore, in a continuous quasi-ordered space $(X, \tau, \leq)$ and any $x \in X$, $\uparrow x = \{a \in X \mid x \leq a\}$ and $\downarrow x = \{a \in X \mid a \leq x\}$ are closed.

The class of continuous quasi-ordered spaces and continuous isotones forms a category which will be denoted by **Wqos**([9]).

Definition 2.1. (1) For a topological quasi-ordered space $(X, \tau, \leq)$, the order $\leq$ is said to be *upper(lower, resp.) regular* if for any closed increasing(decreasing, *resp.*) subset F of X and $a \notin F$, there are a decreasing(an increasing, *resp.*) neighborhood U of a and an increasing(a decreasing, *resp.*) neighborhood V of F such that $U \bigcap V = \emptyset$.

If $\leq$ is both upper and lower regular, then it is called *a regular order.*

(2) A *regularly quasi-ordered space* is a topological quasi-ordered space with a both continuous and regular quasi-order.

EXAMPLE 2.2. *(1) For the two-point chain* $(\{0,1\}, I, \leq)$ *with the indiscrete topology,* $\leq$ *is regular but not continuous. Conversely, for the two-point chain* $(\{0,1\}, \tau, \leq)$ *with* $\tau = \{\emptyset, \{0\}, \{0,1\}\}$, $\leq$ *is continuous but not regular.*

(2) Consider $X = \{a, b, b'\}$, *the topology* $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,b'\}, X\}$ *and the graph* $G_{\leq} = \triangle_X \bigcup \{(a,b), (a,b'), (b,b'), (b',b)\}$, *where* $\triangle_X = \{(x,x) \mid x \in X\}$. *Then* $(X, \tau, \leq)$ *is a regularly quasi-ordered space.*

One has the following immediately from Definition 2.1.

REMARK 2.3. Let $(X, \tau, =)$ be a topological quasi-ordered space with the discrete order. Then one has:

(1) $(X, \tau, =) \in$ **Wqos** if and only if (X, τ) is a Hausdorff space, because the graph $G_{=}$ of the discrete order is Δ_X.

(2) $(X, \tau, =)$ is a regularly quasi-ordered space if and only if (X, τ) is a regular space, because for any $A \subseteq X$, A is both increasing and decreasing. But we note that (X, τ) in Example 2.2 (2) is not a regular space.

(3) If $(X, \tau, \leq)$ is a regularly quasi-ordered space and $\tau \subseteq \tau'$, then $(X, \tau', \leq)$ need not be a regularly quasi-ordered space again(see [6]).

(4) There is a continuous quasi-ordered space which is not a regularly quasi-ordered space. Indeed, consider the set R of real numbers and the discrete order $=$ on R. For any non-zero real number x, let N_x be the filter generated by the set of open intervals in R containing x, and N_0 the filter generated by $\{(-p,p) - \{\frac{1}{n} | n \in N\} | p > 0\}$ containing 0. Then there is a unique topology τ on R such that N_x is precisely the neighborhood filter of x with respect to τ. Since (R, τ) is a Hausdorff space but not a regular space, $(R, \tau, =)$ is a continuous but not regularly quasi-ordered space.

Let **Rqos** be the full subcategory of **Wqos** determined by regularly quasi-ordered spaces and **Rpos** its full subcategory determined by a partial order relation instead of a quasi-order relation. Then **Rpos** $\subset$ **Rqos** $\subset$ **Wqos**.

THEOREM 2.4. *For any topological quasi-ordered space* $(X, \tau, \leq)$, $\leq$ *is upper(lower, resp.) regular if and only if for each* $x \in X$ *and an* *open decreasing(increasing, resp.) neighborhood U of x, there exists a decreasing(an increasing, resp.) neighborhood V of x such that the closure $\overline{V}$ of V is contained in U.*

Proof. Let $x \in X$ and U be an open decreasing neighborhood of x. Then the complement $\complement U$ of U is closed increasing and $x \notin \complement U$. Since $\leq$ is upper regular, there are a decreasing neighborhood V of x and an increasing neighborhood W of $\complement U$ such that $V \bigcap W = \emptyset$. So there is an open G such that $\complement U \subseteq G \subseteq W$. Since $V \subseteq \complement W \subseteq \complement G \subseteq U$, the closure $\overline{V}$ of V is contained in U.

Conversely, let F be a closed increasing set and $a \notin F$. Then $\complement F$ is open decreasing and $a \in \complement F$. By the assumption, there is a decreasing neighborhood V of a with $\overline{V} \subseteq \complement F$. So $\complement V$ is increasing and $F \subseteq \complement \overline{V} \subseteq \complement V$. Hence $\complement V$ is an increasing neighborhood of F, and clearly $V \bigcap \complement V = \emptyset$. Thus $\leq$ is an upper regular quasi-order.

We have dually the result for a lower regular quasi-order. □

A subspace of a regular space is again regular, but the following example exhibits that it is not the case for a regular quasi-order.

EXAMPLE 2.5. *Let X be the three point chain $\{0, 1, 2\}$ endowed with the topology given by the only non-trivial open set $\{0, 2\}$. Then the given order is clearly a regular quasi-order, for increasing or decreasing closed sets in X are trivial, i.e., $\emptyset$ or X. The order on the subspace A of X consisting of $\{0, 1\}$ is not regular as indicated in Example 2.2 (1).*

DEFINITION 2.6. A topological quasi-ordered subspace $(A, \tau_A, \leq_A)$ of $(X, \tau, \leq)$ is said to be *compatibly quasi-ordered* if for each τ_A-closed increasing(decreasing, *resp.*) set F in A, there are a τ-closed increasing(decreasing, *resp.*) set F^* in X such that $F = F^* \bigcap A$.

PROPOSITION 2.7. *Any compatibly quasi-ordered subspace of a regularly quasi-ordered space is also a regularly quasi-ordered space.*

Proof. Let $(A, \tau_A, \leq_A)$ be a compatibly quasi-ordered subspace of a regularly quasi-ordered space $(X, \tau, \leq)$ and F a τ_A-closed increasing subset in A and $a \in A - F$. Then there exists a τ-closed increasing subset F^* in X such that $F = F^* \bigcap A$. Since $a \notin F^*$ and $(X, \tau, \leq)$ is a regularly quasi-ordered space, there are an increasing neighborhood U of F^* and a decreasing neighborhood V of a in X such that $U \bigcap V = \emptyset$. So $U \bigcap A$ and $V \bigcap A$ are an increasing neighborhood of F and a decreasing neighborhood of a, respectively, in A. Moreover, $(U \bigcap A) \bigcap (V \bigcap A) = \emptyset$, and hence $\leq_A$ is upper regular.

Dually one has the proof for the lower regular case.

Since a subspace of a continuous quasi-ordered space is clearly a continuous quasi-ordered space, $(A, \tau_A, \leq_A)$ is also a regularly quasi-ordered space. □

3. Normally quasi-ordered spaces

In this section, we deal with normal quasi-ordered spaces which generalize normal spaces. The following definition is due to Nachbin [7].

Definition 3.1. For a topological quasi-ordered space $(X, \tau, \leq)$, the order $\leq$ is said to be *normal* if for any closed increasing subset F_1 and closed decreasing subset F_2 of X with $F_1 \bigcap F_2 = \emptyset$, there are an increasing neighborhood U_1 of F_1 and a decreasing neighborhood U_2 of F_2 such that $U_1 \bigcap U_2 = \emptyset$.

A normal quasi-order need not be regular. Indeed, consider $X = \{a, a', b, c\}$, the topology $\tau = \{\emptyset, \{a, a'\}, \{b\}, \{a, a', b\}, X\}$ and the graph $G_{\leq} = \triangle_X \bigcup \{(a, a'), (a', a), (a, b), (a', b), (b, c), (a, c), (a', c)\}$. Then $\leq$ is normal but not regular. Also there are topological quasi-ordered spaces with normal orders which are not continuous. For a simple example, the two-point space $(\{0, 1\}, I, \leq)$ in Example 2.2 (1), $\leq$ is normal but not continuous.

Definition 3.2. *A normally quasi-ordered space* is a topological space with a both continuous and normal quasi-order.

Example 3.3. *Consider a set* $X = \{a_1, a_2, b_1, b_2\}$*, the topology* $\tau = \{\emptyset, \{a_1, a_2\}, \{b_1, b_2\}, X\}$ *and the graph* $G_{\leq} = \{(a_i, a_j)|\ 1 \leq i, j \leq 2\} \bigcup \{(b_s, b_k)| 1 \leq s, k \leq 2\} \bigcup \{(a_i, b_s)|\ 1 \leq i \leq 2, 1 \leq s \leq 2\}$*. Then* $(X, \tau, \leq)$ *is a normally quasi-ordered space.*

Let **Nqos**(**Npos**, *resp.*) denote the full subcategory of **Wqos** determined by normally quasi-ordered(normally partially-ordered, *resp.*) spaces. Then **Npos** $\subset$ **Nqos** $\subset$ **Wqos**.

Remark 3.4. (1) A normally quasi-ordered space is a regularly quasi-ordered space, but the converse does not hold, i.e., **Nqos** $\subset$ **Rqos**(see [6]).

(2) $(X, \tau, =) \in$ **Nqos** if and only if (X, τ) is a normal space. But for the space $(X, \tau, \leq)$ in Example 3.3, (X, τ) is not a normal space.

(3) If $(X, \tau, \leq) \in$ **Nqos** and $\tau \subseteq \tau'$, then $(X, \tau', \leq)$ need not be in **Nqos**(see[7]).

THEOREM 3.5. *For any topological quasi-ordered space $(X, \tau, \leq)$, $\leq$ is normal if and only if for any closed increasing(decreasing, resp.) subset F of X and an open increasing(decreasing, resp.) neighborhood U of F, there is an increasing(a decreasing, resp.) neighborhood V of F with $\overline{V} \subseteq U$.*

Proof. Assume that $\leq$ is normal and U is an open increasing neighborhood of a closed increasing set F, then $\mathbb{C}U$ is closed decreasing and $\mathbb{C}U \bigcap F = \emptyset$. Since $\leq$ is normal, there are an increasing neighborhood V of F and a decreasing neighborhood W of $\mathbb{C}U$ such that $V \bigcap W = \emptyset$. Since there is an open set G such that $\mathbb{C}U \subseteq G \subseteq W$, $V \subseteq \mathbb{C}W \subseteq \mathbb{C}G \subseteq U$, and hence $\overline{V} \subseteq U$.

Conversely, for any closed increasing subset F_1 and closed decreasing subset F_2 with $F_1 \bigcap F_2 = \emptyset$, $\mathbb{C}F_2$ is open increasing and $F_1 \subseteq \mathbb{C}F_2$. By the assumption there is an increasing neighborhood V of F_1 such that $\overline{V} \subseteq \mathbb{C}F_2$. And $\mathbb{C}V$ is also a decreasing neighborhood of F_2, and clearly $V \bigcap \mathbb{C}V = \emptyset$. Thus $\leq$ is normal. □

Now, we suggest a good generalization of Urysohn's Separation Theorem on topological spaces to topological quasi-ordered spaces as follows. In the following, $\Re$ denotes the real line endowed with the usual topology and the usual order.

THEOREM 3.6. *A topological quasi-ordered space $(X, \tau, \leq)$ is a normally quasi-ordered space if and only if it satisfies:*

(1) for any closed increasing subset F_1 and any closed decreasing subset F_2 of X with $F_1 \bigcap F_2 = \emptyset$, there is a continuous isotone $f : X \longrightarrow \Re$ such that $f(F_1) \subseteq \{1\}$, $f(F_2) \subseteq \{0\}$ and $0 \leq f \leq 1$, and

(2) for any $x \not\leq y$, there is a continuous isotone $g : X \longrightarrow \Re$ such that $g(x) > g(y)$.

Proof. It is known that $\leq$ is normal if and only if (1) holds(see [7]). So it remains to show that a normal order $\leq$ is continuous if and only if (2) holds.

Assume that $\leq$ is continuous and $x \not\leq y$, $\uparrow x = \{a \in X \mid x \leq a\}$ and $\downarrow y = \{b \in X \mid b \leq y\}$ are disjoint closed increasing and closed decreasing sets, respectively. Moreover, since $\leq$ is normal, there is a continuous isotone $f : X \longrightarrow \Re$ such that $f(\uparrow x) \subseteq \{1\}$, $f(\downarrow y) \subseteq \{0\}$ by (1) and hence $f(x) = 1 > 0 = f(y)$.

Conversely, let $x \not\leq y$, then there is a continuous isotone $g : X \longrightarrow \Re$ with $g(x) > g(y)$. Choose any real number ξ with $g(y) < \xi < g(x)$. Then $V = \{a \in X \mid g(a) > \xi\}$ is clearly an open increasing neighborhood of

x, $W = \{b \in X \mid g(b) < \xi\}$ an open decreasing neighborhood of y, and $V \bigcap W = \emptyset$. This completes the proof. □

REMARK 3.7. Any subspace of a normally quasi-ordered space need not be normally quasi-ordered(see [6]). But any compatibly quasi-ordered subspace of a normally quasi-ordered space is also a normally quasi-ordered space, by the exactly same argument as that in Proposition 2.7.

4. Reflective subcategories between **Rqos**(**Nqos**, *resp.*) and **Rpos**(**Npos**, *resp.*)

The category $\mathbf{T_0Rqos}$($\mathbf{T_1Rqos}$, $\mathbf{T_2Rqos}$, *resp.*) denotes the full subcategory of **Rqos** consisting of those objects $(X, \tau, \leq)$ such that (X, τ) is a T_0-space (T_1-space, T_2-space, *resp.*). Similarly we define the full subcategories $\mathbf{T_0Nqos}$, $\mathbf{T_1Nqos}$ and $\mathbf{T_2Nqos}$ of **Nqos**.

THEOREM 4.1. *The category* $\mathbf{T_0Rqos}$*(*$\mathbf{T_0Nqos}$*, resp.) is an epireflective subcategory of the category* **Rqos***(***Nqos***, resp.). Moreover,* $\mathbf{T_0Rqos}$*(*$\mathbf{T_0Nqos}$*, resp.) is initially dense in* **Rqos***(***Nqos***, resp.).*

Proof. For any $(X, \tau, \leq) \in \mathbf{Rqos}$, the $\mathbf{T_0Rqos}$-reflection of $(X, \tau, \leq)$ can be constructed as follows:

Let $E : \mathbf{T_0Rqos} \hookrightarrow \mathbf{Rqos}$ be the embedding functor. For any $(X, \tau, \leq) \in \mathbf{Rqos}$, we define a relation $\mathcal{R} = \{(x, y) \in X \times X \mid \mathcal{N}(x) = \mathcal{N}(y)\}$, where $\mathcal{N}(x)$ is the neighborhood filter of x ($x \in X$). Then $\mathcal{R}$ is an equivalence relation. On the quotient set $X/\mathcal{R}$, let $\tau_{\mathcal{R}}$ be the quotient topology on $X/\mathcal{R}$, i.e., the final topology with respect to the quotient map $q : X \longrightarrow X/\mathcal{R}$ defined by $q(x) = [x]$, and the graph $G_{\leq_{\mathcal{R}}} = \{([x], [y]) \in X/\mathcal{R} \times X/\mathcal{R} \mid$ there are a, $b \in X$ such that $a \leq b$, $q(a) = [x]$ and $q(b) = [y]\}$. Clearly $\leq_{\mathcal{R}}$ is a quasi-order and $(X/\mathcal{R}, \tau_{\mathcal{R}})$ is a T_0-space. Moreover, since $q : (X, \tau, \leq) \longrightarrow (X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ is an onto open final map and every open set in $(X, \tau, \leq)$ is saturated with respect to $\tau_{\mathcal{R}}$, $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}}) \in \mathbf{T_0Rqos}$.

For any $(Y, \tau', \leq') \in \mathbf{T_0Rqos}$ and $f : (X, \tau, \leq) \longrightarrow E((Y, \tau', \leq'))$ in **Rqos**, we have the relation $\mathcal{R} \subseteq ker(f)$. By the property of topological quasi-ordered spaces(see [10]), there is a unique continuous isotone $\overline{f}$: $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}}) \longrightarrow (Y, \tau', \leq')$ defined by $\overline{f}([x]) = f(x)$ with $E(\overline{f}) \circ q = f$. Thus $(q, (X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}}))$ is the $\mathbf{T_0Rqos}$-reflection of $(X, \tau, \leq)$.

For the second statement, we note that the graph $G_{\leq} = (q \times q)^{-1}(G_{\leq_{\mathcal{R}}})$, for $\leq$ is a continuous order on $(X, \tau, \leq)$ and hence $\leq$ is the initial order for q. Also the topology τ on X is initial for q, because for any open U

in X, $U = q^{-1}(q(U))$ and hence the reflection q is initial. Thus $\mathbf{T_0Rqos}$ is initially dense in $\mathbf{Rqos}$.

With the similar process we have the result for $\mathbf{T_0Nqos}$ and $\mathbf{Nqos}$. □

We define the following full subcategories of $\mathbf{Rqos}$:

DEFINITION 4.2. (1) $\mathbf{R_0Rqos}$ consists of those objects X in $\mathbf{Rqos}$ which satisfy the condition: for any $x, y \in X$, if there is an open neighborhood U of x with $y \notin U$, then there is an open neighborhood V of y with $x \notin V$.

(2) $\mathbf{R_1Rqos}$ consists of those objects X in $\mathbf{Rqos}$ which satisfy the condition: for any $x, y \in X$, if there is an open neighborhood U of x with $y \notin U$, then there are open neighborhoods U' and V' of x and y, respectively, such that $U' \bigcap V' = \emptyset$.

(3) $\mathbf{T_0^{-1}Rqos}$ consists of those objects X in $\mathbf{Rqos}$ which satisfy the condition: for any $x, y \in X$, $x \leq y$ and $y \leq x$ if and only if $\mathcal{N}(x) = \mathcal{N}(y)$.

From the above definitions, we have the following inclusion between them immediately.

REMARK 4.3. (1) $\mathbf{Rpos} \subset \mathbf{T_2Rqos} \subset \mathbf{T_1Rqos} \subset \mathbf{T_0Rqos} \subset \mathbf{Rqos}$
(2) $\mathbf{Rpos} \subset \mathbf{T_0^{-1}Rqos} \subset \mathbf{R_1Rqos} \subset \mathbf{R_0Rqos} \subset \mathbf{Rqos}$

PROPOSITION 4.4. *(1)* $\mathbf{R_0Rqos} = \{(X, \tau, \leq) \mid \mathbf{T_0Rqos}$*-reflection of* $(X, \tau, \leq)$ *is a* T_1*-space*$\}$

(2) $\mathbf{R_1Rqos} = \{(X, \tau, \leq) \mid \mathbf{T_0Rqos}$*-reflection of* $(X, \tau, \leq)$ *is a* T_2*-space*$\}$

(3) $\mathbf{T_0^{-1}Rqos} = \{(X, \tau, \leq) \mid \mathbf{T_0Rqos}$*-reflection of* $(X, \tau, \leq)$ *is a partially ordered space*$\}$

Proof. (1) and (3) are similar to the proof of Propositions 2.14 and 2.4, respectively, in [9].

(2) For any $(X, \tau, \leq) \in \mathbf{R_1Rqos}$, it is enough to show that the $\mathbf{T_0Rqos}$-reflection $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ is a T_2-space. Suppose that $[x] \neq [y]$ in $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$. Then $\mathcal{N}(x) \neq \mathcal{N}(y)$ so that we may assume that there is an open neighborhood U of x with $y \notin U$. Since $(X, \tau, \leq) \in \mathbf{R_1Rqos}$, there are open neighborhoods V and W of x and y, respectively, such that $V \bigcap W = \emptyset$. So $q(V)$ and $q(W)$ are disjoint open neighborhoods of $[x]$ and $[y]$, respectively. Hence $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ is a T_2-space.

For the converse, take any $(X, \tau, \leq)$ in the right hand side. Suppose that for x, y in X, there is an open neighborhood U of x with $y \notin U$. Then $\mathcal{N}(x) \neq \mathcal{N}(y)$, and hence $[x] \neq [y]$. Since $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ is a

T_2-space, there are disjoint open neighborhoods V and W of $[x]$ and $[y]$, respectively. So $q^{-1}(V)$ and $q^{-1}(W)$ are disjoint open neighborhoods of x and y, respectively. Hence $(X, \tau, \leq) \in \mathbf{R_1Rqos}$. □

COROLLARY 4.5. *$\mathbf{T_1Rqos}$, $\mathbf{T_2Rqos}$ and $\mathbf{Rpos}$ are epireflective subcategories and initially dense in $\mathbf{R_0Rqos}$, $\mathbf{R_1Rqos}$ and $\mathbf{T_0^{-1}Rqos}$, respectively.*

Using the fact that $\mathbf{Rpos}$ is finally dense in $\mathbf{Wqos}$, we have $\mathbf{T_0Rqos}$, $\mathbf{T_1Rqos}$, $\mathbf{T_2Rqos}$ and $\mathbf{Rpos}$ are finally dense in $\mathbf{Rqos}$, $\mathbf{R_0Rqos}$, $\mathbf{R_1Rqos}$ and $\mathbf{T_0^{-1}Rqos}$, respectively.

Replacing $\mathbf{Rqos}$ by $\mathbf{Nqos}$ in the above Definition 4.2 through Corollary 4.5, we have the analogous results, namely $\mathbf{T_0Nqos}$, $\mathbf{T_1Nqos}$, $\mathbf{T_2Nqos}$ and $\mathbf{Npos}$ are epireflective subcategories, initially dense and finally dense in $\mathbf{Nqos}$, $\mathbf{R_0Nqos}$, $\mathbf{R_1Nqos}$ and $\mathbf{T_0^{-1}Nqos}$, respectively.

Collecting the above facts, we have the following diagram of reflective subcategories between the category $\mathbf{Rqos}$($\mathbf{Nqos}$, *resp.*) and $\mathbf{Rpos}$($\mathbf{Npos}$, *resp.*) :

$$\begin{array}{ccccc}
 & & \mathbf{Rqos} & & \\
 & & (\mathbf{Nqos}) & & \\
 & r\nearrow & & \nwarrow d & \\
\mathbf{R_0Rqos}(\mathbf{R_0Nqos}) & & & & \mathbf{T_0Rqos}(\mathbf{T_0Nqos}) \\
r\uparrow & & \nwarrow d & & r\uparrow \\
\mathbf{R_1Rqos}(\mathbf{R_1Nqos}) & & & & \mathbf{T_1Rqos}(\mathbf{T_1Nqos}) \\
r\uparrow & & \nwarrow d & & r\uparrow \\
\mathbf{T_0^{-1}Rqos}(\mathbf{T_0^{-1}Nqos}) & & & & \mathbf{T_2Rqos}(\mathbf{T_2Nqos}) \\
 & \nwarrow d & & r\nearrow & \\
 & & \mathbf{Rpos} & & \\
 & & (\mathbf{Npos}) & , &
\end{array}$$

where d is the inclusion functor with initial and final dense reflection and r is the inclusion functor with epireflection.

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