JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **23**, No. 3, September 2010

EXISTENCE OF PERIODIC SOLUTIONS IN FERROELECTRIC LIQUID CRYSTALS

Jinhae Park*

Abstract. We introduce the Landau-de Gennes model in order to understand molecular structures in ferroelectric liquid crystals. We investigate equilibrium configurations of the governing energy functional by means of bifurcation analysis. In particular, we obtain periodic solutions of the functional, which is a signature of a rich variety of applications of ferroelectric materials.

1. Introduction

In this article, we analyze the structure of liquid crystals focusing on polarizations. We assume that the liquid crystals are of ferroelectric, i.e. possessing the spontaneous polarization. The presence of the polarization has a significant effect on the development of technology, such as a fast switching between the active and inactive state. Mathematically, the structure of such a material can be understood by equilibrium states of the Ginzburg-Landau type of energy functional.

Molecules in nematic liquid crystals are described by a traceless symmetric second order tensor $\mathbf{Q}$. Shapes of molecules are characterized by three eigenvalues of $\mathbf{Q}$ and the direction field $\mathbf{n}$ of a molecule is defined by the unit eigenvector whose corresponding eigenvalue has the largest magnitude. The order tensor $\mathbf{Q}$ is a measure of the local degree of orientational order in liquid crystals. The liquid crystal is said to be *uniaxial* if two eigenvalues of $\mathbf{Q}$ are equal, and it is *biaxial* when $\mathbf{Q}$ has three distinct eigenvalues. The tensor $\mathbf{Q}$ is zero in the isotropic phase. In smectic liquid crystals, centers of mass of molecules are arranged locally in one-dimensional layers described by a complex field $\Psi = \rho e^{i\omega}$;

Received August 09, 2010; Accepted September 06, 2010.
2010 Mathematics Subject Classification: Primary 46T99, 34A34; Secondary 34K18, 49J99.
Key words and phrases: bifurcation, ferroelectric, liquid crystal, periodic solutions, smectic.

level sets of the phase function ω denote layer locations, with $\nabla\omega$ being parallel to the layer normal. The tensor $\mathbf{Q}$ and complex field Ψ describe molecules of smectic liquid crystals.

As a novel state of matter, many of liquid crystals exhibit *ferroelectricity* which refers to the permanent or spontaneous polarization [12]. Due to the appearance of the polarization, the structures of these materials are so complicated that a great deal of effort has been invested from the viewpoints of both mathematics and physics. A various different patterns of the polarization have been observed in the physics literature [12, 16]. Among many ferroelectric materials, there are two typical types, which are called *ferroelectric* and *antiferroelectric phases*. In the ferroelectric phase, the polarization prefers to be aligned to one direction so that the net polarization is not zero. But if the material in the antiferroelectric phase, the net polarization is zero because two opposite neighboring polarizations are canceled. Most of the known antiferroelectric phases appear at lower temperature than ferroelectric ones, which is opposite to the situation in solid state [12]. However, several different types of local polarization arrangements, i.e. mixed states of ferro and antiferroelectric phases are possible. This explains the observations of multiple periodic phases found in some materials [10, 19]. If an electric field is applied in such phases, the dipoles which are already pointed in the direction of the applied field will remain so aligned, but those which are oriented in the opposite direction to the field will tend to reverse their orientation. This results in nucleations and domain walls, and a relation between polarization and applied field, which is the most important characteristic of ferroelectric materials.

In order to study such phenomena, we introduce a generalized Landau-de Gennes energy. With a special geometry, we simplify the problem into a one-dimensional problem governed by the Ginzburg-Landau energy

$$\int_\Omega \left\{ \frac{\varepsilon^2}{2}|\nabla u|^2 + f(u) - Eu \right\} dx, \tag{1.1}$$

where $\varepsilon > 0$ is a fixed number and

$$f(u) = \frac{1}{6}u^2[(u^2-1)^2 - \alpha], \tag{1.2}$$

with a constant α depending on the temperature and an applied field E. As a macroscopic model, the triple well potential energy $f(u)$ allows for mixed phases of ferro-and antiferroelectric phases. In other words, $u = 0$ corresponds to the antiferroelectric phase and $u = \pm 1$ represent two uniform ferroelectric phases. In this paper, we apply local

and global bifurcation theories to the Euler-Lagrange equation corresponding to (1.1) in order to obtain periodic equilibrium configurations. In particular, application of an electric field yields a nonlocal problem and it enables us to introduce a bifurcation parameter. The Ginzburg-Landau energy with a double well potential, $f(u) = \frac{1}{4}(u^2-1)^2$, have been studied extensively by many authors. Although it is impossible to list all of them, J. Carr, M. Gurtin, and M. Slemrod [3] studied global minimizers of the energy functional, and $\Gamma-$ convergence results of the energy minimizers are found in [13, 14, 20]. To the best knowledge of the author, results related with $f(u)$ given in (1.2) are not available in the literature.

This paper is organized as follows. In section 2, we present Landau-de Gennes theory to discuss the governing energy functional for the system. Some basic properties are presented in section 3. In section 4, we discuss a one-dimensional problem and also introduce a nonlocal problem with an applied electric field. Periodic solutions of the nonlocal problem obtained in section 4 are discussed in section 5.

2. An extended Landau - de Gennes energy

In this section, we introduce energy densities for ferroelectric liquid crystals in the Landau-de Gennes theory. Since the second order tensor, i.e. 3×3 matrix $\mathbf{Q}$ is a symmetric matrix, all eigenvalues of $\mathbf{Q}$ are real and there exists an orthogonal matrix $\mathcal{O}$ such that $\mathcal{O}^T\mathbf{Q}\mathcal{O}$ is diagonal. This implies that $\mathbf{Q}$ can be expressed by

$$\mathbf{Q} = S_1\left(\mathbf{m}\otimes\mathbf{m} - \frac{1}{3}\mathbf{I}\right) + S_2\left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathbf{I}\right), \tag{2.1}$$

where $\{\mathbf{m}, \mathbf{n}, \mathbf{m}\times\mathbf{n}\}$ is an orthonormal basis for $\mathbf{R}^3$ consisting of unit eigenvectors of $\mathbf{Q}$. Three eigenvalues of $\mathbf{Q}$ are

$$\frac{1}{3}(2S_1 - S_2), \quad -\frac{1}{3}(S_1+S_2), \quad \frac{1}{3}(2S_2 - S_1).$$

In the Landau-de Gennes theory, neglecting the higher derivatives and powers of $\mathbf{Q}$, the free energy density $\mathcal{F}_{sm}$ for smectic liquid crystals [17] is given by

$$\begin{aligned}\mathcal{F}_{sm} &= \frac{1}{2}\left(L_1 Q_{\alpha\beta,\gamma}Q_{\alpha\beta,\gamma} + L_2 Q_{\alpha\beta,\beta}Q_{\alpha\gamma,\gamma} + L_3 Q_{\alpha\beta,\gamma}Q_{\alpha\gamma,\beta}\right)\\ &+ f_{bulk}(\mathbf{Q}) + \frac{B_0}{2}|\Delta\Psi|^2 + \frac{B_1}{2}|\nabla\Psi|^2 + \frac{B_2}{4}|\nabla\Psi|^4 - \frac{\delta}{2}|\Psi|^2 \mathrm{tr}\,\mathbf{Q}^2\end{aligned} \tag{2.2}$$

$$-\frac{\gamma}{2}(\mathbf{Q}\nabla\Psi)\cdot\nabla\Psi^* + \frac{\alpha}{4}(|\Psi|^2-\rho_0^2)^2,$$

where $\mathbf{Q}=(Q_{\alpha\beta})$, $f_{bulk}(\mathbf{Q}) = \frac{A}{2}\mathrm{tr}\,\mathbf{Q}^2 - \frac{B}{3}\mathrm{tr}\,\mathbf{Q}^3 + \frac{C}{4}\left(\mathrm{tr}\,\mathbf{Q}^2\right)^2$, and $Q_{\alpha\beta,\gamma}$ denoting the partial derivative of $Q_{\alpha\beta}$ with respect to x_γ. The bulk energy f_{bulk} is a potential function for uniaxial nematic liquid crystals, meaning that it favors molecules to be uniaxial [15]. For more details about terms involved in Ψ, we refer the reader to [17]. In order to study behavior of biaxial liquid crystals, one has to add higher powers of $\mathbf{Q}$ into f_{bulk}. Due to the presence of the polarization field $\mathbf{P}$, we include the polarization and electrostatic energies into the system

$$\mathcal{F}_{pol} = \frac{1}{2}\left[D_1(\nabla\cdot\mathbf{P})^2 + D_2|\nabla\times\mathbf{P}|^2 + a|\mathbf{P}|^2 + b|\mathbf{P}|^4 + c|\mathbf{P}|^6\right], \tag{2.3}$$

$$\mathcal{F}_{elec} = -\frac{1}{2}(\epsilon(\mathbf{Q})\mathbf{E})\cdot\mathbf{E} - (\mathbf{P}^f+\mathbf{P})\cdot\mathbf{E}. \tag{2.4}$$

subject to the Maxwell's equations

$$\nabla\cdot(\boldsymbol{\epsilon}(\mathbf{Q})\mathbf{E}) = -\nabla\cdot(\mathbf{P}^f+\mathbf{P}), \quad \nabla\times\mathbf{E}=0, \tag{2.5}$$

where $\mathbf{P}^f=(P_1^f,P_2^f,P_3^f)$ with

$$P_i^f = \epsilon_0 Q_{i,\gamma\gamma} + \epsilon_1 Q_{\beta\gamma}Q_{i\beta,\gamma} + \epsilon_2 Q_{i\alpha}Q_{\alpha\gamma,\gamma}$$

is the flexoelectric polarization vector induced by a splay and bending distortion [2] being analogous to piezoelectric polarization in solids, and the dielectric permittivity tensor is given by [2]

$$\epsilon(\mathbf{Q}) = \epsilon_0\mathbf{I} + \epsilon_1\mathbf{Q} + \epsilon_2\mathbf{Q}^2. \tag{2.6}$$

In particular, the order tensor $\mathbf{Q}$ can be written as

$$\mathbf{Q} = \frac{3}{2}S(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathbf{I})$$

for uniaxial nematic liquid crystals. In this case, we have

$$\epsilon(\mathbf{Q})\mathbf{E}\cdot\mathbf{E} = \left(\epsilon_0 - \frac{\epsilon_1}{2}S + \frac{\epsilon_2}{4}S^2\right)|\mathbf{E}|^2 + \frac{3}{2}S\left(\epsilon_1+\frac{\epsilon_2}{2}S\right)(\mathbf{n}\cdot\mathbf{E})^2,$$

$$\mathbf{P}^f = e_{11}(\nabla\cdot\mathbf{n})\mathbf{n} + e_{33}\mathbf{n}\times\nabla\times\mathbf{n}, \tag{2.7}$$

$$e_{11} = \frac{3}{2}\epsilon_3 S + \frac{3}{4}(2\epsilon_5-\epsilon_4)S^2, \;\; e_{33} = \frac{3}{2}\epsilon_3 S + \frac{3}{4}(2\epsilon_4-\epsilon_5)S^2.$$

Then the permittivity $\epsilon_\perp$ and dielectric anisotropic constant ϵ_a are defined by

$$\epsilon_\perp = \epsilon_0 - \frac{\epsilon_1}{2}S + \frac{\epsilon_2}{4}S^2, \;\; \epsilon_a = \frac{3}{2}S\left(\epsilon_1+\frac{\epsilon_2}{2}S\right). \tag{2.8}$$

Since some material can have $\epsilon_1 > 0$ and $S > 0$, we have to include ϵ_2-term in order to satisfy solvability condition $\epsilon_\perp > |\epsilon_a|$. The expression for $\mathbf{P}^f$ in (2.7) agrees with the one in [7, pp. 136].

The total energy functional $\mathcal{E}$ is given by

$$\mathcal{E} = \int_\Omega \{\mathcal{F}_{sm} + \mathcal{F}_{pol} + \mathcal{F}_{elec}\} \tag{2.9}$$

subject to the Maxwell's equation (2.5). We assume the constitutive coefficients satisfy

$$\begin{aligned} &L_1 > 0, \quad L_1 + L_2 + L_3 > 0, \quad B_0 > 0, \\ &\alpha > 0, \quad C > 0, \quad D_1 > 0, \quad D_2 > 0, \quad c > 0. \end{aligned} \tag{2.10}$$

3. Basic properties

LEMMA 1. *If* $\mathbf{Q}$ *is a traceless, symmetric second order tensor, then* $6(tr\,\mathbf{Q}^3)^2 \le (tr\,\mathbf{Q}^2)^3$ *and eigenvalues of* $\mathbf{Q}$ *are represented by*

$$\begin{cases} \lambda_1 = \frac{2\sqrt{tr\,\mathbf{Q}^2}}{\sqrt{6}} \cos\alpha, \\ \lambda_2 = \frac{2\sqrt{tr\,\mathbf{Q}^2}}{\sqrt{6}} \left(-\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha\right), \\ \lambda_3 = \frac{2\sqrt{tr\,\mathbf{Q}^2}}{\sqrt{6}} \left(-\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha\right), \end{cases} \tag{3.1}$$

where

$$\begin{aligned} \cos(3\alpha) &= -\frac{\sqrt{6}tr\,\mathbf{Q}^3}{tr\,\mathbf{Q}^2\sqrt{tr\mathbf{Q}^2}}, \\ \sin(3\alpha) &= \sqrt{1 - \frac{6(tr\,\mathbf{Q}^3)^2}{(tr\,\mathbf{Q}^2)^3}}, \quad \alpha \in \left[0, \frac{\pi}{3}\right]. \end{aligned} \tag{3.2}$$

Proof. If λ is an eigenvalue of $\mathbf{Q}$, then

$$\det(\lambda\mathbf{I} - \mathbf{Q}) = \lambda^3 - \operatorname{tr}\mathbf{Q}\lambda^2 + \frac{1}{2}\left\{(\operatorname{tr}\mathbf{Q})^2 - \operatorname{tr}\mathbf{Q}^2\right\}\lambda - \det\mathbf{Q} = 0.$$

Since $\operatorname{tr}\mathbf{Q} = 0$, we have

$$\lambda^3 - \frac{\operatorname{tr}\mathbf{Q}^2}{2}\lambda - \det\mathbf{Q} = 0,$$

and $\operatorname{tr}\mathbf{Q}^3 = 3\det\mathbf{Q}$. Hence the eigenvalues of $\mathbf{Q}$ are roots of

$$f(\lambda) := \lambda^3 - \frac{\operatorname{tr}\mathbf{Q}^2}{2}\lambda - \frac{1}{3}\operatorname{tr}\mathbf{Q}^3.$$

Since $\mathbf{Q}$ is symmetric, $\mathbf{Q}$ has real eigenvalues. The cubic polynomial $f(\lambda)$ must have real roots. In order for eigenvalues to be real, $f(\lambda)$ satisfies

$$\begin{aligned} f(\lambda^*) &= -\frac{1}{6\sqrt{6}}\mathrm{tr}\,\mathbf{Q}^2\sqrt{\mathrm{tr}\,\mathbf{Q}^2} + \frac{1}{2\sqrt{6}}\mathrm{tr}\,\mathbf{Q}^2\sqrt{\mathrm{tr}\,\mathbf{Q}^2} \\ &\quad - \frac{1}{3}\mathrm{tr}\,\mathbf{Q}^3 \geq 0 \qquad \text{if } \mathrm{tr}\,\mathbf{Q}^3 \leq 0, \\ f(-\lambda^*) &= -\frac{1}{6\sqrt{6}}\mathrm{tr}\,\mathbf{Q}^2\sqrt{\mathrm{tr}\,\mathbf{Q}^2} + \frac{1}{2\sqrt{6}}\mathrm{tr}\,\mathbf{Q}^2\sqrt{\mathrm{tr}\,\mathbf{Q}^2} \\ &\quad - \frac{1}{3}\mathrm{tr}\,\mathbf{Q}^3 \leq 0 \qquad \text{if } \mathrm{tr}\,\mathbf{Q}^3 \geq 0, \end{aligned}$$

where $\lambda^* = -\sqrt{\frac{\mathrm{tr}\,\mathbf{Q}^2}{6}}$.

This implies that $6(\mathrm{tr}\,\mathbf{Q}^3)^2 \leq (\mathrm{tr}\,\mathbf{Q}^2)^3$.

In order to find explicit forms of eigenvalues, we apply Cardano's method. Let $p = -\frac{\mathrm{tr}\mathbf{Q}^2}{2}, q = \frac{1}{3}\mathrm{tr}\,\mathbf{Q}^3$. Then the roots are

$$\text{(3.3)} \qquad \begin{cases} \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \\ \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \\ \qquad\qquad + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \\ \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \\ \qquad\qquad + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \end{cases}$$

Let

$$\beta^2 = \frac{6(\mathrm{tr}\,\mathbf{Q}^3)^2}{(\mathrm{tr}\,\mathbf{Q}^2)^3}, 0 \leq \beta \leq 1.$$

Then

$$\begin{aligned} -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} &= \frac{1}{6\sqrt{6}}\mathrm{tr}\,\mathbf{Q}^2\sqrt{\mathrm{tr}\,\mathbf{Q}^2}\left(-\frac{\sqrt{6}\mathrm{tr}\,\mathbf{Q}^3}{\mathrm{tr}\,\mathbf{Q}^2\sqrt{\mathrm{tr}\,\mathbf{Q}^2}} + i\sqrt{1-\beta^2}\right), \\ -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} &= \frac{1}{6\sqrt{6}}\mathrm{tr}\,\mathbf{Q}^2\sqrt{\mathrm{tr}\,\mathbf{Q}^2}\left(-\frac{\sqrt{6}\mathrm{tr}\,\mathbf{Q}^3}{\mathrm{tr}\,\mathbf{Q}^2\sqrt{\mathrm{tr}\,\mathbf{Q}^2}} - i\sqrt{1-\beta^2}\right) \end{aligned}$$

Let α be such that

$$\cos(3\alpha) = -\frac{\sqrt{6}\mathrm{tr}\,\mathbf{Q}^3}{\mathrm{tr}\,\mathbf{Q}^2\sqrt{\mathrm{tr}\,\mathbf{Q}^2}},\ \ \sin(3\alpha) = \sqrt{1-\beta^2},\ \ 3\alpha \in [0,\pi].$$

Then

$$\begin{aligned}\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} &= \frac{1}{\sqrt{6}}\sqrt{\mathrm{tr}\,\mathbf{Q}^2}e^{i\alpha},\\ \sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}} &= \frac{1}{\sqrt{6}}\sqrt{\mathrm{tr}\,\mathbf{Q}^2}e^{-i\alpha}.\end{aligned} \tag{3.4}$$

Plugging (3.4) into (3.3) we obtain (3.1). This completes the proof. □

COROLLARY 2. *The tensor* $\mathbf{Q}$ *is uniaxial if and only if* $6(tr\,\mathbf{Q}^3)^2 = (tr\,\mathbf{Q}^2)^3$.

Proof. If $6(\mathrm{tr}\,\mathbf{Q}^3)^2 = (\mathrm{tr}\,\mathbf{Q}^2)^3$, then $\sin\alpha = 0$ and thus $\mathbf{Q}$ is uniaxial. Suppose that $6(\mathrm{tr}\,\mathbf{Q}^3)^2 \neq (\mathrm{tr}\,\mathbf{Q}^2)^3$. Then $\sin 3\alpha \neq 0$, i.e. $\alpha \neq 0, \frac{\pi}{3}$, and thus $\alpha \in (0, \frac{\pi}{3})$ by Lemma 1 and $\lambda_2 \neq \lambda_3$. If $\mathbf{Q}$ is uniaxial, then either $\lambda_1 = \lambda_2$ or $\lambda_1 = \lambda_3$. It follows that α satisfies either $\tan\alpha = -\sqrt{3}$ or $\tan\alpha = \sqrt{3}$. This is a contradiction because α should be an angle in $\left(0, \frac{\pi}{3}\right)$. □

Let $\mathcal{A}$ be the set of $(\mathbf{Q}, \mathbf{P}, \Psi, \varphi)$ such that

$$\mathbf{Q} \in W^{1,2}(\Omega, \mathcal{S}), \mathbf{P} \in W^{1,2}(\Omega, \mathbb{R}^3), \Psi \in W^{2,2}(\Omega, \mathbb{C}), \varphi \in W^{1,2}(\Omega, \mathbb{R}),$$

satisfying (2.5) with $\mathbf{E} = \nabla\varphi$ where $\mathcal{S}$ is the set of 3×3 traceless symmetric real matrices. By the standard theory of calculus of variations together with the same arguments in [18], the energy functional $\mathcal{E}$ allows a minimizer in $\mathcal{A}$.

THEOREM 3. *There exists a minimizer of the energy functional* $\mathcal{E}$ *in* $\mathcal{A}$.

4. A one-dimensional problem

In this section, we consider a system of polar uniaxial smectic liquid crystals with uniform smectic layers $\nabla\omega = (1,0,0)$ in which the direction field takes only two vectors $\pm\mathbf{n}, \mathbf{n} = (n_1, n_2, 0)$ and $\mathbf{P} = (0,0,P)$ where $\mathbf{n}$ is a fixed unit vector. Since $\mathbf{Q} = \frac{3}{2}S\left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathbf{I}\right)$ for uniaxial liquid crystals, the values of the order parameter $\mathbf{Q}$ corresponding to $\pm\mathbf{n}$ are the same. Roughly speaking, such a system can be considered as a

limiting problem when $L = L_1 = L_2 = L_3 \to 0$ and $C \to \infty$. Hence, we shall drop the energy terms involved with $\mathbf{Q}$. We further assume that $D_1 = D_2 = D$ and the system depends only on x. The structure of the system is determined by the energy associated with the polarization field

$$\frac{1}{2}\int_0^L \left\{ D\left(\frac{dP}{dx}\right)^2 + aP^2 + bP^4 + cP^6 - \tilde{E}P \right\} dx, \tag{4.1}$$

where $(0, 0, \tilde{E})$ is an applied electric field.

4.1. Periodic solutions in the absence of electric field

Let us assume that $\tilde{E} = 0$. We seek periodic solutions of the Euler-Lagrange equation corresponding to (4.1)

$$\frac{d^2P}{dx^2} = \beta P + \gamma P^3 + \delta P^5 \text{ in } \mathbf{R} \tag{4.2}$$

where $\beta = \frac{a}{D}, \gamma = \frac{2b}{D}, \delta = \frac{3c}{D} > 0$. In order to obtain periodic solutions, we introduce the following Morse lemma [21].

LEMMA 4. *Suppose that* $\mathbf{x} = \mathbf{0}$ *is a non-degenerate critical point, i.e.* $\det\left(\nabla^2 H(\mathbf{0})\right) \neq 0$, *of the* C^∞ *function*

$$H(\mathbf{x}) = H(\mathbf{0}) - c_1x_1^2 - c_2x_2^2 - \cdots - c_ix_i^2 + c_{i+1}x_{i+1}^2 + \cdots + c_nx_n^2 + \text{ higher order term},$$

with coefficients $c_l > 0, l = 1, 2, \cdots, n$. *Then there is a diffeomorphism* ψ *from* U *to* V *which transforms* $H(\mathbf{x})$ *to the form*

$$H(\psi^{-1}(\mathbf{y})) = G(\mathbf{y}) = G(\mathbf{0}) - y_1^2 - \cdots - y_i^2 + y_{i+1}^2 + \cdots + y_n^2,$$

where U *and* V *are open neighborhoods of* $\mathbf{0}$.

The equation (4.2) can be written as a system of first order equations

$$\begin{cases} P' = Q, \\ Q' = \beta P + \gamma P^3 + \delta P^5. \end{cases} \tag{4.3}$$

In order to apply lemma 4, we define

$$H(Q, P) = \frac{1}{2}Q^2 - \frac{\beta}{2}P^2 - \frac{\gamma}{4}P^4 - \frac{\delta}{6}P^6.$$

If $\beta < 0$, then $(Q, P) = (0, 0), (0, \pm P_0)$ are critical points, where

$$P_0 = \sqrt{\frac{1}{2}\left(-\frac{\gamma}{\delta} + \sqrt{\frac{\gamma^2 - 4\beta\delta}{\delta^2}}\right)}.$$

At $(0,0)$, by lemma 4 we get a diffeomorphism that transforms H into $y_1^2+y_2^2$. This results in closed orbits in the phase plane because $(0,0)$ is a center. Hence we obtain a one-parameter family of periodic solutions. Similarly, we can prove that there are two one-parameter families of periodic solutions for $\gamma^2 > 4\beta\delta > 0, \gamma < 0$, and there is no periodic solutions for $\gamma^2 - 4\beta\delta < 0$ or $\beta > 0, \gamma > 0$.

THEOREM 5. *If either $\beta < 0$ or $\gamma^2 > 4\beta\delta > 0, \gamma < 0$ is satisfied, then (4.2) has periodic solutions. Moreover, for such a solution P, the period L is given by*

$$L = 2\int_{P_1}^{P_2} \frac{1}{\sqrt{2\left(\frac{\beta}{2}P^2 + \frac{\gamma}{4}P^4 + \frac{\delta}{6}P^6 - W\right)}}\, dP, \tag{4.4}$$

where P_1 and P_2 ($P_1 < P_2$) are the values of intersection of the periodic orbit with the P-axis in the phase plane satisfying the first integral equation

$$\frac{\beta}{2}P^2 + \frac{\gamma}{4}P^4 + \frac{\delta}{6}P^6 = \frac{Q^2}{2} + W.$$

Proof. See [21]. □

4.2. A nonlocal problem in the presence of the electric field

For the remaining part of this paper, we study periodic solutions of the energy (4.1) with a constant electric field $\tilde{E}$ and $b < 0$ subject to the Neumann boundary condition. We introduce the scalings

$$P_0^2 = \frac{|b|}{2c}, \alpha = \frac{b^2-4ac}{b^2},\ \varepsilon^2 = \frac{D}{3cL^2P_0^4},\ \tilde{x} = \frac{x}{L},\ u(\tilde{x}) = \frac{P(x)}{P_0},\ \ \tilde{E} = \frac{E}{3cP_0^5}.$$

Writing the energy (4.1) in terms of u and $\tilde{x}$ and dividing by $3cLP_0^6$ followed by dropping the tilde on $\tilde{x}$, we get a new energy

$$\int_0^1 \left\{ \frac{\varepsilon^2}{2}(u'(x))^2 + f(u) - Eu \right\} dx,$$

and the corresponding Euler-Lagrange equation is

$$\begin{cases} -\varepsilon^2 u''(x) + f'(u) = E \text{ in } [0,1], \\ u'(0) = u'(1) = 0, \end{cases} \tag{4.5}$$

where

$$f(u) = \frac{1}{6}u^2[(u^2-1)^2 - \alpha].$$

In order for (4.5) to be solvable, the following equality should hold

$$E = \int_0^1 f'(u(x))\,dx. \tag{4.6}$$

It is immediate that for any real $\lambda \in \mathbf{R}$, $u = \lambda$ is a solution. Among many other solutions, we choose solutions satisfying $\int_0^1 u = \lambda$ for each $\lambda \in \mathbf{R}$. If u is such a solution, then u satisfies

$$\begin{cases} -\varepsilon^2 u''(x) + f'(u) = \int_0^1 f'(u)\,dx \text{ in } [0,1], \\ u'(0) = u'(1) = 0,\ \int_0^1 u\,dx = \lambda. \end{cases} \tag{4.7}$$

The equation (4.7) is a nonlocal equation and corresponds to the Euler-Lagrange equation for the problem

$$\begin{cases} \mathbb{E}(u) = \int_0^1 \left\{ \frac{\varepsilon^2}{2}(u'(x))^2 + f(u) \right\} dx \\ \text{subject to} \\ u'(0) = u'(1) = 0,\ \int_0^1 u\,dx = \lambda, \quad \lambda \in \mathbf{R}. \end{cases} \tag{4.8}$$

5. Special periodic solutions

Throughout this paper, we assume that a sufficiently small number $\varepsilon > 0$ is fixed.

First, we recall that $u = \lambda$ is a solution of (4.7) for any $\lambda \in \mathbf{R}$. Introducing a new variable w by $u = w + \lambda$, the equation (4.7) is equivalent to

$$\begin{cases} -\varepsilon^2 w''(x) + f'(w(x) + \lambda) = \int_0^1 f'(w(s) + \lambda)\,ds \text{ in } [0,1], \\ w'(0) = w'(1) = 0,\ \int_0^1 w\,dx = 0. \end{cases} \tag{5.1}$$

Let

$$\mathcal{X} = \left\{ w \in C^2[0,1] : w'(0) = w'(1) = 0, \int_0^1 w\,dx = 0 \right\},$$

$$\mathcal{Z} = \left\{ z \in C^0[0,1] : \int_0^1 z\,dx = 0 \right\},$$

where $C^n[0,1]$ is the set of all functions from $[0,1]$ from $\mathbf{R}$ whose all jth derivatives $(0 \le j \le n)$ are continuous. Clearly, $\mathcal{X}$ and $\mathcal{Y}$ are not empty and Hilbert spaces with the inner product $<,>_2$, defined by

$$< u, v >_2 = \int_0^1 u(s)v(s)\,ds$$

for two functions u and v.

Define $G : \mathcal{X} \times \mathbf{R} \to \mathcal{Z}$ by

$$G(w,\lambda) = -\varepsilon^2 w'' + f'(w+\lambda) - \int_0^1 f'(w(s)+\lambda)\,ds. \tag{5.2}$$

Then $G(0,\lambda) = 0$ for any $\lambda \in \mathbf{R}$.

The linearized equation of (5.1) at $(0,\lambda)$ is given by

$$\begin{cases} -\varepsilon^2 w'' + f''(\lambda)w = 0 \text{ in } [0,1], \\ w'(0) = w'(1) = 0, \ \ \int_0^1 w\,dx = 0. \end{cases} \tag{5.3}$$

The solution space of the linearized equation is spanned by $\cos k\pi x$ if there exists $k \in \mathbb{N}$ satisfying

$$\varepsilon^2 k^2 \pi^2 = -f''(\lambda). \tag{5.4}$$

Since ε is sufficiently small, there always exists a pair of λ and k satisfying (5.4).

The number of real roots of (5.4) depends on the values of α. We summarize them as follows.

1. If $\alpha \le -\frac{7}{5}$, then there is no root.
2. Let $-\frac{7}{5} < \alpha < 1$. If K is the largest integer satisfying

$$\varepsilon^2 k^2 \pi^2 < \frac{7}{15} + \frac{\alpha}{3}, \text{ for } 1 \le k \le K, \tag{5.5}$$

then for each $1 \le k \le K$, there are four real roots of (5.4), two pairs of which have opposite signs. We denote them by

$$\pm\lambda_k^i, (i = 1,2), \ \ 0 < \lambda_k^1 < \sqrt{\frac{2}{5}} < \lambda_k^2. \tag{5.6}$$

3. For $\alpha > 1$, let K_0 be the largest integer satisfying

$$\varepsilon^2 k^2 \pi^2 < \frac{1}{3}(\alpha - 1) \text{ for } 1 \le k \le K_0. \tag{5.7}$$

   (a) If $1 \le k \le K_0$, then the equation (5.4) has two roots with opposite signs, which we denote them by $\pm\lambda_k$, $\lambda_k > \sqrt{\frac{2}{5}}$.
   (b) If $K_0 < k \le K$, then the equation (5.4) has four roots satisfying (5.6). We keep the same notation as in (5.6).

From now on, we follow all notations for roots discussed above.

5.1. Local bifurcation analysis

Now, let λ_0 denote one of values of λ satisfying (5.4) with k, and L_{λ_0} be the linearized operator, $D_wG(0,\lambda_0)$, of G at $(0,\lambda_0)$. Then we obtain

$$\dim \ker L_{\lambda_0} = \operatorname{codim} L_{\lambda_0} = 1.$$

In particular, $L_{\lambda_0} = -\varepsilon^2 w'' + f''(\lambda_0)w$ is a self-adjoint operator. By Lyapunov-Schmidt reduction, there exist spaces $\mathcal{X}_0$ and $\mathcal{Z}_0$ such that

$$\mathcal{X} = \ker L_{\lambda_0} \oplus \mathcal{X}_0, \qquad \mathcal{Z} = \mathcal{R}(L_{\lambda_0}) \oplus \mathcal{Z}_0.$$

where $P : \mathcal{X} \to \ker L_{\lambda_0}$ and $Q : \mathcal{Z} \to \mathcal{Z}_0$ are continuous projection maps. By direct computations, we establish

$$\begin{aligned} &\ker L_{\lambda_0} = \operatorname{span}\{\phi_k\}, \qquad \phi_k(x) = \sqrt{2}\cos k\pi x, \\ &D^2_{w\lambda}F(0,\lambda_0)\phi_k \notin \mathcal{R}(L_{\lambda_0}). \end{aligned}$$

It is also easy to see that

$$\begin{aligned} \mathcal{R}(L_{\lambda_0}) &= \left\{ v \in \mathcal{Z} : \int_0^1 v(s)\phi_k(s)\,ds = 0 \right\}, \\ \mathcal{Z}_0 &= \operatorname{span}\{v_0^*\} = \mathcal{R}(L_{\lambda_0})^\perp = \ker L^*_{\lambda_0}, \end{aligned}$$

where $v_0^* = \phi_k$. Furthermore, by Lyapunov Reduction, there exists a continuous map $\phi : \mathcal{U} \times (a,b) \to \mathcal{W}$ satisfying

$$QG(w + \phi(w,\lambda),\lambda) = 0, \tag{5.8}$$

for all $(w,\lambda) \in \mathcal{U} \times (a,b)$, $\phi(0,\lambda_0) = 0$ where $\mathcal{U}, (a,b)$ and $\mathcal{W}$ are open neighborhoods of $0, \lambda_0$, and w_0 in $\ker L_{\lambda_0}, \mathbf{R}$, and $\mathcal{X}_0$ respectively.

For any $(w,\lambda) \in \mathcal{U} \times (a,b)$, we define Φ by

$$\Phi(w,\lambda) = QG(w + \phi(w,\lambda),\lambda). \tag{5.9}$$

By local bifurcation theorem [4], there exists a local nontrivial solution curve $\mathcal{S}$ in $\mathcal{X} \times \mathbf{R}$ emanating from $(0,\lambda_0)$ such that all solutions of $G(w,\lambda) = 0$ in a neighborhood of $(0,\lambda_0)$ are either on the trivial line or on the nontrivial curve $\mathcal{S}$. If $\mathcal{S}$ is parameterized by s near $(0,\lambda_0)$ as

$$\mathcal{S} : \{(w(s),\lambda(s)) : s \in (-\delta,\delta)\} \text{ for some } \delta > 0, \tag{5.10}$$

then $w(s)$ behaves like $s\phi_k + \omega(s)$ for some ω satisfying

$$\omega(0) = 0, \quad \dot{\omega}(0) = \frac{d\omega}{ds}\Big|_{s=0} = 0.$$

LEMMA 6. *Let $\mathcal{S}$ be parameterized by s in (5.10). Then*

$$\dot\lambda(0)=0,\qquad \ddot\lambda(0)=-\frac{1}{6}\left[\frac{3f^{(4)}(\lambda_0)}{f'''(\lambda_0)}+\frac{f'''(\lambda_0)}{f''(\lambda_0)}\right]. \tag{5.11}$$

Proof. By direct computations using formulas in [9, pp. 31], we obtain

$$\begin{aligned}
&< D^2_{w\lambda}G(0,\lambda)v_0, v_0^* >= f'''(\lambda),\\
&D^2_{ww}G(0,\lambda)[v_0,v_0]=f'''(\lambda)v_0^2-f'''(\lambda)\int_0^1 v_0^2\,dx=f'''(\lambda)(v_0^2-1),\\
&D^3_{www}G(0,\lambda)[v_0,v_0,v_0]=f^{(4)}(\lambda)v_0^3.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&< D_{ww}G(0,\lambda)[v_0,v_0], v_0^* >=\int_0^1\left\{f'''(\lambda)v_0^3-f'''(\lambda)v_0\right\}dx=0,\\
&\dot\lambda(0)=-\frac{< D^2_{ww}G(0,\lambda)[v_0,v_0], v_0^* >}{< D_{w\lambda}G(0,\lambda)v_0, v_0^* >}=0,
\end{aligned}$$

so that $D^2_{ww}G(0,\lambda)[v_0,v_0]\in\mathcal{R}(L)$, where $L=D_wG(0,\lambda)$. By Lyapunov-Schmidt reduction, we have

$$D^2_{ww}G(0,\lambda)[v_0,v_0]=(I-Q)D^2_{ww}G(0,\lambda)[v_0,v_0].$$

Next, we recall (see [11, pp. 19] or [9, pp. 33]) that

$$\ddot\lambda(0)=-\frac{1}{3}\frac{< D^3_{vvv}\Phi(0,\lambda)[v_0,v_0,v_0], v_0^* >}{< D^2_{x\lambda}G(0,\lambda)v_0, v_0^* >},$$

where Φ is defined in (5.9) and

$$\begin{aligned}
&D^3_{vvv}\Phi(0,\lambda_0)[v_0,v_0,v_0]=QD^3_{vvv}G(0,\lambda_0)[v_0,v_0,v_0]\\
&-3QD^2_{xx}G(0,\lambda_0)\left[v_0,(I-P)L^{-1}(I-Q)D^2_{vv}G(0,\lambda)[v_0,v_0]\right].
\end{aligned}$$

In order to compute $\ddot\lambda(0)$, we need to find $L^{-1}D^2_{ww}G(0,\lambda)[v_0,v_0]=L^{-1}[f'''(\lambda)(v_0^2-1)]$ so that $(I-P)L^{-1}D^2_{ww}G(0,\lambda)[v_0,v_0]$ can be calculated. Let w be a solution in $\mathcal{X}$ satisfying

$$Lw=f'''(\lambda)(v_0^2-1), w'(0)=w'(1)=0, \int_0^1 w\,dx=0.$$

Then

$$w(x)=-\frac{f'''(\lambda)}{3f''(\lambda)}\cos 2k\pi x.$$

Since $w(x) = (I-P)w(x)$, we have

$$(I-P)L^{-1}(I-Q)D^2_{ww}G(0,\lambda)[v_0,v_0] = \frac{f'''(\lambda)}{3\varepsilon^2k^2\pi^2}\cos 2k\pi x,$$

$$QD^2_{ww}G(0,\lambda)[v_0,w] = \frac{[f'''(\lambda)]^2}{3\varepsilon^2k^2\pi^2}v_0\cos 2k\pi x.$$

We then obtain

$$D^3_{vvv}\Phi(0,\lambda_0)[v_0,v_0,v_0] = f^{(4)}(\lambda)v_0^3 - \frac{[f'''(\lambda)]^2}{\varepsilon^2k^2\pi^2}v_0\cos 2k\pi x,$$

$$< D^3_{vvv}\Phi(0,\lambda_0)[v_0,v_0,v_0], v_0^* >= \frac{3f^{(4)}(\lambda)}{2} - \frac{[f'''(\lambda)]^2}{2\varepsilon^2k^2\pi^2}.$$

Since $\varepsilon^2k^2\pi^2 = -f''(\lambda)$, we get

$$\ddot{\lambda}(0) = -\frac{1}{6}\left[\frac{3f^{(4)}(\lambda)}{f'''(\lambda)} + \frac{f'''(\lambda)}{f''(\lambda)}\right].$$

□

Using the formulae in lemma 6, we can evaluate $\ddot{\lambda}(0)$ at each point to decide types of bifurcation points. If $(0,\lambda)$ satisfies $\ddot{\lambda}(0) > 0$ ($\ddot{\lambda}(0) < 0$), then it is called *supercritical* (*subcritical*).

Corollary 7. *All bifurcation points are pitchforks, i.e.* $\dot{\lambda}(0) = 0$, $\ddot{\lambda}(0) \neq 0$. *In particular, if* $(0,\lambda_0)$ *is a bifurcation point, then it is*

$$\begin{cases} \textit{subcritical} & \textit{if } \lambda_0 \in \{\lambda_k, -\lambda_k^1, \lambda_k^2\}, \\ \textit{supercritical} & \textit{if } \lambda_0 \in \{-\lambda_k, \lambda_k^1, -\lambda_k^2\}. \end{cases} \tag{5.12}$$

It is not hard to check the following properties of $D_wG(0,\lambda)$, which are used to determine local stabilities of nontrivial solution branches.

1. If $-\frac{7}{5} < \alpha < 1$, then kth eigenvalue of $D_wG(0,\lambda)$ is

   $$\begin{cases} \text{negative} & \text{if } \lambda \in (-\infty,-\lambda_k^2)\cup(-\lambda_k^1,\lambda_k^1)\cup(\lambda_k^2,\infty), \\ \text{positive} & \text{if } \lambda \in (-\lambda_k^2,-\lambda_k^1)\cup(\lambda_k^1,\lambda_k^2), \end{cases}$$

   where $1 \le k \le K$.
2. Let $\alpha > 1$.
   (a) If $1 \le k \le K_0$, then kth eigenvalue of $D_wG(0,\lambda)$ is

   $$\begin{cases} \text{negative} & \text{if } \lambda \in (-\infty,-\lambda_k)\cup(\lambda_k,\infty), \\ \text{positive} & \text{if } \lambda \in (-\lambda_k,\lambda_k). \end{cases}$$

   (b) If $K_0 < k \le K$, then the kth eigenvalue of $D_wG(0,\lambda)$ is

   $$\begin{cases} \text{negative} & \text{if } \lambda \in (-\infty,-\lambda_k^2)\cup(-\lambda_k^1,\lambda_k^1)\cup(\lambda_k^2,\infty), \\ \text{positive} & \text{if } \lambda \in (-\lambda_k^2,-\lambda_k^1)\cup(\lambda_k^1,\lambda_k^2). \end{cases}$$

5.2. Global bifurcation analysis

In this section, we study the global behavior of nontrivial solution branches emanating from bifurcation points. Specifically, we prove existence of finitely many closed curves consisting of these nontrivial branches. Moreover, the number of these closed solutions curves goes to ∞ as $\varepsilon \to 0$.

THEOREM 8. *Let $\mathcal{M}_k$ be a maximal connected subset of the closure of the nontrivial solution branch bifurcating from $(0, \lambda_0)$ where $\lambda_0 \in \{\pm\lambda_k, \pm\lambda_k^i (i = 1, 2)\}$. Then $\mathcal{M}_k$ is bounded and $\mathcal{M}_l \cap \mathcal{M}_l = \emptyset$ if $k \neq l$.*

Proof. By properties of the function f, there exists $M_1 > 0$ such that

$$\begin{cases} f' \text{ is increasing on } (-\infty, -M_1] \cup [M_1, \infty), \\ f'(-M_1) < f'(s) < f'(M_1) \text{ for all } -M_1 < s < M_1. \end{cases} \tag{5.13}$$

Let (w, λ) be any solution pair in $\mathcal{X} \times \mathbf{R}$ which lies on $\mathcal{S}$. Since $w \in C^2[0, 1]$, w has the absolute minimum at x_0 and absolute maximum at x_1 for some $x_0, x_1 \in [0, 1]$. Since $\int_0^1 w(s)\, ds = 0$, we get $w(x_0) < 0 < w(x_1)$ and w satisfies the Euler-Lagrange equation

$$-\varepsilon^2 w''(x) + f'(w(x) + \lambda) = \int_0^1 f'(w(s) + \lambda)\, ds \text{ for all } x \in [0, 1].$$

Since the right-hand side of the above equation depends only on λ, we have

$$-\varepsilon^2 w''(x_0) + f'(w(x_0) + \lambda) = -\varepsilon^2 w''(x_1) + f'(w(x_1) + \lambda).$$

The minimality and maximality of w at x_0 and x_1 (respectively) yield

$$f'(w(x_1) + \lambda) \leq f'(w(x_0) + \lambda).$$

From (5.13), we get $-M_1 \leq w(x_0) + \lambda \leq \lambda$ and $\lambda \leq w(x_1) + \lambda \leq M_1$ so that $|\lambda| \leq M_1$ and $||w||_\infty \leq M_1$. From the Euler-Lagrange equation, $||w'||_\infty$ and $||w''||_\infty$ are bounded by $\frac{C_1}{\varepsilon^2}$ where C_1 depending on M_1. Hence, $|\lambda| + ||w||_{C^2[0,1]} \leq D$ where D depends only on M_1, ε, and therefore $\mathcal{M}_k$ is bounded. Now, any element w in $\mathcal{S}$ has exactly k zeros, each of which is simple (characterized by $\cos k\pi x$). Let $\mathcal{Z}_j$ be the set of all functions in $\mathcal{X}$ which have exactly j zeros which are simple. By the standard argument [1, pp.164], the sets $\mathcal{Z}_j$ are disjoint. This completes the proof. □

Let $(0, \lambda)$ be a bifurcation point corresponding to k. We rewrite the Euler-Lagrange equation (5.1) by

$$\varepsilon^2 w - K(w, \lambda) = 0, \tag{5.14}$$

where

$$\frac{d^2}{dx^2}K(w,\lambda) = f'(w+\lambda) - \int_0^1 f'(w(s)+\lambda)ds. \tag{5.15}$$

Choose $\mathcal{Y} = \mathcal{Z}$ and define $F : \mathcal{Y} \times \mathbf{R} \to \mathcal{Y}$ by

$$F(w,\lambda) = w - \varepsilon^{-2}K(w,\lambda).$$

Then K is a compact operator, and $D_wF(0,\lambda)$ has an odd crossing number at λ because its kernel has the dimension 1. So we can apply the Global Bifurcation Theorem [5, 8].

Now, we observe how the number of real roots of (5.4) changes when the value of α varies. From (5.5)-(5.7), the critical case occurs when $\alpha = 1$. For a fixed ε, the number of real roots of (5.4) corresponding to $k = K_0$ jumps from 2 to 4 by decreasing the value of α because of (5.7). The new roots are nucleated at 0 in the transition. This implies that two roots $\pm\lambda_k^1$ appear or disappear at the same time with varying values of α.

For $1 \le k \le K_0$, let $\mathcal{M}_k$ be the maximal connected subset of the closure of the nontrivial solution curve bifurcating from $(0,\lambda_k)$. By the Global bifurcation theorem, the curve $\mathcal{M}_k$ connects with $(0,-\lambda_k)$ because $\mathcal{M}_k$ cannot contain $\pm\lambda_j (j \ne k)$. Remember that the absolute value of these roots is always greater than $\sqrt{\frac{2}{5}}$. By the principle of exchange of stability [6], two bifurcation points $(0,\pm\lambda_j^1)(K_0 < j \le K)$ cannot be connected. This implies that a single closed curve composed of two nontrivial solution branches bifurcating from $(0,\pm\lambda_{K_0})$ is split into two branches when we decrease the value of α. As a summary, we conclude the following corollary

Corollary 9. *Let a sufficiently small ε be given. If $1 \le k \le K_0$, then there is a single closed nontrivial solution curve, which connects $(0,\lambda_k)$ with $(0,-\lambda_k)$. If $K_0 < k \le K$, then there are two closed nontrivial solution curves, which connect $(0,\pm\lambda_k^1)$ with $(0,\pm\lambda_k^2)$ respectively.*

Schematic shapes of the nontrivial solution branches obtained in this section are described in Figure 1. It should be addressed that figures drawn here are not real solution curves. Real solution curves may be more complicated than the ones in Figure 1. As we see from Figure 1(R), there exists a pair of two closed curves inside a single closed curve when ε is very small and $\alpha > 1$.

In Figure 1(L) and 1(M), we take $K = 3$ and $-\frac{7}{5} < \alpha < 1$. The Figure 1(L) is a schematic curve for α being away from 1, while Figure

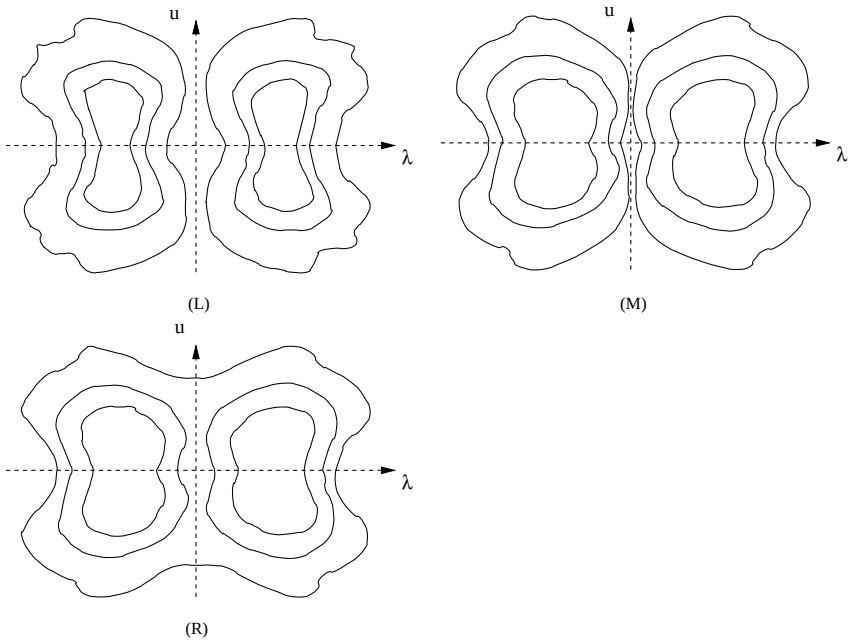

FIGURE 1. Schematic solution branches

1(M) is drawn when α is close to 1. We take $k_0 = 1, K = 3$ and $\alpha(> 1)$ close to 1 in Figure 1(R).

References

[1] S. S. Antman, *Nonlinear Problems of Elasticity*, Applied Math. Sci. 107, Springer-Verlag, 1995.

[2] G. Barbero and L. R. Evangelista, *An elementary course on the continuum theory for nematic liquid crystals*, World Scientific, 2001.

[3] J. Carr, M. E. Gurtin, and M. Slemrod, *Structured phase transitions on a finite interval*, Arch. Ration. Mech. Anal. **86** (1984), 317–351.

[4] M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Funct. Anal. **8** (1971), 321–340.

[5] ———, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7** (1971), 487–513.

[6] M. G. Crandall and P. H. Rabinowitz, *The principle of exchange of stability*, Dynamical systems, Proc. Internat. Sympos., Univ. Florida, Gainesville, Fla., (1976), 27–41.

[7] P. G. de Gennes and J. Prost, *The Physics of Liquid Crystals*, Oxford University Press, 1993.

[8] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1984.

[9] M. Golubitsky and D. G. Schaffer, *Singularities and Groups in Bifurcation Theory Vol. I*, Springer-Verlag, New York, 1985.
[10] A. Jákli, T. Kósa, A. Vajda, E. Benkler, I. Jánossy, and P. Palffy-Muhoray, *Optically induced periodic structures in smectic-C liquid crystals*, Phys. Rev. E, **63** (2000), 011705–5.
[11] H. Kielhofer, *Bifurcation Theory: An introduction with applications to PDEs*, Springer-Verlag, New York, 2004.
[12] S. T. Lagerwall, *Ferroelectric and Antiferroelectric Liquid Crystals*, Wiley-VCH, 1999.
[13] L. Modica, *The Gradient Theory of Phase Transitions and the Minimal Interface Criterion*, Arch. Ration. Mech. Anal. **98** (1987), 123–142.
[14] L. Modica and S. Mortola, *Un esempio di* Γ^-*-convergenza (italian)*, Boll. Un. Mat. Ital. A, 14-B (1977), 285–299.
[15] N. J. Mottram and C. Newton, *Introduction to Q-tensor theory*, University of Strathclyde, Department of Mathematics research report, 2004:10 (2004).
[16] I. Muševič, R. Blinc, and B. Žekš, *The Physics of Ferroelectric and Antiferroelectric Liquid Crystals*, World-Scientific, Singapore, New Jersey, London, Hong Kong, 2000.
[17] P. K. Mukherjee, H. R. Brand, and H. Pleiner, *Landau Model of the Smectic C - Isotropic Phase Transition*, Physica A, **312** (2002).
[18] J. Park and M. C. Calderer, *Analysis of nonlocal electrostatic effects in chiral smectic c liquid crystals*, SIAM J. Appl. Math. **66** (2006), 2107–2126.
[19] S. Pikin, *Structural Transformations in Liquid Crystals*, Gordon and Breach Science Publishers, New York, 1991.
[20] P. Sternberg, *The effect of a singular perturbation on nonconvex variational problems*, Arch. Ration. Mech. Anal. **101** (1988), 206–260.
[21] F. Verhurst, *Nonlinear Differential Equations and Dynamical Systems*, Springer-Verlag, New York, 2000.

*

PDE and Functional Analysis Research Center
Seoul National University
Seoul 151-747, Republic of Korea
E-mail: jhpark2003@snu.ac.kr or jhpark2003@gmail.com