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ON THE STABILITY OF GENERALIZED DERIVATION IN FUZZY BANACH ALGEBRA

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Abstract. In this article, we take account of the fuzzy stability for generalized derivation in fuzzy Banach algebra.

1. Introduction and preliminaries

The stability problem of functional equations has originally been formulated by S. M. Ulam [25] : *Under what condition does there exists a homomorphism near an approximate homomorphism*? As an answer to the problem of Ulam, D. H. Hyers has proved the stability of the additive functional equation [13], which states that *if* $\varepsilon > 0$ *and* $f : \mathcal{X} \to \mathcal{Y}$ *is a mapping with* $\mathcal{X}, \mathcal{Y}$ *Banach spaces, such that*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \tag{1.1}$$

for all $x, y \in \mathcal{X}$, *then there exists a unique additive mapping* $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x) - \mathcal{L}(x)\| \le \varepsilon$$

for all $x \in \mathcal{X}$. This stability phenomenon is called the *Hyers-Ulam stability* of the additive functional equation $f(x + y) = f(x) + f(y)$.

A generalized version of the theorem of Hyers for additive mappings was given by T. Aoki [1] (cf. also [7]) and for linear mappings was presented by Th. M. Rassias [22] by considering the case when the inequality (1.1) is unbounded. Due to the fact, the additive functional equation is said to have the *generalized Hyers-Ulam stability* (*Hyers-Ulam-Rassias stability*) property. Since then, a great deal of work has been done by a

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number of authors and the problems concerned with the generalizations and the applications of the stability to functional equations have been developed as well (for instant, [8, 11, 12, 21]). In particular, the stability concerning derivations between operator algebras was first obtained by P. Šemrl [24]. Recently, R. Badora [2] gave a generalization of the Bourgin's result [6]. He also dealt with the Hyers-Ulam stability and the Bourgin-type superstability of derivations in [3].

Definition 1.1. Let $\mathcal{A}$ be an algebra over the real or complex field $\mathbb{F}$. An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a *derivation* if the functional equation $d(xy) = xd(y) + d(x)y$ holds for all $x, y \in \mathcal{A}$. An additive mapping $g : \mathcal{A} \to \mathcal{A}$ is said to be a *generalized derivation* if there exists a derivation $d : \mathcal{A} \to \mathcal{A}$ such that the functional equation $g(xy) = xg(y) + d(x)y$ is fulfilled for all $x, y \in \mathcal{A}$.

A. K. Katrasas [14] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Later, some authors have defined fuzzy norms on a vector space from various points of view [10, 16, 26]. In particular, T. Bag and S. K. Samanta [4], following S. C. Cheng and J. N. Mordeson [9], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [15]. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [5].

Following [4], we give the notation of fuzzy norm as follows (cf. [17, 18, 19, 20]).

Definition 1.2. Let $\mathcal{X}$ be a real vector space. A function $N : \mathcal{X} \times \mathbb{R} \to [0, 1]$ is said to be a *fuzzy norm* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and $a, b \in \mathbb{R}$,

(N1) $N(x, a) = 0$ if $a \leq 0$;
(N2) $x = 0$ if and only if $N(x, a) = 1$ for all $a > 0$;
(N3) $N(ax, b) = N(x, \frac{b}{|a|})$ if $a \neq 0$;
(N4) $N(x + y, a + b) \geq \min\{N(x, a), N(y, b)\}$
(N5) $N(x, .)$ is non-decreasing function on $\mathbb{R}$ and $\lim_{a\to\infty} N(x, a) = 1$;
(N6) For $x \neq 0$, $N(x, .)$ is (upper semi) continuous on $\mathbb{R}$.

In this case, the pair $(\mathcal{X}, N)$ is called a *fuzzy normed vector space.*

The examples of fuzzy norms and the properties of fuzzy normed vector spaces are given in [17, 18, 19, 20].

Definition 1.3. Let $(\mathcal{X}, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be *convergent* if there exists $x \in \mathcal{X}$ such that $\lim_{n\to\infty} N(x_n - x, a) = 1$ for all $a > 0$. In this case, x is the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

DEFINITION 1.4. Let $(\mathcal{X}, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $\mathcal{X}$ is called *Cauchy* if for each ε and each $a > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, a) > 1 - \varepsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

DEFINITION 1.5. ([23]) Let $\mathcal{X}$ be an algebra and $(\mathcal{X}, N)$ be a fuzzy Banach space. The pair $(\mathcal{X}, N)$ is said to be a *fuzzy Banach algebra* if for every $x, y \in \mathcal{X}$, $a, b \in \mathbb{R}$,

$$N(xy, ab) \geq \min\{N(x, a), N(y, b)\}.$$

EXAMPLE 1.6. *Let $(\mathcal{X}, \|.\|)$ be a Banach algebra. Then*

$$N(x, a) = \begin{cases} 0 & \text{if } a \leq \|x\|, \\ 1 & \text{if } a > \|x\| \end{cases}$$

is a fuzzy norm and so $(\mathcal{X}, N)$ is a fuzzy Banach algebra (See [23]).

In the present article, we use the definition of fuzzy normed spaces given in [17, 18, 19, 20] to establish the stability of generalized derivation in fuzzy Banach algebra.

2. Main results

Throughout this paper, the element e of an algebra will denote a unit.

THEOREM 2.1. *Let $(\mathcal{A}, N)$ be a fuzzy Banach algebra with unit and let $(\mathcal{C}, N')$ be a fuzzy normed space. Assume that $\varphi : \mathcal{A}^2 \to \mathcal{C}$ and $\phi : \mathcal{A}^2 \to \mathcal{C}$ are functions such that for some $0 < \alpha < 1$ and some $0 < \beta < 1$,*

(2.1)
$$N'(\varphi(2x, 2y), t) \geq N'(\alpha\varphi(x, y), t) \quad \text{and} \quad N'(\phi(2x, y), t) \geq N'(\beta\varphi(x, y), t)$$

for all $x, y \in \mathcal{A}$ and all $t > 0$. Suppose that $f, g : \mathcal{A} \to \mathcal{A}$ are mappings such that

$$N(f(x + y) - f(x) - f(y), t) \geq N'(\varphi(x, y), t) \tag{2.2}$$

and

$$N(f(xy) - xf(y) - g(x)y, t) \geq N'(\phi(x, y), t) \tag{2.3}$$

for all $x, y \in \mathcal{A}$ *and all* $t > 0$. *Then* $f : \mathcal{A} \to \mathcal{A}$ *is a generalized derivation.*

Proof. It follows from (2.2) and Theorem 3.1 [18] that there exists a unique additive mapping $h : \mathcal{A} \to \mathcal{A}$ satisfying

$$N(f(x) - h(x), t) \geq N'\Big(\frac{2\varphi(x, x)}{2 - \alpha}, t\Big) \tag{2.4}$$

for all $x \in \mathcal{A}$ and all $t > 0$, where

$$h(x) = N - \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{2.5}$$

for all $x \in \mathcal{A}$.

Let us assume that $\Delta : \mathcal{A}^2 \to \mathcal{A}$ is a function defined by

$$\Delta(x, y) = f(xy) - xf(y) - g(x)y$$

for all $x, y \in \mathcal{A}$. By letting $x := 2^n x$ and $t := 2^n s$ in (2.3), then we get

$$N\Big(\frac{f(2^n xy)}{2^n} - xf(y) - \frac{g(2^n x)}{2^n} y, s\Big) \geq N'\Big(\phi(x, y), \frac{2^n}{\beta^n} s\Big),$$

for all $x \in \mathcal{A}$ and all $s > 0$, which implies

$$N - \lim_{n\to\infty} \frac{\Delta(2^n x, y)}{2^n} = 0 \tag{2.6}$$

for all $x, y \in \mathcal{A}$. Putting $x := 2^n x, y := e$ and $t := 2^n s$ in (2.3), then we have

$$N\Big(\frac{g(2^n x)}{2^n} - \frac{f(2^n x)}{2^n} + xf(e), s\Big) \geq N'\Big(\phi(x, e), \frac{2^n}{\beta^n} s\Big)$$

for all $x \in \mathcal{A}$. So we deduce that

$$\lim_{n\to\infty} N\Big(\frac{g(2^n x)}{2^n} - \frac{f(2^n x)}{2^n} + xf(e), s\Big) = 1 \tag{2.7}$$

for all $x \in \mathcal{A}$ and all $s > 0$. Note that

$$\begin{aligned} &N\Big(\frac{g(2^n x)}{2^n} - (h(x) - xf(e)), s\Big) \\ &\geq \min\Big\{N\Big(\frac{g(2^n x)}{2^n} - \frac{f(2^n x)}{2^n} + xf(e), \frac{s}{2}\Big),\ N\Big(\frac{f(2^n x)}{2^n} - h(x), \frac{s}{2}\Big)\Big\} \end{aligned}$$

for all $x \in \mathcal{A}$ and all $s > 0$. By (2.5) and (2.7), we get

$$h(x) - xf(e) = N - \lim_{n\to\infty} \frac{g(2^n x)}{2^n}$$

for all $x \in \mathcal{A}$. If we define a mapping $\delta : \mathcal{A} \to \mathcal{A}$ by $\delta(x) = h(x) - xf(e)$ for all $x \in \mathcal{A}$, then, by the additivity of h that δ is additive.

Letting $x := 2^n x$ and $y := e$ in (2.3), then we have

$$N(g(2^n x) - f(2^n x) + 2^n x f(e), s) \geq N'\Big(\phi(x, e), \frac{1}{\beta^n} s\Big) \tag{2.8}$$

for all $x \in \mathcal{A}$ and all $s > 0$. In (2.4), setting $x := 2^n x$, we obtain

$$N(f(2^n x) - h(2^n x), t) \geq N'\Big(\frac{2\varphi(x, x)}{2 - \alpha}, \frac{1}{\alpha^n} t\Big) \tag{2.9}$$

for all $x \in \mathcal{A}$ and all $t > 0$. The conditions (2.8) and (2.9) imply that

$$\lim_{n\to\infty} N(g(2^n x) - f(2^n x) + 2^n x f(e), s) = 1 \tag{2.10}$$

and

$$\lim_{n\to\infty} N(f(2^n x) - h(2^n x), t) = 1 \tag{2.11}$$

for all $x \in \mathcal{A}$ and all $s, t > 0$. In particular, we see that

$$\begin{aligned}
&N(g(2^n x) - f(2^n x) + 2^n x f(e) + f(2^n x) - h(2^n x), s) \\
&\geq \min\Big\{N\Big(g(2^n x) - f(2^n x) + 2^n x f(e), \frac{s}{2}\Big), N\Big(f(2^n x) - h(2^n x), \frac{s}{2}\Big)\Big\}
\end{aligned}$$

for all $x \in \mathcal{A}$ and all $s > 0$. Thus, by virtue of (2.10) and (2.11), we have

$$\lim_{n\to\infty} N(g(2^n x) - f(2^n x) + 2^n x f(e) + f(2^n x) - h(2^n x), s) = 1 \tag{2.12}$$

for all $x \in \mathcal{A}$ and all $s > 0$. It follows from the additivity of h that

$$\begin{aligned}
&N\Big(\Big(\frac{g(2^n x)}{2^n} - \delta(x)\Big)y, s\Big) = N\Big(\Big(\frac{g(2^n x)}{2^n} - (h(x) - xf(e))\Big)y, s\Big) \\
&= N((g(2^n x) - f(2^n x) + 2^n x f(e) + f(2^n x) - h(2^n x))y, 2^n s) \\
&\geq \min\{N(g(2^n x) - f(2^n x) + 2^n x f(e) + f(2^n x) - h(2^n x), s), N(y, 2^n)\}
\end{aligned}$$

for all $x \in \mathcal{A}$ and all $s > 0$. So we have by (2.12)

$$\lim_{n\to\infty} N\Big(\Big(\frac{g(2^n x)}{2^n} - \delta(x)\Big)y, s\Big) = 1 \tag{2.13}$$

for all $x \in \mathcal{A}$ and all $s > 0$. We now note that

$$\begin{aligned}
&N\Big(xf(y) + \frac{g(2^n x)}{2^n} y + \frac{\Delta(2^n x, y)}{2^n} - xf(y) - \delta(x)y, s\Big) \\
&\geq \min\Big\{N\Big(\Big(\frac{g(2^n x)}{2^n} - \delta(x)\Big)y, \frac{s}{2}\Big), N\Big(\frac{\Delta(2^n x, y)}{2^n}, \frac{s}{2}\Big)\Big\}
\end{aligned}$$

for all $x \in \mathcal{A}$ and all $s > 0$. In view of (2.6) and (2.13), we see that

$$\lim_{n\to\infty} N\Big(xf(y) + \frac{g(2^n x)}{2^n}y + \frac{\Delta(2^n x, y)}{2^n} - xf(y) - \delta(x)y, s\Big) = 1 \tag{2.14}$$

for all $x, y \in \mathcal{A}$ and all $s > 0$. Now, using (2.5) and (2.14), we get

$$\begin{aligned} h(xy) &= N - \lim_{n\to\infty} \frac{f(2^n x \cdot y)}{2^n} \\ &= N - \lim_{n\to\infty} \Big[xf(y) + \frac{g(2^n x)}{2^n}y + \frac{\Delta(2^n x, y)}{2^n}\Big] \\ &= xf(y) + \delta(x)y \end{aligned} \tag{2.15}$$

for all $x, y \in \mathcal{A}$. Applying (2.15) and the additivity of δ, we obtain

$$xf(2^n y) + \delta(x)\cdot 2^n y = h(x\cdot 2^n y) = h(2^n x \cdot y) = 2^n xf(y) + \delta(x)\cdot 2^n y,$$

which means that

$$x\frac{f(2^n y)}{2^n} = xf(y).$$

for all $x, y \in \mathcal{A}$. Hence we get

$$\lim_{n\to\infty} N\Big(x\frac{f(2^n y)}{2^n} - xf(y), s\Big) = 1 \tag{2.16}$$

for all $x, y \in \mathcal{A}$ and all $s > 0$. In particular, we have by the additivity of h,

$$\begin{aligned} &N\Big(x\frac{f(2^n y)}{2^n} - xh(y), s\Big) \\ &= N(xf(2^n y) - xh(2^n y), 2^n s) \\ &\geq \min\{N(x, 2^n), N(f(2^n y) - h(2^n y), s)\} \end{aligned}$$

for all $x, y \in \mathcal{A}$ and all $s > 0$. This inequality and (2.11) guarantee the following

$$\lim_{n\to\infty} N\Big(x\frac{f(2^n y)}{2^n} - xh(y), s\Big) = 1 \tag{2.17}$$

for all $x, y \in \mathcal{A}$ and all $s > 0$.

Now it follows by (2.16) and (2.17) that

$$xf(y) = N - \lim_{n\to\infty} x\frac{f(2^n y)}{2^n} = xh(y) \tag{2.18}$$

for all $x, y \in \mathcal{A}$. Consequently, (2.15) becomes

$$h(xy) = xh(y) + \delta(x)y$$

for all $x, y \in \mathcal{A}$.

On the other hand,

$$\delta(xy) = h(xy) - xyf(e) = x(h(y) - yf(e)) + \delta(x)y = x\delta(y) + \delta(x)y,$$

for all $x, y \in \mathcal{A}$. That is, δ is a derivation.

Letting $x = e$ in (2.18), we have $f = h$. Therefore, we conclude f is a generalized derivation, which completes the proof. □

COROLLARY 2.2. *Let $(\mathcal{A}, N)$ be a fuzzy Banach algebra with unit and let $(\mathcal{C}, N')$ be a fuzzy normed space. Assume that $\varphi : \mathcal{A}^2 \to \mathcal{C}$ and $\phi : \mathcal{A}^2 \to \mathcal{C}$ are functions such that for some $0 < \alpha < 1$ and some $0 < \beta < 1$,*

$$N'(\varphi(2x, 2y), t) \geq N'(\alpha\varphi(x, y), t) \quad \text{and} \quad N'(\phi(2x, y), t) \geq N'(\beta\varphi(x, y), t)$$

for all $x, y \in \mathcal{A}$ and all $t > 0$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping such that

$$N(f(x + y) - f(x) - f(y), t) \geq N'(\varphi(x, y), t)$$

and

$$N(f(xy) - xf(y) - f(x)y, t) \geq N'(\phi(x, y), t)$$

for all $x, y \in \mathcal{A}$ and all $t > 0$. Then $f : \mathcal{A} \to \mathcal{A}$ is a derivation.

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