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A POSITIVE PRESENTATION FOR THE PURE BRAID GROUP

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Abstract. For the pure braid groups, this paper gives a positive finite presentation whose generators are the standard Artin generators, and determines whether the submonoid generated by the Artin generators is a Garside monoid or not.

1. Introduction

The braid groups B_n, $n \geqslant 2$, have two well-known group presentations. The first one is the Artin presentation discovered by Emil Artin [1]. It has generators $\sigma_1, \ldots, \sigma_{n-1}$ and defining relations:

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geqslant 2, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1. \end{cases}$$

The second one is the dual presentation discovered by Birman, Ko and Lee [4]. It has generators a_{ij} for $1 \leqslant j < i \leqslant n$ and defining relations:

$$\begin{cases} a_{rs}a_{ij} = a_{ij}a_{rs} & \text{if } (r-i)(r-j)(s-i)(s-j) > 0, \\ a_{ij}a_{jk} = a_{jk}a_{ik} = a_{ik}a_{ij} & \text{if } 1 \leqslant k < j < i \leqslant n. \end{cases}$$

See Figure 1(a, b) for the generators σ_i and a_{ij}. They are related as follows:

$$\begin{array}{ll} \sigma_i = a_{(i+1)i} & \text{for } i = 1, \ldots, n-1, \\ a_{ij} = (\sigma_{i-1}\cdots\sigma_{j+1})\sigma_j(\sigma_{j+1}^{-1}\cdots\sigma_{i-1}^{-1}) & \text{for } 1 \leqslant j < i \leqslant n. \end{array}$$

The Artin and dual presentations of B_n are *positive presentations*: all the defining relations are described only in terms of positive words. (A word is called a *positive word* if it only involves positive powers of the generators.) Therefore these presentations define not only the group

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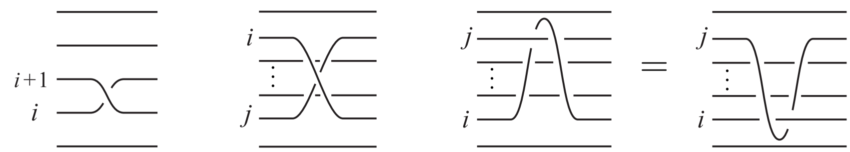


(a) σ_i (b) a_{ij} (c) A_{ij}

FIGURE 1. generators

B_n, but also monoids. In the late sixties, Garside [9] studied the monoid B_n^+ consisting of braids represented by positive words in $\sigma_1, \ldots, \sigma_{n-1}$, and solved the word and conjugacy problems. Later, Birman, Ko and Lee [4] showed that all the machineries developed by Garside can be applied to the monoid arising from the dual presentation. Dehornoy and Paris [8] introduced the notion of *Garside monoid* and *Garside group*, and showed that the monoids arising from the Artin and dual presentations are Garside monoids.

There is a natural epimorphism from the braid group B_n to the symmetric group Σ_n on n objects, defined by sending $\sigma_i \in B_n$ to the transposition $(i, i+1) \in \Sigma_n$. The kernel of this homomorphism is called the *pure braid group*, denoted by P_n.

For $1 \leqslant i < j \leqslant n$, let $A_{ij} = A_{ji} = a_{ji}^2$. See Figure 1(c). In [1], Artin found a group presentation for P_n. It has generators A_{ij} for $1 \leqslant i < j \leqslant n$ and defining relations:

(A1) $A_{rs}^{-1} A_{ij} A_{rs} = A_{ij}$ if $r < s < i < j$ or $i < r < s < j$;
(A2) $A_{ri}^{-1} A_{ij} A_{ri} = A_{rj} A_{ij} A_{rj}^{-1}$ if $r < i < j$;
(A3) $A_{is}^{-1} A_{ij} A_{is} = (A_{ij} A_{sj}) A_{ij} (A_{ij} A_{sj})^{-1}$ if $i < s < j$;
(A4) $A_{rs}^{-1} A_{ij} A_{rs} = (A_{rj} A_{sj} A_{rj}^{-1} A_{sj}^{-1}) A_{ij} (A_{rj} A_{sj} A_{rj}^{-1} A_{sj}^{-1})^{-1}$ if $r < i < s < j$.

Comparing with the presentations of B_n, the above presentation of P_n is much more complicated. Moreover, it is not a positive presentation, hence it does not define a monoid. Recently, Margalit and McCammond [10] proposed positive presentations for P_n, by using new generators.

In this paper, we are interested in the standard Artin generators for P_n. First, we give a positive presentation for P_n, using the Artin generators.

THEOREM 1.1. *The pure braid group P_n is generated by $\{A_{ij} \mid 1 \leqslant i < j \leqslant n\}$ subject to the following three types of relations:*

(P1) $A_{ij}A_{rs} = A_{rs}A_{ij}$ *if* $r < s < i < j$ *or* $i < r < s < j$;
(P2) $A_{ji}A_{ir}A_{rj} = A_{ir}A_{rj}A_{ji} = A_{rj}A_{ji}A_{ir}$ *if* $r < i < j$;
(P3) $A_{rs}(A_{jr}A_{ji}A_{js}) = (A_{jr}A_{ji}A_{js})A_{rs}$ *if* $r < i < s < j$.

Next, we decide whether the submonoid of P_n generated by the Artin generators is a Garside monoid or not.

THEOREM 1.2. *Let* P_n^+ *be the submonoid of* P_n *generated by the Artin generators* A_{ij} *for* $1 \leqslant i < j \leqslant n$. *Then the monoid* P_n^+ *is a Garside monoid for* $n = 2, 3$, *but it is not for* $n \geqslant 4$.

2. Garside monoids and groups

This section briefly reviews Garside monoids and Garside groups. See [8, 7] for details.

For a monoid M, let 1 denote the identity element. An element $a \in M \setminus \{1\}$ is called an *atom* if $a = bc$ for $b, c \in M$ implies either $b = 1$ or $c = 1$. For $a \in M$, let $\|a\|$ be the supremum of the lengths of all expressions of a in terms of atoms. The monoid M is said to be *atomic* if it is generated by its atoms and $\|a\| < \infty$ for any element a of M. In an atomic monoid M, there are partial orders $\leqslant_L$ and $\leqslant_R$: $a \leqslant_L b$ if $ac = b$ for some $c \in M$; $a \leqslant_R b$ if $ca = b$ for some $c \in M$.

DEFINITION 2.1. An atomic monoid M is called a *Garside monoid* if it satisfies the following properties:

(i) M is left and right cancellative.
(ii) $(M, \leqslant_L)$ and $(M, \leqslant_R)$ are lattices. That is, for every $a, b \in M$ there are a unique least common multiple $a \vee_L b$ (resp. $a \vee_R b$) and a unique greatest common divisor $a \wedge_L b$ (resp. $a \wedge_R b$) with respect to $\leqslant_L$ (resp. $\leqslant_R$).
(iii) M contains an element Δ, called a *Garside element*, satisfying the following:
  (a) for each $a \in M$, $a \leqslant_L \Delta$ if and only if $a \leqslant_R \Delta$;
  (b) the set $\{a \in M \mid a \leqslant_L \Delta\}$ is finite and generates M.

A *Garside group* is defined as the group of fractions of a Garside monoid. When M is a Garside monoid and G is the group of fractions of M, we identify the elements of M and their images in G. The ordered pair (M, Δ) is called a *Garside structure* on G. We remark that a given group G may admit more than one Garside structure.

Definition 2.2. Let M be a Garside monoid, and let $a, b, c \in M$. The element c is called a *left common multiple* of a and b if $a \leqslant_L c$ and $b \leqslant_L c$. And the element $a \vee_L b$ is called a *left lcm* of a and b.

3. Proof of Theorem 1.1

Note that the generators of the presentation in Theorem 1.1 are the same as those of the Artin presentation. Since it is straightforward to check that the relations (P1)–(P3) can be derived from the Artin relations, it suffices to show that the Artin relations can be derived from the relations (P1)–(P3). Recall that $A_{pq} = A_{qp}$ for any $1 \leqslant p < q \leqslant n$.

(A1) is the same as (P1).

After rearranging, (A2) is equivalent to $A_{ij}A_{ri}A_{rj} = A_{ri}A_{rj}A_{ij}$ with $r < i < j$, which is the first identity of (P2).

After rearranging, (A3) is equivalent to

$$\text{(A3}'\text{)} \qquad A_{ij}A_{is}A_{ij}A_{sj} = A_{is}A_{ij}A_{sj}A_{ij} \qquad \text{with } i < s < j.$$

This relation can be derived from (P2) as follows:

$$\begin{aligned}\text{LHS of (A3}'\text{)} &= A_{ij}(A_{si}A_{ij}A_{js}) \overset{\text{(P2)}}{=} (A_{ij}A_{js}A_{si})A_{ij} \overset{\text{(P2)}}{=} A_{si}A_{ij}A_{js}A_{ij} \\ &= \text{RHS of (A3}'\text{)}.\end{aligned}$$

The relation (A4) is

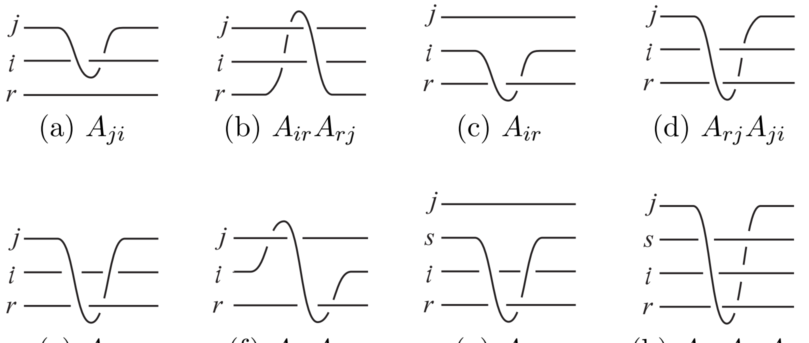
$$A_{rs}^{-1}A_{ij}A_{rs} = (A_{rj}A_{sj}A_{rj}^{-1}A_{sj}^{-1})A_{ij}(A_{sj}A_{rj}A_{sj}^{-1}A_{rj}^{-1})$$

with $r < i < s < j$. After rearranging, this is equivalent to

$$\text{(A4}'\text{)} \qquad A_{ij}A_{rs}A_{rj}A_{sj}A_{rj}^{-1} = A_{rs}A_{rj}A_{sj}A_{rj}^{-1}A_{sj}^{-1}A_{ij}A_{sj}.$$

Since

$$\begin{aligned}\text{LHS of (A4}'\text{)} &= A_{ij}(A_{sr}A_{rj}A_{js})A_{rj}^{-1} \\ &\overset{\text{(P2)}}{=} A_{ij}(A_{js}A_{sr}A_{rj})A_{rj}^{-1} \\ &= A_{ij}A_{js}A_{sr}, \\ \text{RHS of (A4}'\text{)} &= (A_{sr}A_{rj}A_{js})A_{rj}^{-1}A_{sj}^{-1}A_{ij}A_{sj} \\ &\overset{\text{(P2)}}{=} (A_{js}A_{sr}A_{rj})A_{rj}^{-1}A_{sj}^{-1}A_{ij}A_{sj} \\ &= A_{js}A_{sr}A_{sj}^{-1}A_{ij}A_{sj},\end{aligned}$$



(a) A_{ji} (b) $A_{ir}A_{rj}$ (c) A_{ir} (d) $A_{rj}A_{ji}$

(e) A_{rj} (f) $A_{ji}A_{ir}$ (g) A_{rs} (h) $A_{jr}A_{ji}A_{js}$

FIGURE 2. The pairs of braids which commute each other.

(A4) is equivalent to $A_{ij}A_{js}A_{sr} = A_{js}A_{sr}A_{sj}^{-1}A_{ij}A_{sj}$. By multiplying A_{rj} to the left, this is equivalent to

(A4″) $\quad A_{rj}A_{ij}A_{js}A_{sr} = A_{rj}A_{js}A_{sr}A_{sj}^{-1}A_{ij}A_{sj}$ with $r < i < s < j$.

This relation can be derived from (P2) and (P3) as follows:

$$\begin{aligned}\text{RHS of (A4}'') &= (A_{rj}A_{js}A_{sr})A_{sj}^{-1}A_{ij}A_{sj}\\ &\overset{\text{(P2)}}{=} (A_{sr}A_{rj}A_{js})A_{sj}^{-1}A_{ij}A_{sj} = A_{rs}(A_{jr}A_{ji}A_{js})\\ &\overset{\text{(P3)}}{=} (A_{jr}A_{ji}A_{js})A_{rs} = \text{LHS of (A4}'').\end{aligned}$$

This completes the proof.

REMARK 3.1. Like (P1), the relations (P2) and (P3) can be viewed as commutativity relations:

$$\begin{aligned}A_{ji}(A_{ir}A_{rj}) &= (A_{ir}A_{rj})A_{ji};\\ A_{ir}(A_{rj}A_{ji}) &= (A_{rj}A_{ji})A_{ir};\\ A_{rj}(A_{ji}A_{ir}) &= (A_{ji}A_{ir})A_{rj};\\ A_{rs}(A_{jr}A_{ji}A_{js}) &= (A_{jr}A_{ji}A_{js})A_{rs}.\end{aligned}$$

It is easy to see from Figure 2 that the pairs of braids in the above identities commute.

4. Proof of Theorem 1.2

For $n = 2$, the pure braid group P_n is an infinite cyclic group generated by A_{12}, hence we are done because $\mathbb{Z}$ is a Garside group with Garside monoid $\mathbb{Z}_{\geqslant 0}$ [3, Example 2].

Case 1. $n = 3$

If we set $x = A_{12}$, $y = A_{31}$ and $z = A_{23}$, then the presentation in Theorem 1.1 becomes

$$P_3 = \langle x, y, z \mid xyz = yzx = zxy \rangle.$$

In this case, the monoid P_3^+ is known to be a Garside monoid [8, Example 5].

Case 2. $n \geqslant 4$

Fix $n \geqslant 4$. Assume that P_n^+ is a Garside monoid. From the relations (P2) and (P3), we have

$$A_{31}A_{14}A_{43} = A_{14}A_{43}A_{31},$$
$$A_{13}(A_{41}A_{42}A_{43}) = (A_{41}A_{42}A_{43})A_{13}.$$

Therefore both $A_{31}A_{14}A_{43}$ and $A_{13}A_{41}A_{42}A_{43}$ are left common multiples of A_{13} and A_{14}.

Let $\alpha \in P_n^+$ be the left lcm of A_{13} and A_{14}. Because left common multiples of A_{13} and A_{14} have word-length at least 2, the word-length of α is either 2 or 3.

Subcase 2.1. α has word-length 3.

Since we have assumed that P_n^+ is a Garside monoid, by the uniqueness of left lcm, one has $\alpha = A_{31}A_{14}A_{43}$. Hence

$$A_{13}A_{41}A_{42}A_{43} = \alpha X = A_{31}A_{14}A_{43}X$$

for some Artin generator X. From the above equations, one has $X = A_{43}^{-1}A_{42}A_{43}$.

Meanwhile, it is well-known that the set $\{A_{i4} \mid i \in \{1, \ldots, n\} \backslash \{4\}\}$ generates the free group F_{n-1} [2]. Therefore X cannot be a generator, which is a contradiction.

Subcase 2.2. α has word-length 2.

In this case, there are Artin generators X and Y such that

$$\alpha = A_{13}X = A_{14}Y.$$

Let $\phi : P_n \to P_3$ be the homomorphism obtained by deleting all the k-th strands for $k \neq 1, 3, 4$. Using the convention of $P_3 = \langle x, y, z \mid xyz = yzx = zxy \rangle$ in Case 1, ϕ sends the monoid P_n^+ to P_3^+ such that

$$\phi(A_{13}) = A_{12} = x, \quad \phi(A_{14}) = A_{13} = y.$$

Hence we have

$$\phi(\alpha) = x\phi(X) = y\phi(Y).$$

Thus $\phi(\alpha) \in P_3^+$ is a left common multiple of x and y with word-length at most 2 because each of $\phi(X)$ and $\phi(Y)$ is an Artin generator or the identity. This is a contradiction because in the Garside monoid P_3^+ the left lcm of x and y is xyz.

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