JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **23**, No. 3, September 2010

WEAK DISTANCES

Se Hwa Chung*

Abstract. In this paper, we introduce a concept of a weak distance and show that AP is a bireflective subcategory of WD. Moreover, we introduce a concept of a co-tower which is an equivalent description of approach spaces.

1. Introduction

In order to study the "approximation structure," various structures such as topological structures, quasi-metric spaces, quasi-uniform spaces, uniform spaces, convergence structures have been introduced. The natural question is whether there is one setting under which the above structures can be studied. There have been at least three conceptual approaches, namely syntopogenous structures introduced by Császár[2], nearness structures by Herrlich[4] and approach structures by Lowen [7]. These new structures or constructs should contain old ones as well-behaved subconstructs like reflective or coreflective subconstructs.

The main idea behind approach system is to axiomatize the notion of distance between points and sets in such a way as to generalize both the ∞-metric and topological situations. The concept of distance is most closely related to the concept of a κ-metric as introduced by Shchepin[9]. In [8], seven distinct but equivalent ways to axiomatize an approach space are given and in [7] Lowen showed that Top is bireflective and bicorefective in AP and $pqMET^{\infty}$ is bicoreflective in AP.

In this paper, we introduce a concept of a weak distance, which is a generalization of a concept of approach space and show that AP is a

Received June 28, 2010; Accepted September 01, 2010.
2010 Mathematics Subject Classification: Primary 54A05, 54B05, 54B30, 54E25; Secondary 54E35.
Key words and phrases: approrach space, distance, weak distance, bireflective subcategory, tower, co-tower.
This research was supported by the Kyungwon University Research Fund in 2010.

bireflective subcategory of WD. Moreover, we introduce a concept of a co-tower which is an equivalent description of approach spaces.

The rest of this paper is organized as follows. In section 2, we investigate approach spaces from the view of conceptual aspects. In section 3, we introduce a concept of a weak distance and investigate approach spaces from the view of categorical aspects. In section 4, we characterize approach space via interior operators and introduce a concept of co-towers which is equivalent with the concept of towers.

For the general background of approach spaces and ∞pq-metric spaces, we refer to [8] and for the category theory, we refer to [1] and [5]. In what follows, given a set X we denote its power set by 2^X. $\vee$ stands for supremum and $\wedge$ stands for infimum.

2. Preliminaries

In this section, we investigate approach spaces from the view of conceptual aspects.

For a set X and a function

$$\delta : X \times 2^X \longrightarrow [0, \infty]$$

consider the following properties;

(A1) for each $x \in X$, $\delta(x, x) = 0$,

(A2) $\delta(x, \emptyset) = \infty$,

(A3) for each $x \in X$ and $\vartheta \subseteq 2^X$, $\delta(x, \cup\vartheta) = \wedge\{\delta(x, A) : A \in \vartheta\}$,

(A4) for each $x \in X$ and $A, B \subseteq 2^X$ $\delta_\mathcal{T}(x, A \cup B) = \delta_\mathcal{T}(x, A) \wedge \delta_\mathcal{T}(x, B)$,

(A5) for $x, y, z \in X$, $\delta(x, \{y\}) \leq \delta(x, \{z\}) + \delta(z, \{y\})$,

(A6) $\forall x \in X, \forall A \in 2^X, \forall k \in [0, \infty]$: $\delta(x, A) \leq \delta(x, A^{(k)}) + k$, where

$$A^{(k)} = \{x \in X : \delta(x, A) \leq k\},$$

(A7) $\forall x \in X, \forall A, B \in 2^X$, $\delta(x, A) \leq \delta(x, B) + \vee_{b \in B}\delta(b, A)$,

(A8) if $\delta(x, A) > 0$, then there is a subset $G \subseteq X$ such that $x \in G$ and for any $g \in G$, $\delta(g, X - G \cup A) > 0$,

(A9) if $B \subseteq A \subseteq X$, then for any $x \in X$,

$$\delta(x, A) \leq \delta(x, B).$$

The following proposition contains some simple but fundamental properties which we will use implicitly in the sequel.

Proposition 2.1. *For a function $\delta : X \times 2^X \longrightarrow [0, \infty]$, one has the following:*

(1) *(A5) and (A3) imply (A7).*

(2) *(A7) implies (A5).*
(3) *(A6) and (A9) imply (A7).*
(4) *(A7) implies (A6).*

Proof. (1) Take any $x \in X$ and $A, B \subseteq 2^X$, then for each $z \in A$ and $y \in B$,

$$\delta(x, z) \leq \delta(x, y) + \delta(y, z)$$

and then

$$\delta(x, A) \leq \delta(x, y) + \delta(y, A)$$

and hence

$$\delta(x, A) \leq \delta(x, B) + \vee_{y \in B} \delta(y, A).$$

(2) it is founded in [7].
(3) it is founded in [8].
(4) Take any $k \in [0, \infty]$, $A \in 2^X$, then by assumption,

$$\delta(x, A) \leq \delta(x, A^{(k)}) + \vee_{a \in A^{(k)}} \delta(a, A)$$

and so

$$\delta(x, A) \leq \delta(x, A^{(k)}) + \epsilon$$

for $\vee\{\delta(a, A) : a \in A^{(k)}\} \leq k$. □

By the above proposition if a function $\delta : X \times 2^X \to [0, \infty]$ satisfies (A3), then (A5), (A6) and (A7) are equivalent with each other, because δ satisfies (A3) and hence it satisfies (A9). It is clear that (A8) is equivalent with the property: for any $x \in X$ and $A \in 2^X$,

$$\delta(x, A) = \delta(x, A^{(0)})$$

Hence (A6) implies (A8). Finally, a function $\delta : X \times 2^X \to [0, \infty]$ satisfies (A7) if and only if it satisfies the following : for any $x \in X$ and $A, B \subseteq 2^X$,

$$\wedge_{y \in B} [\delta(x, A) - \delta(y, A)] \leq \delta(x, B).$$

Definition 2.2. ([8]) A function $d : X \times X \to [0, \infty]$ is called an extended pseudo-quasi-metric (shortly ∞pq-metric) on X if it satisfies the following properties: for all $x, y, z \in X$,
(M1) $d(x, x) = 0$,
(M2) $d(x, y) \leq d(x, z) + d(z, y)$.

Proposition 2.3. ([8]) *Let (X, d_X) and (Y, d_Y) be ∞pq-metric spaces, then a function $f : X \to Y$ is said to be nonexpansive if it fulfills the property*

$$d_Y(f(x), f(y)) \leq d_X(x, y) \ \forall x, y \in X.$$

$pqMET^{\infty}$ denotes the category of ∞pq-metric spaces and nonexpansive functions.

The following is an internal characterization of ∞pq-metric.

PROPOSITION 2.4. *A function $d : X \times X \to [0, \infty]$ is a ∞pq-metric if and only if there is a function*

$$\delta_d : X \times 2^X \to [0, \infty]$$

satisfying properties: (A1), (A2), (A3), (A6) and for any $x, y \in X$, $\delta_d(x, y) = d(x, y)$

Proof. For any $x \in X$ and $A \in 2^X$, let $\delta_d(x, A) = \wedge\{d(x, a) : a \in A\}$ then it is easy to show that it satisfies all the properties. The converse is clear. □

The following is an internal characterization of topology:

PROPOSITION 2.5. *A subfamily $\tau \subseteq 2^X$ is a topology on X if and only if there is a function*

$$\delta_\tau : X \times 2^X \to \{0, \infty\}$$

satisfying properties: (A1), (A2), (A4), (A6) and $G \in \tau$ if and only if for any $g \in G$, $\delta_\tau(g, X - G) > 0$

Proof. The function $\delta_\tau : X \times 2^X \to \{0, \infty\}$ defined by

$$\delta_\tau(x, A) = \begin{cases} 0 & x \in cl_\tau(A) \\ \infty & x \notin cl_\tau(A). \end{cases}$$

satisfies all the properties. The converse is clear. □

REMARK 2.6. (1) A subfamily $\tau \subseteq 2^X$ is a finitely generated topology on X if and only if there is a function

$$\delta_\tau : X \times 2^X \to \{0, \infty\}$$

satisfying properties: (A1), (A2), (A3), (A6) and $G \in \tau$ if and only if for any $g \in G$, $\delta_\tau(g, X - G) > 0$.

(2) Every ∞pq-metric satisfies the property (A3), but not topology.

(3) Topology and ∞pq-metric have properties (A1), (A2), (A4) and (A6) in common with each other.

The following is due to Lowen.

DEFINITION 2.7. A function $\delta : X \times 2^X \to [0, \infty]$ is called a distance on X if it satisfies properties (A1), (A2), (A4) and (A6).

By the preceding remark, every distance is both topology and ∞pq-metric.

3. Relationship between AP and WD

In this section, we introduce a concept of a weak distance and investigate approach spaces from the view of categorical aspects. In the last section we observed that topology and ∞pq-metric have the properties (A1), (A2), (A4), (A5) and (A8) in common with each other. From this observation, we have the following definition:

Definition 3.1. A function $\delta : X \times 2^X \to [0, \infty]$ is said to be a **weak distance** on X if it satisfies properties (A1), (A2), (A4), (A5) and (A8). The pair (X, δ) is called a **weak distance space** (or simply, *wd*-space).

By Proposition 2.1 and the preceding remark, every weak distance is also both topology and ∞pq-metric and the concept of weak distances is weaker than that of distances.

Theorem 3.2. *Every approach space is a wd-space.*

Proof. It is immediate from Definitions 2.7 and 3.1. □

Remark 3.3. If X is a finite set, then a function $\delta : X \times 2^X \to [0, \infty]$ is a distance on X if and only if it is a weak distance on X.

Definition 3.4. Let (X, δ_X) and (Y, δ_Y) be *wd*-spaces, then a function $f : X \to Y$ is said to be:

(1) a **contraction** if it fulfills the property:

$$\delta_Y(f(x), f(A)) \leq \delta_X(x, A) \qquad \forall x \in X, A \in 2^X$$

(2) an **isomorphism** if it is a 1-1 correspondence and both f and the inverse function of f are contractions.

Theorem 3.5. *For wd-spaces X, Y and Z, the following holds:*

(1) The identity function $id_X : X \to X$ is a contraction.

(2) For contractions $f : X \to Y$ and $g : Y \to Z$, the composition $g \circ f : X \to Z$ is again a contraction.

Theorem 3.6. *Let X be a set and (Y, α) a wd-space and $f : X \to Y$ a function. Then the function*

$$\delta_\alpha : X \times 2^X \to [0, \infty] : \ (x, A) \to \alpha(f(x), f(A))$$

satisfies the following:

(1) δ is a weak distance on X

(2) For any weak distance space (Z, η), a function $g : (Z, \eta) \to (X, \delta)$ is a contraction if and only if $f \circ g : (Z, \eta) \to (Y, \alpha)$ is a contraction.

Proof. (1) For any $x \in X$, $\alpha(f(x), f(x)) = 0$ and so $\delta(x, x) = 0$. Take any $x, y, z \in X$, then $\alpha(f(x), f(y)) \leq \alpha(f(x), f(z)) + \alpha(f(z), f(y))$ and so $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$. Since $f(\emptyset) = \emptyset$ and $\alpha(f(x), f(\emptyset)) = \infty$, $\delta(x, \emptyset) = \infty$. Take any $A, B \in 2^X$, then

$$\begin{aligned} & \delta(x, A \cup B) \\ = \; & \alpha(f(x), f(A \cup B)) \\ = \; & \delta(x, A) \wedge \delta(x, B). \end{aligned}$$

Suppose $\delta(x, A) > 0$, then $\alpha(f(x), f(A)) > 0$ and so there is a subset $G \supseteq Y$ such that $f(x) \in G$ and for any $g \in G$, $\alpha(g, f(A)) > 0$ and $\alpha(g, Y - G) > 0$. Take any $a \in f^{-1}(G)$, then $f(a) \in G$ and so $\alpha(f(a), f(A)) > 0$. Thus $\delta(a, A) > 0$. Since $f(X - f^{-1}(G)) \subseteq Y - G$, for any $a \in f^{-1}(G)$, one has the following:

$$0 < \alpha(f(a), Y - G) \leq \alpha(f(a), f(X - f^{-1}(G))) = \delta(a, X - f^{-1}(G))$$

Hence $0 < \delta(a, X - f^{-1}(G))$. In all, δ is a weak distance on X.

(2) Suppose $g : (Z, \eta) \to (X, \delta)$ is a contraction, Since $f : (X, \delta) \to (Y, \alpha)$ is a contraction, by Theorem 3.5 $f \circ g : (Z, \eta) \to (Y, \alpha)$ is a contraction. Suppose $f \circ g : (Z, \eta) \to (Y, \alpha)$ is a contraction, then $\alpha(f \circ g(z), f \circ g(A)) \leq \eta(z, A)$. Since $\alpha(f \circ g(z), f \circ g(A)) = \delta(g(z), g(A))$, $\delta(g(z), g(A)) \leq \eta(z, A)$ and hence g is a contraction. □

WD denotes the category of wd-spaces and contractions.

AP denotes the category of approach spaces and contractions.

The following is immediate from Theorem 3.2:

THEOREM 3.7. *AP is a full isomorphism-closed subcategory of WD.*

$pqMET^\infty(X)$ denotes the set of all ∞pq-metric on a set X.

DEFINITION 3.8. ([8]) A subset ς of $pqMET^\infty(X)$ is called a gauge if it ia an ideal in $pqMET^\infty(X)$ which fulfills following propetry:
If $\forall x \in X, \forall k > 0, \forall \omega < \infty : \exists d_x^{k,\omega} \in \varsigma$ sucht that $d(x, \cdot) \wedge \omega \leq d_x^{k,\omega}(x, \cdot) + k$, then $d \in \varsigma$.

DEFINITION 3.9. ([8]) A subset φ of $pqMET^\infty(X)$ is called a gauge basis if it is an ideal bais in $pqMET^\infty(X)$.

THEOREM 3.10. ([8]) *If $\delta : X \times 2^X \to [0, \infty]$ is a distance, then*

$$\varsigma_\delta = \{d \in pqMET^\infty(X) : \forall A \subset X, \forall x \in X : \wedge_{a \in A} d(x, a) \leq \delta(x, A)\}$$

is a gauge on X.

The ς_δ in the above theorem is called the associated gauge with δ.

In the following lemma, we will show that the associated gauge ς_δ with a weak distance δ is a non-empty set.

LEMMA 3.11. *(1) If $\delta : X \times 2^X \to [0, \infty]$ is a weak distance, then the following ∞pq-metrics is in ς_δ.*

$d^l_Y(x, y) = [\delta(x, Y) \wedge (\wedge_{u\in X-Y}\delta(u, Y)) - \delta(y, Y) \wedge (\wedge_{u\in X-Y}\delta(u, Y))] \vee 0$

(2) If $\delta : X\times 2^X \to [0, \infty]$ is a distance, then the following ∞pq-metric is in ς_δ

$$d^u_Y(x, y) = [\delta(x, Y) - \delta(y, Y)] \vee 0$$

In particular, $\wedge_{a\in Y} d^u_Y(x, a) = \delta(x, Y)$

Proof. (1) For any $A \in 2^X$ and $x \in X$,

$$\wedge_{a\in A}d^l_Y(x, a) = \begin{cases} 0 & \text{if } x \in Y \\ \wedge_{u\in X-Y}\delta(u, Y) & \text{if } x \notin Y \text{ and } A \subseteq Y \\ 0 & \text{if } x \notin Y \text{ and } A \subseteq X - Y \\ 0 & \text{if } x \notin Y \text{ and } A \neq \emptyset. \end{cases}$$

Hence $\wedge_{a\in A}d^l_Y(x, a) \leq \delta(x, A)$ for $A \subseteq Y$. Thus d^l_Y is in ς_δ.

(2) It follows from Proposition 2.1 that for any $A \in 2^X$ and $x \in X$,

$$\wedge_{a\in A}d^u_Y(x, a) = [\wedge_{a\in A}[\delta(x, Y) - \delta(a, Y)]] \vee 0 \leq \delta(x, A).$$

Thus d^u_Y is in ς_δ. The particular part is immediate from the fact that for any $a \in Y$, $\delta(a, Y) = 0$. □

Note that Lowen proved Theorem 3.10 without using the axiom (A6) in [7]. Using the above lemma, one has the following theorem.

THEOREM 3.12. $\delta : X \times 2^X \to [0, \infty]$ *is a weak distance, then*

$$\varsigma = \{d \in pqM^\infty(X) : \forall A \subset X, \forall x \in X : \wedge_{a\in A}d(x, a) \leq \delta(x, A)\}$$

is a gauge on X.

Proof. This goes along the same lines as Theorem 3.10. □

The following is now an immediate consequent of Lemma 3.11 and the above theorem.

THEOREM 3.13. ([8]) *If $\delta : X \times 2^X \to [0, \infty]$ is a distance and ς is the associated gauge, then we have*

$$\delta(x, A) = \vee_{d\in\varsigma} \wedge_{a\in A} d(x, a) \forall x \in X, \forall A \subset X$$

THEOREM 3.14. ([8]) *If a subset ς of $pqMET^{\infty}(X)$ is a gauge on X and δ is the associated distance, then*

$$\varsigma = \{d \in pqM^{\infty}(X) : \forall A \subset X, \forall x \in X : \wedge_{a \in A} d(x,a) \leq \delta(x,A)\}.$$

THEOREM 3.15. ([8]) *If (X,δ) and (Y,α) are distance spaces, then a function $f : X \to Y$ is a contraction if and only if $\forall d \in \varsigma_{\alpha} : d \circ (f \times f) \in \varsigma_{\delta}$.*

δ^g denotes the distance associated with the gauge ς_δ. The combined results of 3.10, 3.11, 3.13, and 3.14 prove that distance and gauge are equivalent with each other and that $\delta^g = \delta$ if δ is a distance, and the weak distance associated with gauge is a distance. Using this notations, we have the following:

THEOREM 3.16. *AP is a bireflective subcategory of WD. For any weak distance space (X,δ), its AP-bireflection is given by*

$$id_X : (X,\delta) \to (X,\delta^g).$$

Proof. It is easily verified that $id_X : (X,\delta) \to (X,\delta^g)$ is a contraction. Now suppose that (Y,η) is a approach space and $f : (X,\delta) \to (Y,\eta)$ is a contraction. Firstly, we show that $f : (X,\varsigma_\eta) \to (Y,\varsigma_\delta)$ is a contraction. Indeed, take any $d \in \varsigma_\eta$ then

$$\vee_{d \in \varsigma_\eta} \wedge_{a \in A} d(f(x), f(a)) \leq \eta(f(x), f(A)).$$

Since $f : (X,\delta) \to (Y,\eta)$ is a contraction,$\vee_{d \in \varsigma_\eta} \wedge_{a \in A} d(f(x), f(a)) \leq \delta(x,A)$ and hence for any $x \in X$ and $A \in 2^X$

$$\wedge_{a \in A} d(f(x), f(a)) \leq \delta(x,A).$$

Thus $d \circ (f \times f) \in \varsigma_\delta$. By Theorem 3.15, $f : (X,\delta^g) \to (Y,\eta^g)$ is a contraction and so $f : (X,\delta^g) \to (Y,\eta)$ is a contraction. □

Collecting the above theorems, we have the following:

THEOREM 3.17. *A weak distance δ on a set X is a distance if and only if the identity function $id_X : (X,\delta) \to (X,\delta^g)$ is an isomorphism.*

4. Co-towers

In this section, we characterize approach space via interior operator and introduce a concept of co-towers, which is equivalent with the concept of towers. It is worth noting that in topological spaces, the complement of the interior of a set is equal to the closure of the complement of the set and in an approach space (X,δ), $A^{(0)}$ is interpreted as the closure of the set A.

Theorem 4.1. *Let* (X, δ) *and* (Y, α) *be distance spaces and* $f : X \to Y$ *be a function then following are equivalent:*
(1) $f : (X, \delta) \to (Y, \alpha)$ *is a contraction.*
(2) $\forall x \in X, \forall G \in 2^Y : \alpha(f(x), Y - G) \leq \delta(x, X - f^{-1}(G))$.

Proof. (1) $\Rightarrow$ (2) Take any $x \in X$ and $G \in 2^Y$, then

$$\begin{aligned} & \alpha(f(x), Y - G) \\ \leq \;& \alpha(f(x), f(X - f^{-1}(G))) \\ \leq \;& \delta(x, X - f^{-1}(G)), \end{aligned}$$

because $f : (X, \delta) \to (Y, \alpha)$ is a contraction and $f(X - f^{-1}(G)) \subseteq Y - G$. Thus

$$\alpha(f(x), Y - G) \leq \delta(x, X - f^{-1}(G))$$

(2) $\Rightarrow$ (1) Take any $x \in X$ and $A \in 2^X$, then

$$\begin{aligned} & \alpha(f(x), f(A)) \\ \leq \;& \delta(x, X - f^{-1}(Y - f(A)) \\ \leq \;& \delta(x, A), \end{aligned}$$

because $f^{-1}(Y - f(A)) \subseteq X - A$. Thus $\alpha(f(x), f(A)) \leq \delta(x, A)$ and so f is a contraction. □

Theorem 4.2. *A function* $\delta : X \times 2^X \to [0, \infty]$ *is a distance if and only if it satisfies the following :*
(I1) $\delta(x, \emptyset) = \infty$
(I2) $0 < \delta(x, A)$ *implies* $x \in X - A$,
(I3) *for any* $x \in X$ *and* $A, B \in 2^X$, $\delta(x, A \cup B) = \delta(x, A) \wedge \delta(x, B)$,
(I4) *If* $k < \delta(x, A)$ *then there is a subset* $G \subseteq X$ *such that* $\delta(x, A) \leq \delta(x, X - G) + k$ *and for any* $g \in G$, $k < \delta(g, A)$.

Proof. To show the only-if part it is suffices to show that it satisfies axioms (I2) and (I4). To show that it satisfies (A1), take any $x \in X$ and suppose $\delta(x, x) > 0$ then by (I2), $x \in X - x$, which is a contraction, Hence $\delta(x, x) = 0$ and so δ satisfies (A1). To show that it satisfies (A6), suppose $k \in [0, \infty)$ and $k < \delta(x, A)$, then, by (I4), there is a subset G of X such that $\delta(x, A) \leq \delta(x, X - G) + k$ and for any $g \in G$, $k < \delta(g, A)$. Then $G \subseteq X - A^k$ and so by the above lemma

$$\delta(x, X - G) \leq \delta(x, A^{(k)}).$$

Since $\delta(x, A) \leq \delta(x, X - G) + k$, $\delta(x, A) \leq \delta(x, A^{(k)}) + k$. To show that if part, take any $x \in X$ and $A \in 2^X$ and suppose $0 < \delta(x, A)$ and $x \in A$, then by axioms (A1) and (A5), $\delta(x, A) = 0$, which is a contraction.

Hence $x \in X - A$ and so δ satisfies (I2). Suppose $k < \delta(x, A)$ then by (A6), $\delta(x, A) \leq \delta(x, A^{(k)}) + k$, then $\delta(x, A) \leq \delta(x, X - G) + k$. If $g \in G$, then $g \notin A^k$ and hence $k < \delta(g, A)$. □

We already mentioned that there are several equivalent descriptions of distance. The following is one of them.

Definition 4.3. ([8]) A family of functions

$$t_k : 2^X \to 2^X (k \in [0, \infty))$$

is called a tower on X if it satisfies the following axioms:

(T1) $\forall A \in 2^X, \forall k \in [0, \infty) : A \subseteq t_k(A)$,
(T2) $\forall k \in [0, \infty) : t_k(\emptyset) = \emptyset$,
(T3) $\forall A, B \in 2^X, \forall k \in [0, \infty) : t_k(A \cup B) = t_k(A) \cup t_k(B)$,
(T4) $\forall A \in 2^X, \forall k, l \in [0, \infty) : t_{k+l}(A) = t_k(t_l(A))$,
(T5) $\forall A \in 2^X, \forall k \in [0, \infty) : t_k(A) = \cap_{k<l} t_l(A)$.

Let X be a set and $\delta : X \times 2^X \to [0, \infty]$ a function. Then for any $A \in 2^X$ and $k \in [0, \infty)$,

$$X - A^{(k)} = (X - A)_{(k)}$$

where $A_{(k)} = \{x \in A : \delta(x, X - A) > k\}$. Using this fact and notation, we have the following definition:

Definition 4.4. A family of functions

$$i_k : 2^X \to 2^X (k \in [0, \infty))$$

is called a **co-tower** on X if it satisfies the following axioms:

(CT1) $\forall A \in 2^X, \forall k \in [0, \infty) : i_k(A) \subseteq A$
(CT2) $\forall k \in [0, \infty) : i_k(X) = X$
(CT3) $\forall A, B \in 2^X, \forall k \in [0, \infty) : i_k(A \cap B) = i_k(A) \cap i_k(B)$
(CT4) $\forall A \in 2^X, \forall k, l \in [0, \infty) : i_{k+l}(A) = i_k(i_l(A))$
(CT5) $\forall A \in 2^X, \forall k \in [0, \infty) : i_k(A) = \cup_{k<l} i_l(A)$

Notice that by (CT3) and (CT5) we have

$$\forall A \subseteq B \subseteq X, \forall l \leq k : i_k(A) \subseteq i_l(B).$$

Theorem 4.5. *$(t_k)_{k \in [0,\infty)}$ is a tower on a set X, then the family $(i_k)_{k \in [0,\infty)}$ defined by*

$$i_k : 2^X \to 2^X : A \to X - t_k(X - A)$$

is a co-tower on X.

Proof. It is immediate from Definitions 4.3 and 4.4. □

THEOREM 4.6. *$(i_k)_{k\in[0,\infty)}$ is a co-tower on a set X, then the family $(t_k)_{k\in[0,\infty)}$ defined by*

$$t_k : 2^X \to 2^X : A \to X - i_k(X - A)$$

is a tower on X.

Proof. It is immediate from Definitions 4.3 and 4.4. □

THEOREM 4.7. ([8]) *Let (X, δ) and (Y, α) be distance spaces and $f : X \to Y$ be a function, then the following are equivalent:*
(1) $f : (X, \delta) \to (Y, \alpha)$ is a contraction.
(2) $\forall A \in 2^X, \forall k \in [0, \infty) : f(t_k^\delta(A)) \subseteq t_k^\alpha(f(A))$.

THEOREM 4.8. *A function $f : (X, \delta) \to (Y, \alpha)$ is a contraction if and only if for any $A \in 2^X$ and $k \in [0, \infty) : f(X - i_k^\delta(A)) \subseteq Y - i_k^\alpha(f(A))$.*

Proof. Take any $A \in 2^X$ and, then it follows from the above theorems that $k \in [0, \infty)$.

$$\begin{aligned} & f(X - i_k^\delta(A)) \\ \subseteq \; & f(t_k^\delta(X - A)) \subseteq t_k^\alpha(f(X - A)) \subseteq t_k^\alpha(Y - f(A)) \\ = \; & Y - i_k^\alpha(f(A)). \end{aligned}$$

The if-part is similar to the only-if part. □

Collecting all the above, we have the following:

COROLLARY 4.9. *Towers and co-towers are equivalent systems and hence the concept of co-towers is equivalent with that of distances.*

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Department of Mathematics and Information
Kyungwon University
Seongnam 461-701, Republic of Korea
E-mail: sehwa@kyungwon.ac.kr