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ON DUALITY OF WEIGHTED BLOCH SPACES IN $\mathbb{C}^n$

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Abstract. In this paper, we consider the weighted Bloch spaces $\mathcal{B}_q(q > 0)$ on the open unit ball in $\mathbb{C}^n$. We prove a certain integral representation theorem that is used to determine the degree of growth of the functions in the space $\mathcal{B}_q$ for $q > 0$. This means that for each $q > 0$, the Banach dual of L^1_a is $\mathcal{B}_q$ and the Banach dual of $\mathcal{B}_{q,0}$ is L^1_a for each $q \geq 1$.

1. Introduction

Throughout this paper, let $\mathbb{C}^n$ be the Cartesian product of n copies of $\mathbb{C}$. For two elements $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ of $\mathbb{C}^n$, we define the inner product $< z, w >= \sum_{j=1}^n z_j \overline{w_j}$ and the norm $\| z \| = \sqrt{< z, z >}$. Let N denote the set of natural numbers. A multi-index α is an ordered n-tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\alpha_j \in N, j = 1, 2, \cdots, n$. For a multi-index α and $z \in \mathbb{C}^n$, set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}.$$

Let B be the open unit ball in $\mathbb{C}^n$ and S the boundary of B. For $z \in B, \xi \in \mathbb{C}^n$, set

$${b_B}^2(z,\xi) = \frac{n+1}{(1-\| z \|^2)^2}[\,(1-\| z \|^2)\| \xi \|^2 + |< z, \xi >|^2\,]\ .$$

If $\gamma : [0, 1] \to B$ is a C^1-curve, the Bergman length of γ is defined by $|\gamma|_B = \int_0^1 b_B(\gamma(t), \gamma'(t))dt$. For $z, w \in B$, we define $\beta(z, w) = \inf\{|\gamma|_B : \gamma(0) = z, \gamma(1) = w\}$ where the infimum is taken over all C^1-curves from z to w. β is called the Bergman metric on B.

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If $f \in H(B)$, where $H(B)$ is the set of holomorphic functions on B, then the quantity Qf is defined by

$$Qf(z) = \sup_{\|\xi\|=1} \frac{|\nabla f(z) \cdot \xi|}{b_B(z,\xi)}, \quad z \in B, \quad \xi \in \mathbb{C}^n$$

where $\nabla f(z) = (\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n})$ is the holomorphic gradient of f. The quantity Qf is invariant under the group $Aut(B)$ of holomorphic automorphisms of B. Namely, $Q(f \circ \varphi) = (Qf) \circ \varphi$ for all $\varphi \in Aut(B)$. A holomorphic function $f : B \to \mathbb{C}$ is called a Bloch function if $\sup_{z \in B} Qf(z) < \infty$.

Bloch functions on bounded homogeneous domains were first studied in [10]. In [16], Timoney showed that the linear space of all holomorphic functions $f : B \to \mathbb{C}$ which satisfy $\sup_{z \in B}(1 - \| z \|^2) \| \nabla f(z) \| < \infty$ is equivalent to the space $\mathcal{B}$ of Bloch functions on B. The little Bloch space $\mathcal{B}_0$ is the subspace of $\mathcal{B}$ consisting of those functions $f : B \to \mathbb{C}$ which satisfy $\lim_{\|z\| \to 1}(1 - \| z \|^2) \| \nabla f(z) \| = 0$.

For each $q > 0$, the weighted Bloch space of B, denoted by $\mathcal{B}_q$, consists of holomorphic functions $f : B \to \mathbb{C}$ which satisfy $\sup_{z \in B}(1 - \| z \|^2)^q \| \nabla f(z) \| < \infty$. The corresponding little Bloch space $\mathcal{B}_{q,0}$ is defined by the functions f in $\mathcal{B}_q$ such that $\lim_{\|z\| \to 1}(1 - \| z \|^2)^q \| \nabla f(z) \| = 0$. In particular, $\mathcal{B}_1 = \mathcal{B}$ and $\mathcal{B}_{1,0} = \mathcal{B}_0$.

Let us define a norm on $\mathcal{B}_q$ as follows;

$$\| f \|_q = |f(0)| + \sup\{(1 - \| w \|^2)^q \| \nabla f(w) \| : w \in B\}.$$

In [9], it was shown that the space $\mathcal{B}_q$ is a Banach space with respect to the above norm for each $q > 0$. It is also shown in [9] that the weighted little Bloch space $\mathcal{B}_{q,0}$ is the closure of the set of polynomials in the norm topology of $\mathcal{B}_q$ for each $q \geq 1$. The properties of the space $\mathcal{B}_q$ were investigated in [5], [6], [7] and [8]. In §2, we will show that the functions in the weighted Bloch space $\mathcal{B}_q$ can be extended cotinuously to the closed ball $\bar{B}$.

Let ν be the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(B) = 1$. For $q > -1$, the measure μ_q is the weighted Lebesgue measure such that $d\mu_q = c_q(1 - \| z \|^2)^q d\nu(z)$ where c_q is a normalization constant such that $\mu_q(B) = 1$. For any $q > 0$, let I_q denote the operator defined by

$$I_q f(z) = c_{q-1} \int_B \frac{f(w)}{(1 - < z, w >)^{n+q}} d\nu(w), \quad z \in B$$

where c_{q-1} is a normalization constant.

For each $q > 0$, we let J_q denote the operator defined by

$$J_q f(z) = \frac{c_2}{c_{q-1}} \int_B \frac{(1- \| z \|^2)^2 f(w)}{(1- < z, w >)^{n+3}} d\mu_{q-1}(w)$$

where c_2 is a normalization constant. In §3, we will prove that I_q maps $L^\infty(B)$ boundedly onto $\mathcal{B}_q$ and J_q maps $\mathcal{B}_q$ (in particular, $\mathcal{B}_{q,0}$) boundedly into $L^\infty(B)$.

The space $L^1_a = \{f \in H(B) \mid \| f \|_{L^1_a} = \int_B |f(z)| d\nu(z) < \infty\}$ is a Banach space with the above norm $\| \cdot \|_{L^1_a}$. Using these operators I_q and J_q, in §4, we will show that for each $q > 0$, the Banach dual of L^1_a can be identified with $\mathcal{B}_q$, while the Banach dual of $\mathcal{B}_{q,0}$ can be identified with L^1_a for $q \geq 1$.

2. Some integral representation in weighted Bloch spaces

Theorem 2.1. *If $f \in L^1_{\mu_q}(B) \cap H(B), q > -1$, then*

$$f(z) = \int_B \frac{f(w)}{(1- < z, w >)^{n+q+1}} d\mu_q(w).$$

Proof. See [9, Theorem 2]. □

Theorem 2.2. *For $z \in B$, $c \in \mathbb{R}$ and $t > -1$, we define*

$$I_{c,t}(z) = \int_B \frac{(1 - \| w \|^2)^t}{|1- < z, w > |^{n+1+c+t}} d\nu(w), \quad z \in B.$$

Then,

(i) $I_{c,t}(z)$ is bounded in B if $c < 0$;

(ii) $I_{0,t}(z) \sim -\log(1 - \| z \|^2)$ as $\| z \| \to 1^-$;

(iii) $I_{c,t}(z) \sim (1 - \| z \|^2)^{-c}$ as $\| z \| \to 1^-$ if $c > 0$.

Proof. See [14, Proposition 1.4.10]. □

Theorem 2.3. *Suppose $q > 1$. Then f is in $\mathcal{B}_q$ if and only if f is holomorphic and $(1 - \| z \|^2)^{q-1} |f(z)|$ is bounded on B.*

Proof. See [9, Theorem 6]. □

Let $0 < p < \infty$ and $s \in \mathbb{R}$. The holomorphic Besov p-spaces $\mathcal{B}^s_p(B)$ with weight s is defined by the space of all holomorphic functions f on the unit ball B such that

$$\| f \|_{p,s} = \left\{ \int_B (Qf)^p(z)(1- \| z \|^2)^s d\lambda(z) \right\}^{\frac{1}{p}} < \infty.$$

Here $d\lambda(z) = (1- \| z \|^2)^{-n-1} d\nu(z)$ is an invariant volume measure with respect to the Bergman metric on B(See [11]).

THEOREM 2.4. *Let $0 < p < \infty$ and $s \in \mathbb{R}$. For $q < 1 + \frac{s}{p}$,*

$$\| f \|_{p,s} \leq C \| f \|_q$$

for some constant C.

Proof. From the fact that $Qf(z)$ and $(1- \| z \|^2) \| \nabla f(z) \|$ behave the same within constants as $\| z \| \to 1$, we may replace $Qf(z)$ by $(1- \| z \|^2) \| \nabla f(z) \|$ in the definition of $\| f \|_{p,s}$. Namely,

$$\begin{aligned} \| f \|_{p,s}^p &= \int_B (Qf)^p(z)(1 - \| z \|^2)^s d\lambda(z) \\ &\leq C \int_B \left[(1- \| z \|^2) \| \nabla f(z) \|\right]^p (1 - \| z \|^2)^s d\lambda(z) \\ &\leq C \int_B \left[\frac{(1- \| z \|^2)^q \| \nabla f(z) \|}{(1- \| z \|^2)^{q-1}}\right]^p (1 - \| z \|^2)^s d\lambda(z) \\ &\leq C \| f \|_q^p \int_B (1- \| z \|^2)^{-pq+p+s-n-1} d\nu(z). \end{aligned}$$

The above calculation implies the result if $q < 1 + \frac{s}{p}$ by Theorem 2.2. □

THEOREM 2.5. *Let $q \in (0,1)$ and f in $\mathcal{B}_q$, then the functions in $\mathcal{B}_q$ can be extended continuously to the closed ball $\bar{B}$.*

Proof. If $q \in (0,1)$ and f in $\mathcal{B}_q$, by Theorem 1.4 in [12] and Theorem 2.5, there exists a constant $C > 0$ such that $|f(z) - f(w)| \leq C \| z - w \|^{1-q} \| f \|_q$ for all $z, w \in B$. This implies that the functions in $\mathcal{B}_q$ can be extended continuously to the closed ball $\bar{B}$. □

3. Some operators in weighted Bloch spaces.

Let $C_0(B)$ be the subspace of complex-valued continuous functions on B which vanish on the boundary, $C(\overline{B})$ the space of complex-valued continuous functions on the closed unit ball $\overline{B}$, and $BC(B)$ the space of bounded complex-valued continuous functions on B.

THEOREM 3.1. *For each $q > 0$, the operator I_q maps $C_0(B)$ boundedly onto $\mathcal{B}_q$.*

Proof. Let $f(z) = I_q g(z)$ where $g \in C_0(B)$. Then

$$f(z) = c_{q-1} \int_B \frac{g(w)}{(1- < z, w >)^{n+q}} d\nu(w).$$

Differentiating under the integral sign, we obtain

$$\frac{\partial f}{\partial z_j}(z) = (n+q) c_{q-1} \int_B \frac{g(w)(-\overline{w}_j)}{(1- < z, w >)^{n+q+1}} d\nu(w)$$

for $j = 1, 2, \cdots, n$. This shows that

$$\| \nabla f(z) \| \leq (n+q) c_{q-1} \| g \|_\infty \int_B \frac{d\nu(w)}{|1- < z, w > |^{n+q+1}} .$$

By Theorem 2.3,

$$\| \nabla f(z) \| \leq (n+q) c_{q-1} \| g \|_\infty (1- \| z \|^2)^{-q},$$

$$(1- \| z \|^2)^q \| \nabla f(z) \| \leq C \| g \|_\infty .$$

It is also clear that $|f(0)| \leq c_{q-1} \| g \|_\infty$. Thus,

$$\begin{aligned} \| f \|_q = & \ |f(0)| + \sup\{(1- \| z \|^2)^q \| \nabla f(z) \| : z \in B\} \\ \leq & \ (C + c_{q-1}) \| g \|_\infty . \end{aligned}$$

Hence, I_q maps $C_0(B)$ boundedly into $\mathcal{B}_q$.

If $f \in \mathcal{B}_q$, then $(1- \| z \|^2)^{q-1}|f(z)|$ is bounded in B by Theorem 2.4. By Theorem 2.1,

$$f(z) = c_{q-1} \int_B \frac{(1- \| w \|^2)^{q-1} f(w)}{(1- < z, w >)^{n+q}} d\nu(w) = I_q h(z)$$

where $h(w) = (1- \| w \|^2)^{q-1} f(w)$ is in $C_0(B)$. Therefore, I_q maps $C_0(B)$ onto $\mathcal{B}_q$. □

THEOREM 3.2. *For each $q > 0$, the opertor I_q maps each function of the form $w^\alpha \overline{w}^\beta$ to a monomial.*

Proof. Let $I = (i_1, i_2, \ldots, i_n)$ and $u = (z_1\overline{w}_1, z_2\overline{w}_2, \ldots, z_n\overline{w}_n)$. Since

$$< z, w >^m = (z_1\overline{w}_1 + z_2\overline{w}_2 + \cdots + z_n\overline{w}_n)^m = \sum_{|I|=m} \frac{m!}{I!} u^I,$$

$$\begin{aligned}
&\frac{1}{(1-<z,w>)^{n+q}}\\
&=1+(n+q)<z,w>+\frac{(n+q)(n+q+1)}{2!}<z,w>^2+\cdots\\
&=1+\sum_{m=1}^{\infty}\frac{(n+q+m-1)!}{m!(n+q-1)!}<z,w>^m\\
&=1+\sum_{m=1}^{\infty}\sum_{|I|=m}\frac{(n+q+m-1)!}{m!(n+q-1)!}\frac{m!}{I!}u^I.
\end{aligned}$$

$$\begin{aligned}
I_q(z^\alpha\overline{z}^\beta)&=c_{q-1}\int_B\frac{w^\alpha\overline{w}^\beta}{(1-<z,w>)^{n+q}}d\nu(w)\\
&=c_{q-1}\int_B w^\alpha\overline{w}^\beta d\nu(w)+c_{q-1}\sum_{m=1}^{\infty}\sum_{|I|=m}\\
&\qquad\qquad\frac{(n+q+m-1)!}{m!(n+q-1)!}\frac{m!}{I!}z^I\int_B w^\alpha\overline{w}^\beta\overline{w}^I d\nu(w)\\
&=C_Jz^J
\end{aligned}$$

for some $J=(i_1,\ldots,j_n)$ and constant C_J by $[14, Prop. 1.4.9]$. □

THEOREM 3.3. *For each $q>0$, the operator I_q maps $C(\overline{B})$ boundedly onto $\mathcal{B}_{q,0}$. I_q also maps $C_0(B)$ onto $\mathcal{B}_{q,0}$.*

Proof. By the Stone-Weierstrass approximation theorem, each function in $C(\overline{B})$ can be uniformly approximated by finite linear combinations of functions of the form $z^\alpha\overline{z}^\beta$, which are mapped by I_q to polynomials (finite linear combination of monomials) by Theorem 3.2. Since I_q maps $L^\infty(B)$ boundedly into $\mathcal{B}_q$ and $\mathcal{B}_{q,0}$ is closed in $\mathcal{B}_q$, I_q maps $C(\overline{B})$ boundedly into $\mathcal{B}_{q,0}$. The proof of "onto "part follows from the proof of Theorem 3.1. □

Let E be a normed linear space and M a closed linear subspace of E. If we define linear operations on $E/M=\{x+M : x\in E\}$ by $(x+M)+(y+M)=(x+y)+M$ and $a(x+M)=ax+M$, then $\| x+M\|=\inf\{\| x+m\| : m\in M\}$ is a quotient norm on E/M. If E is a Banach space, so is E/M under this quotient norm.

THEOREM 3.4. *For each $q>1$, there exists a positive real number C such that*

$$C^{-1}\| f\|_q\le \inf\{\| g\|_\infty : f=I_qg, g\in L^\infty(B)\}\le C\ \| f\|_q$$

for all f in $\mathcal{B}_q$ and

$$C^{-1} \parallel f \parallel_q \leq \inf\{\parallel g \parallel_\infty : f = I_q g, g \in C_0(B)\} \leq C \parallel f \parallel_q$$

for all f in $\mathcal{B}_{q,0}$.

Proof. Let us define an equivalence relation on L^∞ such that $g_1 \sim g_2 \Leftrightarrow I_q g_1 = I_q g_2$. Then $L^\infty / \sim$ is the family of equivalence class $[g]$ of g. Let us define a linear operator T from $L^\infty / \sim$ to $\mathcal{B}_q$ such that $T([g]) = R_q g$. Then T is a bounded linear operator on $L^\infty / \sim$ onto $\mathcal{B}_q$. Also T is 1-1. By the open mapping theorem, T^{-1} is continuous. Hence there exists constant C such that $\parallel T^{-1} I_q g \parallel_\infty \leq C \parallel I_q g \parallel_q$. i.e. $\parallel g \parallel_\infty \leq C \parallel f \parallel_q$. □

THEOREM 3.5. *For each $q > 1$, the operator J_q maps $\mathcal{B}_q$ boundedly into $L^\infty(B)$, and for $q \geq 1$, the operator J_q maps $\mathcal{B}_{q,0}$ boundedly into $C_0(B)$. For all $f \in \mathcal{B}_q$, there is a positive real number C (depending only on q) such that*

$$C^{-1} \parallel f \parallel_q \leq \parallel J_q f \parallel_\infty \leq C \parallel f \parallel_q .$$

Proof. If $f \in \mathcal{B}_q$, then there exists $g \in L^\infty(B)$ such that $f = I_q g$. Then, by Fubini's theorem and Theorem 2.1

$$\begin{aligned}
J_q f(z) &= \frac{c_2}{c_{q-1}} \int_B \frac{(1- \parallel z \parallel^2)^2 f(w)}{(1- < z, w >)^{n+3}} d\mu_{q-1}(w) \\
&= c_2 \int_B g(u) \left\{ \int_B \frac{(1- \parallel z \parallel^2)^2 d\mu_{q-1}(w)}{(1- < w, u >)^{n+q}(1- < z, w >)^{n+3}} \right\} d\nu(u) \\
&= c_2 \int_B \frac{(1- \parallel z \parallel^2)^2 g(u)}{(1- < z, u >)^{n+3}} d\nu(u).
\end{aligned}$$

By Theorem 2.2,

$$|J_q f(z)| \leq c_2 \parallel g \parallel_\infty (1- \parallel z \parallel^2)^2 (1- \parallel z \parallel^2)^{-2} \leq c_2 \parallel g \parallel_\infty$$

for all $z \in B$. Therefore, $\parallel J_q f \parallel_\infty \leq c_2 \parallel g \parallel_\infty$ for all $g \in L^\infty(B)$ with $I_q g = f$. By Theorem 3.4, there exists a constant $C > 0$ such that $\parallel J_q f \parallel_\infty \leq c_2 C \parallel f \parallel_q$ for all $f \in \mathcal{B}_q$. Therefore, J_q maps $\mathcal{B}_q$ boundedly into $L^\infty(B)$. For all f in $\mathcal{B}_q$,

$$\begin{aligned}
&I_qJ_qf(z)\\
&= c_{q-1}\int_B \frac{J_qf(w)}{(1-<z,w>)^{n+q}}d\nu(w)\\
&= c_2\int_B \frac{(1-\parallel w\parallel^2)^2}{(1-<z,w>)^{n+q}}\left\{\int_B \frac{f(u)d\mu_{q-1}(u)}{(1-<w,u>)^{n+3}}\right\}d\nu(w)\\
&= \int_B f(u)c_2\left\{\int_B \frac{(1-\parallel w\parallel^2)^2 d\nu(w)}{(1-<w,u>)^{n+3}(1-<z,w>)^{n+q}}\right\}d\mu_{q-1}(u)\\
&= \int_B \frac{f(u)}{(1-<z,u>)^{n+q}}d\mu_{q-1}(u)\\
&= f(z).
\end{aligned}$$

Thus $J_qf \in L^\infty(B)$ implies that $f \in \mathcal{B}_q$ by Theorem 3.1. By Theorem 3.4, $\parallel f\parallel_q = \parallel I_qJ_qf\parallel_q \leq C' \parallel J_qf\parallel_\infty \leq c_2CC' \parallel f\parallel_q$ for all $f \in \mathcal{B}_q$. Also, $J_qf \in C_0(B)$ implies $f \in \mathcal{B}_{q,0}$ by Theorem 3.3.

The operator J_q maps each polynomial to a polynomial times the factor $(1-\parallel z\parallel^2)^2$ which is a function in $C_0(B)$ by Theorem 3.2. Since $\mathcal{B}_{q,0}$ is the closure of the set of polynomials in $\mathcal{B}_q$ for $q \geq 1$ (See [9, Theorem 7]) and $C_0(B)$ is closed in $L^\infty(B)$, J_q maps $\mathcal{B}_{q,0}$ into $C_0(B)$. □

4. Duality in weighted Bloch spaces.

THEOREM 4.1. *Suppose that $f \in L^1_a$ is bounded and $g \in \mathcal{B}_q$. Then*

$$\left|\int_B f(z)\overline{g(z)}(1-\parallel z\parallel^2)^{q-1}d\nu(z)\right| \leq C\parallel f\parallel_{L^1_a}\parallel g\parallel_q$$

for some constant $C > 0$ which is independent of f and g.

Proof. Writing $g = I_q\varphi$ for some $\varphi \in L^\infty(B)$ and applying Fubini's theorem, we have

$$\begin{aligned}
&\int_B f(z)\overline{g(z)}(1-\parallel z\parallel^2)^{q-1}d\nu(z)\\
&= \int_B f(z)\left\{\int_B \frac{\overline{\varphi(w)}}{(1-<w,z>)^{n+q}}d\nu(w)\right\}d\mu_{q-1}(z)\\
&= \int_B \overline{\varphi(w)}\left\{\int_B \frac{f(z)}{(1-<w,z>)^{n+q}}d\mu_{q-1}(z)\right\}d\nu(w)\\
&= \int_B f(w)\overline{\varphi(w)}d\nu(w).
\end{aligned}$$

Hence, $|\int_B f(z)\overline{g(z)}(1-\| z \|^2)^{q-1}d\nu(z)| \leq \| f \|_{L^1_a}\| \varphi \|_\infty$. Taking the infimum over φ and applying Theorem 3.4,

$$|\int_B f(z)\overline{g(z)}(1-\| z \|^2)^{q-1}d\nu(z)| \leq C \| f \|_{L^1_a}\| g \|_q$$

for some constant $C > 0$. □

THEOREM 4.2. *Suppose that $f \in L^1_a$ and $g \in \mathcal{B}_q$. Then*

$$\lim_{r\to 1^-} \int_B f(rz)\overline{g(z)}(1-\| z \|^2)^{q-1}d\nu(z)$$

exists and the absolute value of the above limit is less than or equal to $C \| f \|_{L^1_a}\| g \|_q$ where C is a constant independent of f and g.

Proof. Given g in $\mathcal{B}_q$, Theorem 4.1 shows that

$$F_g(f) = \int_B f(z)\overline{g(z)}(1-\| z \|^2)^{q-1}d\nu(z), \quad f \in H^\infty(B)$$

extends to a bounded linear functional on L^1_a with $\| F_g \| \leq C \| g \|_q$. Fix f in L^1_a and g in $\mathcal{B}_q$. Let $f(rz) = f_r(z), 0 < r < 1$. Each f_r is in $H^\infty(B)$ and $\| f_r - f \|_{L^1_a} \to 0$ as $r \to 1^-$. It follows that

$$\lim_{r\to 1^-} \int_B f(rz)\overline{g(z)}(1-\| z \|^2)^{q-1}d\nu(z) = \lim_{r\to 1^-} F_g(f_r) = F_g(f)$$

and that $|F_g(f)| \leq \| F_g \| \| f \|_{L^1_a} \leq C \| g \|_q \| f \|_{L^1_a}$. □

THEOREM 4.3. *If F is a bounded linear functional on L^1_a, then there exists a function g in $\mathcal{B}_q$ such that*

$$F(f) = \lim_{r\to 1^-} \int_B f(rz)\overline{g(z)}(1-\| z \|^2)^{q-1}d\nu(z), \quad f \in L^1_a.$$

Proof. By the Hahn-Banach extension theorem, F extends to a bounded linear functional on $L^1(B, d\nu)$ without increasing the norm. Since $(L^1)^* = L^\infty$, there is a function $\varphi \in L^\infty(D)$ such that $F(f) = \int_B f(z)\overline{\varphi(z)}d\nu(z), f \in L^1_a$. If we use the inequality in Theorem 4.1,

$$F(f) = \lim_{r\to 1^-} \int_B f_r(z)\overline{\varphi(z)}d\nu(z) = \lim_{r\to 1^-} \int_B f_r(z)\overline{I_q\varphi(z)}(1-\| z \|^2)^{q-1}d\nu(z)$$

for each f in L^1_a. Let $g = I_q\varphi$. Then g is in $\mathcal{B}_q$ and

$$F(f) = \lim_{r\to 1^-} \int_B f(rz)\overline{g(z)}(1-\| z \|^2)^{q-1}d\nu(z)$$

for all f in L^1_a. □

THEOREM 4.4. *For each $q > 0$, the Banach dual of L_a^1 can be identified with $\mathcal{B}_q$ (with equivalent norms) under the pairing*

$$< f, g >= \lim_{r\to 1^-} \int_B f(rz)\overline{g(rz)}(1- \parallel z \parallel^2)^{q-1} d\nu(z),\ \ f \in L_a^1, g \in \mathcal{B}_q.$$

Proof. If F is a bounded linear functional on L_a^1, by Theorem 4.3, there exists a function g in $\mathcal{B}_q$ such that

$$F(f) = \lim_{r\to 1^-} \int_B f(rz)\overline{g(z)}(1- \parallel z \parallel^2)^{q-1} d\nu(z),\ \ f \in L_a^1.$$

By the rotation invariance of the measure $(1- \parallel z \parallel^2)^{q-1} d\nu(z)$, we have

$$\int_B f(rz)\overline{g(z)}(1- \parallel z \parallel^2)^{q-1} d\nu(z) = \int_B f(sz)\overline{g(sz)}(1- \parallel z \parallel^2)^{q-1} d\nu(z)$$

where $s = \sqrt{r}$. This clearly implies that

$$\begin{aligned} &\lim_{r\to 1^-} \int_B f(rz)\overline{g(z)}(1- \parallel z \parallel^2)^{q-1} d\nu(z) \\ &= \lim_{r\to 1^-} \int_B f(rz)\overline{g(rz)}(1- \parallel z \parallel^2)^{q-1} d\nu(z). \end{aligned}$$

□

THEOREM 4.5. *For each $q \geq 1$, the Banach dual of $\mathcal{B}_{q,0}$ can be identified with L_a^1 (with equivalent norms) under the pairing*

$$< f, g >= \lim_{r\to 1^-} \int_B f(rz)\overline{g(rz)}(1- \parallel z \parallel^2)^{q-1} d\nu(z), \quad g \in L_a^1, f \in \mathcal{B}_{q,0}.$$

Proof. Let F be a bounded linear functional on $\mathcal{B}_{q,0}$. By Theorem 3.5, the operator J_q maps $\mathcal{B}_{q,0}$ boundedly into $C_0(B)$. Let X be the image of $\mathcal{B}_{q,0}$ under the mapping J_q. Then X is a closed subspace of $C_0(B)$ and $F \circ J_q^{-1} : X \to C$ is a bounded linear functional. Let $\varphi \in X$. For some a finite Borel measure μ on B, $F \circ J_q^{-1}(\varphi) = \int_B \varphi(z) d\overline{\mu}(z)$ by the Riesz representation theorem. For each f in $\mathcal{B}_{q,0}$,

$$\begin{aligned} F(f) &= \int_B J_q f(z) d\overline{\mu}(z) \\ &= \frac{c_2}{c_{q-1}} \int_B \left\{ \int_B \frac{(1- \parallel z \parallel^2)^2 f(w)}{(1- < z, w >)^{n+3}} d\mu_{q-1}(w) \right\} d\overline{\mu}(z) \\ &= c_2 \lim_{r\to 1^-} \int_B (1- \parallel z \parallel^2)^2 \left\{ \int_B \frac{(1- \parallel w \parallel^2)^{q-1} f(w)}{(1- < rz, w >)^{n+3}} d\nu(w) \right\} d\overline{\mu}(z). \end{aligned}$$

Let g be the holomorphic function defined by

$$g(z) = c_2 \int_B \frac{(1- \parallel w \parallel^2)^2}{(1- < z, w >)^{n+3}} d\mu(w).$$

By Theorem 2.2 and Fubini's Theorem,

$$\int_B |g(z)| d\nu(z) \leq c_2 \int_B (1- \parallel w \parallel^2)^2 \left\{ \int_B \frac{d\nu(z)}{|1- < z, w > |^{n+3}} \right\} d|\mu|(w)$$
$$\leq c_2 \int_B d|\mu|(w) = c_2 \parallel \mu \parallel .$$

This implies that g belongs to L^1_a and

$$F(f) = \lim_{r \to 1^-} \int_B f(rz) \overline{g(rz)} (1- \parallel z \parallel^2)^{q-1} d\nu(z).$$

□

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