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THE MEAN VALUE AND VARIANCE OF ONE–SIDED FUZZY SETS

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Abstract. In this paper, we define the one-sided fuzzy set and we calculate the mean value and variance, defined by C. Carlsson and R. Fullér, of this fuzzy set. And we obtain a result that, in some special case, the mean of the product of two fuzzy sets is the product of means of each fuzzy sets. This result can be considered as the similar result which is well-known in the independence of events in probability theory.

1. Introduction

In 2001, C. Carlsson and R Fullér [1] introduced the concepts of possibilistic mean value and variance of fuzzy numbers. And using these concepts, they defined the interval-valued possibilistic mean, crisp possibilistic mean value and crisp (possibilistic) variance of a continuous possibilistic distribution and they proved some properties of these concepts. In this paper, we define the one-sided fuzzy set and calculate the mean value and variance of various type of oen-sided fuzzy sets. And we obtain a result that, in some special case, the mean of the product of two fuzzy sets is the product of means of each fuzzy sets. This result can be considered as the similar result which is well-known in the independence of events in probability theory.

2. One-sided fuzzy set

Definition 2.1 A *triangular fuzzy set* is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a-\alpha, a+\beta \le x, \\ (x-a+\alpha)/\alpha & \text{if} \quad a-\alpha \le x < a, \\ (a+\beta-x)/\beta & \text{if} \quad a \le x < a+\beta. \end{cases}$$

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The above triangular fuzzy set is denoted by $A = (\alpha, a, \beta)$.

DEFINITION 2.2 A *quadratic fuzzy set* is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < \alpha, \beta \leq x, \\ -a(x-\alpha)(x-\beta) = -a(x-k)^2 + 1 & \text{if} \quad \alpha \leq x < \beta, \end{cases}$$

where $a > 0$. The above quadratic fuzzy set is denoted by $A = [\alpha, k, \beta]$.

In the above definitions, each fuzzy set has a continuous membership function. To develop our calculations, we define new fuzzy sets having discontinuous membership functions.

DEFINITION 2.3 A *left fuzzy set* is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a-\alpha, a < x, \\ f(x) & \text{if} \quad a-\alpha \leq x \leq a, \end{cases}$$

where $f(x)$ is a continuous and increasing function with $f(a-\alpha) = 0, f(a) = 1$. Similarly, a *right fuzzy set* is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a, \alpha+\beta < x, \\ g(x) & \text{if} \quad a \leq x \leq a+\beta, \end{cases}$$

where $g(x)$ is a continuous and decreasing function with $g(a) = 1, g(a+\beta) = 0$. We call these sets one-sided fuzzy sets.

By using this definition, we can define the left triangular fuzzy set as follows.

DEFINITION 2.4 A left triangular fuzzy set is a fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a-\alpha, a < x, \\ (x-a+\alpha)/\alpha & \text{if} \quad a-\alpha \leq x \leq a. \end{cases}$$

Similarly, a right triangular fuzzy set can be defined.

By the same ways, left(right) quadratic fuzzy set can be defined. And, to denote these one-sided fuzzy sets, we will use the notations in Definition 2.1 and 2.2. For example, the left triangular fuzzy set in Definition 2.4 is denoted by $(\alpha, a, 0)$.

Definition 2.5 A γ-level set of a fuzzy set A is defined by $[A]^\gamma = \{t \in \mathbb{R} | \mu_A(t) \geq \gamma\}$ if $\gamma > 0$ and $[A]^\gamma = cl\{t \in \mathbb{R} | \mu_A(t) > \gamma\}$ if $\gamma = 0$.

The addition, multiplication and scalar multiplication of fuzzy sets are defined by the extension principle [4].

Theorem 2.6. *(1) Let A_1 and A_2 be left fuzzy sets. Then $A_1 + A_2$ is a left fuzzy set.*
(2) Let A_1 and A_2 be right fuzzy sets. Then $A_1 + A_2$ is a right fuzzy set.

Proof. (1) Let A_1 and A_2 be left fuzzy sets with membership functions

$$\mu_{A_1}(x) = \begin{cases} 0 & \text{if} \quad x < a_1 - \alpha_1, a_1 < x, \\ f_1(x) & \text{if} \quad a_1 - \alpha_1 \leq x \leq a_1, \end{cases}$$

and

$$\mu_{A_2}(x) = \begin{cases} 0 & \text{if} \quad x < a_2 - \alpha_2, a_2 < x, \\ f_2(x) & \text{if} \quad a_2 - \alpha_2 \leq x \leq a_2, \end{cases}$$

respectively, where $f_1(x)$ and $f_2(x)$ are continuous and increasing functions with $f_1(a_1 - \alpha_1) = 0$, $f_1(a_1) = 1$, $f_2(a_2 - \alpha_2) = 0$ and $f_2(a_2) = 1$. Since $[A_1]^\gamma = [f_1^{-1}(\gamma), a_1]$ and $[A_2]^\gamma = [f_2^{-1}(\gamma), a_2]$, we have $[A_1 + A_2]^\gamma = [f_1^{-1}(\gamma) + f_2^{-1}(\gamma), a_1 + a_2]$. Hence $A_1 + A_2$ is a left fuzzy set.

(2) By the similar manner, (2) can be obtained. □

Example 2.7 Let $A_1 = (2, 4, 0)$ and $A_2 = (3, 7, 0)$ be left triangular fuzzy sets and $B_1 = (0, 5, 3)$ and $B_2 = (0, 7, 2)$ be right triangular fuzzy sets. Let $[A_1]^\gamma$, $[A_2]^\gamma$, $[B_1]^\gamma$ and $[B_2]^\gamma$ be γ-level sets of A_1, A_2, B_1 and B_2, respectively.
(1) Since $[A_1]^\gamma = [2\gamma + 2, 4]$ and $[A_2]^\gamma = [3\gamma + 4, 7]$, we have $[A_1 + A_2]^\gamma = [5\gamma + 6, 11]$. Hence $A_1 + A_2 = (5, 11, 0)$ is a left triangular fuzzy set.
(2) Since $[B_1]^\gamma = [5, 8 - 3\gamma]$ and $[B_2]^\gamma = [7, 9 - 2\gamma]$, we have $[B_1 + B_2]^\gamma = [12, 17 - 5\gamma]$. Hence $B_1 + B_2 = (0, 12, 5)$ is a right triangular fuzzy set.

Example 2.8 Let $A_1 = [1, 2, 0]$ and $A_2 = [3, 4, 0]$ be left quadratic fuzzy sets and $B_1 = [0, 5, 8]$ and $B_2 = [0, 2, 3]$ be right quadratic fuzzy sets. Let $[A_1]^\gamma$, $[A_2]^\gamma$,$[B_1]^\gamma$ and $[B_2]^\gamma$ be γ-level sets of A_1, A_2, B_1 and B_2, respectively.

(1) Since $[A_1]^\gamma = [2-\sqrt{1-\gamma}, 2]$ and $[A_2]^\gamma = [4-\sqrt{1-\gamma}, 4]$, we have $[A_1+A_2]^\gamma = [6-2\sqrt{1-\gamma}, 6]$. Hence $A_1+A_2 = [4, 6, 0]$ is a left quadratic fuzzy set.
(2) Since $[B_1]^\gamma = [5, 5+3\sqrt{1-\gamma}]$ and $[B_2]^\gamma = [2, 2+\sqrt{1-\gamma}]$, we have $[B_1+B_2]^\gamma = [7, 7+4\sqrt{1-\gamma}]$. Hence $B_1+B_2 = [0, 7, 11]$ is a right quadratic fuzzy set.

Remark. Let A and B be left fuzzy set and right fuzzy set, respectively. Then $A+B$ may be not an one-sided fuzzy set.

Example 2.9 Let $A = (2, 4, 0)$ be left triangular fuzzy set and $B = (0, 5, 3)$ be right triangular fuzzy set. Let $[A]^\gamma$ and $[B]^\gamma$ be γ-level of sets A and B, respectively. Since $[A]^\gamma = [2\gamma+2, 4]$ and $[B]^\gamma = [5, 8-3\gamma]$, we have $[A+B]^\gamma = [2\gamma+7, 12-3\gamma]$. Thus $A+B$ is a triangular fuzzy set, but it is not an one-sided fuzzy set.

Example 2.10 Let $A = [1, 2, 0]$ be left quadratic fuzzy set and $B = [0, 5, 8]$ be right quadratic fuzzy set. Let $[A]^\gamma$ and $[B]^\gamma$ be γ-level of sets A and B, respectively. Since $[A]^\gamma = [2-\sqrt{1-\gamma}, 2]$ and $[B]^\gamma = [5, 5+3\sqrt{1-\gamma}]$, we have $[A+B]^\gamma = [7-\sqrt{1-\gamma}, 7+3\sqrt{1-\gamma}]$. Thus $A+B$ is neither an one-sided fuzzy set nor a quadratic fuzzy set.

3. The mean value and variance of one-sided fuzzy sets

In this section, we introduce the notion of possibilistic mean value and variance of fuzzy sets defined by C. Carlsson and B. Fullér. And, we calculate the possibilistic mean value and variance of one-sided fuzzy sets.

Definition 3.1 ([1]) Let A be a fuzzy set. The lower possibilistic mean value of A is defined by

$$M_*(A) = 2\int_0^1 \gamma a_1(\gamma) d\gamma.$$

Similarly, the upper possibilistic mean value of A is defined by

$$M^*(A) = 2\int_0^1 \gamma a_2(\gamma) d\gamma.$$

Note that $M_*(A)$ is the lower possibility-weighted average of the minima of the γ-sets, and $M^*(A)$ is the upper possibility-weighted average of the maxima of the γ-sets.

Definition 3.2 ([1]) Let A be a fuzzy set. The interval-valued possibilistic mean of A is defined by $M(A) = [M_*(A), M^*(A)]$.

Theorem 3.3. ([1]) *Let A and B be two non-interactive fuzzy sets and let $\lambda \in \mathbb{R}$ be a real number. Then $M(A+B) = M(A) + M(B), M(\lambda A) = \lambda M(A)$.*

Definition 3.4 ([1]) The crisp possibilistic mean value of a fuzzy set A is the arithmetic mean of its lower possibilistic and upper possibilistic mean values, i.e.,

$$\overline{M}(A) = \frac{M_*(A) + M^*(A)}{2}.$$

Theorem 3.5. ([1]) *Let A and B be fuzzy sets and $\lambda \in \mathbb{R}$. Then*

$$\overline{M}(A+B) = \overline{M}(A) + \overline{M}(B), \overline{M}(\lambda A) = \lambda \overline{M}(A).$$

Example 3.6 Let $A = (\alpha, a, 0)$ be a left triangular fuzzy set. Then the γ-level set of A is $[A]^\gamma = [a - (1-\gamma)\alpha, a], \gamma \in [0,1]$. Thus

$$M_*(A) = 2\int_0^1 \gamma(a - (1-\gamma)\alpha)d\gamma = a - \frac{\alpha}{3} \text{ and } M^*(A) = 2\int_0^1 \gamma \cdot a d\gamma = a.$$

Hence we have

$$M(A) = \left[a - \frac{\alpha}{3}, a\right] \text{ and } \overline{M}(A) = \int_0^1 \gamma(a - (1-\gamma)\alpha + a)d\gamma = a - \frac{\alpha}{6}.$$

The following theorems show the relations between $M_*(AB)$ and $M_*(A)$ $M_*(B)$ and between $M^*(AB)$ and $M^*(A)M^*(B)$ for triangular fuzzy sets and quadratic fuzzy sets.

Theorem 3.7. *Let $A = (\alpha_1, a, \beta_1)$ and $B = (\alpha_2, b, \beta_2)$ be triangular fuzzy sets. Then $M_*(A)M_*(B) \le M_*(AB)$ and $M^*(A)M^*(B) \le M^*(AB)$.*

Proof. Since $[A]^\gamma = [a - (1-\gamma)\alpha_1, a + (1-\gamma)\beta_1]$ and $[B]^\gamma = [b - (1-\gamma)\alpha_2, b + (1-\gamma)\beta_2]$, we have

$$M_*(A) = 2\int_0^1 \gamma(a - (1-\gamma)\alpha_1)d\gamma = a - \frac{\alpha_1}{3}$$

and

$$M_*(B) = 2\int_0^1 \gamma(b - (1-\gamma)\alpha_2)d\gamma = b - \frac{\alpha_2}{3}.$$

Since

$$\begin{aligned} M_*(AB) &= 2\int_0^1 \gamma(ab - a(1-\gamma)\alpha_2 - b(1-\gamma)\alpha_1 + (1-\gamma)^2\alpha_1\alpha_2)d\gamma \\ &= ab - \frac{a\alpha_2}{3} - \frac{b\alpha_1}{3} + \frac{\alpha_1\alpha_2}{6}, \end{aligned}$$

we have

$$\begin{aligned} M_*(A)M_*(B) &= ab - \frac{a\alpha_2}{3} - \frac{b\alpha_1}{3} + \frac{\alpha_1\alpha_2}{9} \\ &\le ab - \frac{a\alpha_2}{3} - \frac{b\alpha_1}{3} + \frac{\alpha_1\alpha_2}{6} \\ &= M_*(AB). \end{aligned}$$

Similarly, we have $M^*(A)M^*(B) \le M^*(AB)$. $\square$

THEOREM 3.8. *Let* $A = [\alpha_1, k_1, \beta_1]$ *and* $B = [\alpha_2, k_2, \beta_2]$ *be quadratic fuzzy sets. Then* $M_*(A)M_*(B) \le M_*(AB)$ *and* $M^*(A)M^*(B) \le M^*(AB)$.

Proof. Since

$$[A]^\gamma = \left[k_1 - \sqrt{\frac{1-\gamma}{a}}, k_1 + \sqrt{\frac{1-\gamma}{a}}\right], \quad a = \frac{4}{(\alpha_1 - \beta_1)^2}$$

and

$$[B]^\gamma = \left[k_2 - \sqrt{\frac{1-\gamma}{b}}, k_2 + \sqrt{\frac{1-\gamma}{b}}\right], \quad b = \frac{4}{(\alpha_2 - \beta_2)^2},$$

we have

$$M_*(A) = 2\int_0^1 \gamma\left(k_1 - \sqrt{\frac{1-\gamma}{a}}\right)d\gamma = k_1 - \frac{8}{15\sqrt{a}}$$

and

$$M_*(B) = 2\int_0^1 \gamma\left(k_2 - \sqrt{\frac{1-\gamma}{b}}\right)d\gamma = k_2 - \frac{8}{15\sqrt{b}}.$$

Since

$$\begin{aligned} M_*(AB) &= 2\int_0^1 \gamma\left(k_1k_2 - k_1\sqrt{\frac{1-\gamma}{b}} - k_2\sqrt{\frac{1-\gamma}{a}} + \frac{1-\gamma}{\sqrt{ab}}\right)d\gamma \\ &= k_1k_2 - \frac{8k_1}{15\sqrt{b}} - \frac{8k_2}{15\sqrt{a}} + \frac{1}{3\sqrt{ab}}, \end{aligned}$$

we have

$$\begin{aligned} M_*(A)M_*(B) &= k_1k_2 - \frac{8k_1}{15\sqrt{b}} - \frac{8k_2}{15\sqrt{a}} + \frac{64}{225\sqrt{ab}} \\ &\le k_1k_2 - \frac{8k_1}{15\sqrt{b}} - \frac{8k_2}{15\sqrt{a}} + \frac{1}{3\sqrt{ab}} \\ &= M_*(AB). \end{aligned}$$

Similarly, we have $M^*(A)M^*(B) \le M^*(AB)$. □

THEOREM 3.9. *Let A be a left fuzzy set with $[A]^\gamma = [a(\gamma), a]$ and B be a right fuzzy set with $[B]^\gamma = [b, b(\gamma)]$, then $M(AB) = M(A)M(B)$.*

Proof. Since $[A]^\gamma = [a(\gamma), a]$ and $[B]^\gamma = [b, b(\gamma)]$,

$$M_*(A) = 2\int_0^1 \gamma a(\gamma)d\gamma \quad \text{and} \quad M^*(A) = 2\int_0^1 \gamma \cdot a d\gamma = a.$$

Similarly, we have $M_*(B) = b$ and $M^*(B) = 2\int_0^1 \gamma b(\gamma)d\gamma$. Thus

$$M_*(A)M_*(B) = \left(2\int_0^1 \gamma a(\gamma)d\gamma\right) \cdot b = 2b\int_0^1 \gamma a(\gamma)d\gamma$$

and

$$M^*(A)M^*(B) = a \cdot \left(2\int_0^1 \gamma b(\gamma)d\gamma\right) = 2a\int_0^1 \gamma b(\gamma)d\gamma.$$

Since $[AB]^\gamma = [ba(\gamma), ab(\gamma)]$, we have

$$M_*(AB) = 2b\int_0^1 \gamma a(\gamma)d\gamma \quad \text{and} \quad M^*(AB) = 2a\int_0^1 \gamma b(\gamma)d\gamma.$$

Hence $M_*(A)M_*(B) = M_*(AB)$ and $M^*(A)M^*(B) = M^*(AB)$. By the definition of M, $M(A)M(B) = M(AB)$. □

Theorem 3.9 can be considered as the similar result which is well-known in the independence of events in probability theory.

EXAMPLE 3.10 Let A be a left triangular fuzzy set with $[A]^\gamma = [2\gamma+2, 4]$ and B be a right triangular fuzzy set with $[B]^\gamma = [7, 9-2\gamma]$. Then

$$M_*(A) = 2\int_0^1 \gamma(2\gamma+2)d\gamma = \frac{10}{3} \quad \text{and} \quad M^*(A) = 2\int_0^1 4\gamma d\gamma = 4.$$

Similarly, we have $M_*(B) = 7$ and $M^*(B) = \frac{23}{3}$. Thus

$$M_*(A)M_*(B) = \frac{70}{3} \quad \text{and} \quad M^*(A)M^*(B) = \frac{92}{3}.$$

Since $[AB]^\gamma = [14\gamma + 14, 36 - 8\gamma]$, we have

$$M_*(AB) = 2\int_0^1 \gamma(14\gamma + 14)d\gamma = \frac{70}{3}$$

and

$$M^*(AB) = 2\int_0^1 \gamma(36 - 8\gamma)d\gamma = \frac{92}{3}.$$

Hence $M_*(A)M_*(B) = M_*(AB)$ and $M^*(A)M^*(B) = M^*(AB)$. By the definition of M, we have $M(A)M(B) = M(AB)$.

EXAMPLE 3.11 Let A be a left quadratic fuzzy set with $[A]^\gamma = [2 - \sqrt{1-\gamma}, 2]$ and B be a right quadratic fuzzy set with $[B]^\gamma = [5, 5+3\sqrt{1-\gamma}]$. Then

$$M_*(A) = 2\int_0^1 \gamma(2 - \sqrt{1-\gamma})d\gamma = \frac{22}{15} \text{ and } M^*(A) = 2\int_0^1 2\gamma d\gamma = 2.$$

Similarly, we have $M_*(B) = 5$ and $M^*(B) = \frac{33}{5}$. Thus

$$M_*(A)M_*(B) = \frac{22}{3} \text{ and } M^*(A)M^*(B) = \frac{66}{5}.$$

Since $[AB]^\gamma = [10 - 5\sqrt{1-\gamma}, 10 + 6\sqrt{1-\gamma}]$, we have

$$M_*(AB) = 2\int_0^1 \gamma(10 - 5\sqrt{1-\gamma})d\gamma = \frac{22}{3}$$

and

$$M^*(AB) = 2\int_0^1 \gamma(10 + 6\sqrt{1-\gamma})d\gamma = \frac{66}{5}.$$

Hence $M_*(A)M_*(B) = M_*(AB)$ and $M^*(A)M^*(B) = M^*(AB)$. By the definition of M, we have $M(A)M(B) = M(AB)$.

REMARK. Although both A and B are left fuzzy sets (or right fuzzy sets), the equality $M(A)M(B) = M(AB)$ does not always holds.

Definition 3.12 ([1]) The variance of fuzzy set A with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ is defined by

$$Var(A) = \frac{1}{2}\int_0^1 \gamma(a_2(\gamma) - a_1(\gamma))^2 d\gamma.$$

Example 3.13 If $A = (\alpha, a, 0)$ is a left triangular fuzzy set then

$$Var(A) = \frac{1}{2}\int_0^1 \gamma(a - (a - \alpha(1-\gamma)))^2 d\gamma = \frac{\alpha^2}{24}.$$

Definition 3.14 ([1]) The covariance between fuzzy sets A with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and B with $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$) is defined by

$$Cov(A,B) = \frac{1}{2}\int_0^1 \gamma((a_2(\gamma) - a_1(\gamma))((b_2(\gamma) - b_1(\gamma))d\gamma.$$

Proposition 3.15. *Let $A_1 = (\alpha_1, a_1, 0)$ and $A_2 = (\alpha_2, a_2, 0)$ be left triangular fuzzy sets and $B_1 = (0, b_1, \beta_1)$ and $B_2 = (0, b_2, \beta_2)$ be right triangular fuzzy sets. Then $Cov(A_1, A_2) = \frac{\alpha_1\alpha_2}{24}$, $Cov(B_1, B_2) = \frac{\beta_1\beta_2}{24}$, $Cov(A_1, B_2) = \frac{\alpha_1\beta_2}{24}$ and $Cov(A_2, B_1) = \frac{\alpha_2\beta_1}{24}$.*

Proof. Since $[A_1]^\gamma = [a_1 - (1-\gamma)\alpha_1, a_1]$ and $[A_2]^\gamma = [a_2 - (1-\gamma)\alpha_2, a_2]$, we have

$$Cov(A_1, A_2) = \frac{1}{2}\int_0^1 \alpha_1\alpha_2\gamma(1-\gamma)^2 d\gamma = \frac{\alpha_1\alpha_2}{24}.$$

By the similar method, the remaining results can be obtained. □

Example 3.16 Let $A_1 = (2, 4, 0)$ and $A_2 = (3, 7, 0)$ be left triangular fuzzy sets and $B_1 = (0, 5, 3)$ and $B_2 = (0, 7, 2)$ be right triangular fuzzy sets. Then since $[A_1]^\gamma = [2\gamma+2, 4]$ and $[A_2]^\gamma = [3\gamma+4, 7]$, we have $Cov(A_1, A_2) = \frac{1}{4}$. Similarly, $Cov(B_1, B_2) = \frac{1}{4}$, $Cov(A_1, B_2) = \frac{1}{6}$ and $Cov(A_2, B_1) = \frac{3}{8}$.

Proposition 3.17. *Let $A_1 = [\alpha_1, k_1, 0]$ and $A_2 = [\alpha_2, k_2, 0]$ be left quadratic fuzzy sets and $B_1 = [0, m_1, \beta_1]$ and $B_2 = [0, m_2, \beta_2]$ be right quadratic fuzzy sets. Then $Cov(A_1, A_2) = \frac{1}{12\sqrt{a_1 a_2}}$, $Cov(B_1, B_2) = \frac{1}{12\sqrt{b_1 b_2}}$, $Cov(A_1, B_2) = \frac{1}{12\sqrt{a_1 b_2}}$ and*

$Cov(A_2, B_1) = \frac{1}{12\sqrt{a_2 b_1}}$.

Proof. Since

$$[A_1]^\gamma = \left[k_1 - \sqrt{\frac{1-\gamma}{a_1}}, k_1\right], \quad a_1 = \frac{1}{(\alpha_1 - k_1)^2}$$

and

$$[A_2]^\gamma = \left[k_2 - \sqrt{\frac{1-\gamma}{a_2}}, k_2\right], \quad a_2 = \frac{1}{(\alpha_2 - k_2)^2},$$

we have

$$Cov(A_1, A_2) = \frac{1}{2}\int_0^1 \gamma \frac{1-\gamma}{\sqrt{a_1 a_2}} d\gamma = \frac{1}{12\sqrt{a_1 a_2}}.$$

By the similar method, the remaining results can be obtained. □

Example 3.18 Let $A_1 = [1, 2, 0]$ and $A_2 = [3, 4, 0]$ be left quadratic fuzzy sets and $B_1 = [0, 5, 8]$ and $B_2 = [0, 2, 3]$ be right quadratic fuzzy sets. Then since $[A_1]^\gamma = [2 - \sqrt{1-\gamma}, 2]$ and $[A_2]^\gamma = [4 - \sqrt{1-\gamma}, 4]$, we have $Cov(A_1, A_2) = \frac{1}{12}$. Similarly, $Cov(A_1, B_2) = \frac{1}{12}$, $Cov(A_2, B_1) = \frac{1}{4}$ and $Cov(B_1, B_2) = \frac{1}{4}$.

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