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STRONGLY C-CONVEX FUNCTIONS AND ALMOST C-CONVEX FUNCTIONS ON PRECONVEXITY SPACES

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Abstract. In this paper, we introduce the concepts of strongly c-convex(strongly c-concave) function, almost c-convex function on preconvex spaces. We investigate the relationships between such concepts and several types of preconvex sets.

1. Introduction

In [1], Guay introduced the concept of preconvexity spaces defined by a binary relation on the power set $P(X)$ of a set X and investigated some properties. He showed that a preconvexity on a set yields a convexity space in the same manner as a proximity [3] yields a topological space.

The purpose of this paper is to introduce the concepts of strongly c-convex function and almost c-convex function on preconvex spaces. We show that every strongly c-convex function is c-convex and every c-convex function is almost c-convex. In particular, we investigate the relationships between such concepts and several types of preconvex sets.

Definition 1.1 ([1]). Let X be a nonempty set. A binary relation σ on $P(X)$ is called a preconvexity on X if the relation satisfies the following properties; we write $x\sigma A$ for $\{x\}\sigma A$:
(1) If $A \subseteq B$, then $A\sigma B$.
(2) If $A\sigma B$ and $B = \emptyset$, then $A = \emptyset$.
(3) If $A\sigma B$ and $b\sigma C$ for all $b \in B$, then $A\sigma C$.
(4) If $A\sigma B$ and $x \in A$, then $x\sigma B$.

The pair (X, σ) is called a preconvexity space.

In a preconvexity space (X, σ), $G(A) = \{x \in X : x\sigma A\}$ is called the *convexity hull* of a subset A. A is called *convex* [1] if $G(A) = A$.

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$I_\sigma(A) = \{x \in A : x \not\sigma (X - A)\}$ (simply, $I(A)$) is called the *co-convexity hull* [2] of a subset A. A is called a *co-convex set* if $I(A) = A$.

THEOREM 1.2 ([2]). *Let (X, σ) be a preconvexity space and $A \subseteq X$. Then A is a co-convex set iff A^c is a convex set.*

THEOREM 1.3 ([1, 2]). *For a preconvexity space (X, σ),*
(1) $A\sigma B$ iff $A \subseteq G(B)$.
(2) $A\sigma B$ iff $G(A)\sigma G(B)$.
(3) $G(\emptyset) = \emptyset$; $I(X) = X$.
(4) $I(A) \subseteq A \subseteq G(A)$ for all $A \subseteq X$.
(5) If $A \subseteq B$, then $G(A) \subseteq G(B)$ and $I(A) \subseteq I(B)$.
(6) $G(G(A)) = G(A)$ and $I(I(A)) = I(A)$ for $A \subseteq X$.
(7) $I(A) = X - G(X - A)$; $G(A) = X - I(X - A)$.

2. Main results

DEFINITION 2.1. Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : (X, \sigma) \to (Y, \mu)$ is said to be *strongly c-convex* if $A\sigma B$ implies $f(A)\mu I_\mu(f(B))$.

REMARK 2.2. Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be *c-convex* [1] if $A\sigma B$ implies $f(A)\mu f(B)$. Obviously we know that every strongly c-convex function is c-convex.

LEMMA 2.3 ([3]). *Let (X, σ) be a preconvexity space. For all $A \subseteq X$, $G(A)\sigma A$.*

THEOREM 2.4. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then the following things are equivalent:*
(1) f is strongly c-convex.
(2) $f(G_\sigma(A)) \subset G_\mu(I_\mu(f(A)))$ for all $A \subset X$.
(3) $G_\sigma(f^{-1}(B)) \subset f^{-1}(G_\mu(I_\mu(B)))$ for all $B \subset Y$.
(4) $f^{-1}(I_\mu(G_\mu(B))) \subset I_\sigma(f^{-1}(B))$ for all $B \subset Y$.

Proof. (1) $\Rightarrow$ (2) For each $A \subset X$, since $G(A)\sigma A$ and f is strongly c-convex, we have that $f(G(A))\mu I_\mu(f(A))$. Thus from Theorem 1.3(1), it follows $f(G_\sigma(A)) \subset G_\mu(I_\mu(f(A)))$.

(2) $\Rightarrow$ (1) Let $A\sigma B$ for $A, B \subset X$. Then $A \subset G_\sigma(A) \subset G_\sigma(B)$. From (2), it follows $f(A) \subset f(G_\sigma(A)) \subset f(G_\sigma(B)) \subset G_\mu(I_\mu(f(B)))$. So $f(A) \subset G_\mu(I_\mu(f(B)))$. By Theorem 1.3(1), we have $f(A)\mu I_\mu(f(B))$.

(2) $\Rightarrow$ (3) For $B \subset Y$, by (2), $f(G_\sigma(f^{-1}(B))) \subset G_\mu(I_\mu(f(f^{-1}(B)))) \subset G_\mu(I_\mu(B))$. Thus $G_\sigma(f^{-1}(B)) \subset f^{-1}(G_\mu(I_\mu(B)))$.

(3) $\Rightarrow$ (2) For $A \subset X$, $G_\sigma(A) \subset G_\sigma(f^{-1}(f(A))) \subset f^{-1}(G_\mu(I_\mu(f(A))))$. This implies $f(A) \subset G_\mu(I_\mu(f(B)))$.

(3) $\Rightarrow$ (4) For $B \subset Y$, by (3), $G_\sigma(f^{-1}(Y-B)) \subset f^{-1}(G_\mu(I_\mu(Y-B)))$. From Theorem 2.2, it follows $G_\sigma(X - f^{-1}(B)) \subset f^{-1}(Y - I_\mu(G_\mu(B))) = X - f^{-1}(I_\mu(G_\mu(B)))$.

Thus $f^{-1}(I_\mu(G_\mu(B))) \subset X - G_\sigma(X - f^{-1}(B)) = I_\sigma(f^{-1}(B))$.

(4) $\Rightarrow$ (3) It is similar to the proof of (3) $\Rightarrow$ (4) □

Let (X, σ) be a preconvexity space and $A \subset X$. A is said to be *p-preconvex* [4] (resp., *α-preconvex*) [6] if $A \subset I_\sigma(G_\sigma(A))$ (resp., $A \subset I_\sigma(G_\sigma(I_\sigma(A)))$. And A is said to be *cop-preconvex* (resp., *coα-preconvex*) if the complement of A is a p-preconvex (resp., *α-preconvex*) set.

Lemma 2.5. *Let (X, σ) be a preconvexity space and $A \subset X$.*
(1) A is cop-preconvex iff $G_\sigma(I_\sigma(A)) \subset A$ [4].
(2) A is coα-preconvex iff $G_\sigma(I_\sigma(G_\sigma(A))) \subset A$ [6].

Theorem 2.6. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). If f is strongly c-convex, then $f^{-1}(B)$ is convex for every cop-preconvex set B in Y.*

Proof. Let B be a cop-preconvex set in Y, since f is strongly c-convex, by Lemma 2.5, $G_\sigma(f^{-1}(B)) \subset f^{-1}(G_\mu(I_\mu(B))) \subset f^{-1}(B)$. It implies $f^{-1}(B)$ is convex. □

Theorem 2.7. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). If f is strongly c-convex, then $f^{-1}(B)$ is convex for every coα-preconvex set B in Y.*

Proof. Let B be a coα-preconvex set in Y. Then by Lemma 2.5,
$G_\sigma(f^{-1}(B)) \subset f^{-1}(G_\mu(I_\mu(B))) \subset f^{-1}(G_\mu(I_\mu(G_\mu(B)))) \subset f^{-1}(B)$.
Hence $f^{-1}(B)$ is convex. □

Definition 2.8. Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be *strongly c-concave* if for $C, D \subseteq Y$ whenever $C\mu D$, $f^{-1}(C)\sigma I_\sigma(f^{-1}(D))$.

Remark 2.9. Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be *c-concave* [3] if for $C, D \subseteq Y$ whenever $C\mu D$, $f^{-1}(C)\sigma f^{-1}(D)$. Obviously it is that every strongly c-concave function is c-concave.

Theorem 2.10. *Let $f : X \to Y$ be a function on two preconvexities (X, σ) and (Y, μ). Then the following things are equivalent:*
(1) f is strongly c-concave.

(2) $f^{-1}(G_\mu(A)) \subset G_\sigma(I_\sigma(f^{-1}(A)))$ *for all* $A \subset Y$.
(3) $I_\sigma(G_\sigma(f^{-1}(A))) \subset f^{-1}(I_\mu(A))$ *for all* $A \subset Y$.

Proof. (1) $\Rightarrow$ (2) Let f be strongly c-concave and $A \subset Y$. Since $G_\mu(A)\mu A$ and f is strongly c-concave, $f^{-1}(G_\mu(A))\sigma I_\sigma(f^{-1}(A))$. Thus by Theorem 1.3, $f^{-1}(G_\mu(A)) \subset G_\sigma(I_\sigma(f^{-1}(A)))$.

(2) $\Rightarrow$ (1) If $C\mu D$ for $C, D \subset Y$, then $C \subset G_\mu(D)$. By hypothesis, $f^{-1}(C) \subset f^{-1}(G_\mu(D)) \subset G_\sigma I_\sigma(f^{-1}(D)))$. So $f^{-1}(C)\sigma I_\sigma(f^{-1}(D))$.

(2) $\Rightarrow$ (3) For $A \subset Y$, from Theorem 1.3 and the condition (2), it follows $X - f^{-1}(I_\mu(A)) \subset f^{-1}(G_\mu(Y - A)) \subset G_\sigma(I_\sigma(f^{-1}(Y - A))) = X - I_\sigma(G_\sigma(f^{-1}(A)))$. So $I_\sigma(G_\sigma(f^{-1}(A))) \subset f^{-1}(I_\mu(A))$.

Similarly, it is obtained that (3) $\Rightarrow$ (2). □

Definition 2.11. Let (X, σ) be a preconvexity space and $A \subset X$. A is called a *regular-preconvex set* (briefly, *r-preconvex set*) if $A = I_\sigma(G_\sigma(A))$. And A is called a co*regular-preconvex* set (briefly, co*r*-*preconvex* set) if the complement of A is a r-preconvex set.

Definition 2.12. Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be *almost c-convex* if $f^{-1}(A)$ is convex for every co$r$-preconvex set A of X.

Theorem 2.13. *Let* $f : X \to Y$ *be a function on two preconvexity spaces* (X, σ) *and* (Y, μ). *Then the following things are equivalent:*
(1) f *is almost c-convex.*
(2) $G_\sigma(f^{-1}(G_\mu(I_\mu(F)))) \subset f^{-1}(F)$ *for every convex set* $F \subset Y$.
(3) $G_\sigma(f^{-1}(G_\mu(I_\mu(G_\mu(B))))) \subset f^{-1}(G_\mu(B))$ *for every* $B \subset Y$.

Proof. (1) $\Rightarrow$ (2) For a convex subset F of Y, then $G_\mu(I_\mu(G_\mu(I_\mu(F))))$ $= G_\mu(I_\mu(F))$, that is, $G_\mu(I_\mu(F))$ is co$r$-preconvex. By (1) and $G_\mu(I_\mu(F)))$ $\subset F$, $G_\sigma(f^{-1}(G_\mu(I_\mu(F)))) = f^{-1}(G_\mu(I_\mu(F))) \subset f^{-1}(F)$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Let A be co$r$-preconvex in Y. Since $G_\mu(I_\mu(G_\mu(A)) = A$ and $G_\mu(A) = A$, by (3), we have $G_\sigma(f^{-1}(B)) \subset f^{-1}(B)$, and $f^{-1}(B)$ is convex. Hence f is almost c-convex. □

Easily we have the following:

Theorem 2.14. *Let* $f : X \to Y$ *be a function on two preconvexity spaces* (X, σ) *and* (Y, μ). *Then the following things are equivalent:*
(1) f *is almost c-convex.*
(2) $f^{-1}(A)$ *is co-convex for every* r*-preconvex set* A *of* X.
(3) $f^{-1}(U) \subset I_\mu(f^{-1}(I_\mu(G_\mu(U))))$ *for every co-convex set* $U \subset Y$.

Theorem 2.15. *Let $f : X \to Y$ be a function on two preconvexity spaces (X, σ) and (Y, μ). Then if f is strongly c-convex, then it is almost c-convex.*

Proof. Let A be co$r$-preconvex in Y. Then $G_\mu(I_\mu(A)) = A$. From Theorem 2.4 (3), it follows $G_\sigma(f^{-1}(A)) \subset f^{-1}(G_\mu(I_\mu(A))) = f^{-1}(A)$. It implies $f^{-1}(A)$ is convex, and hence f is almost c-convex. □

Let (X, σ) be a preconvexity space and $A \subset X$. A is said to be *semi-preconvex* [5] (resp., *β-preconvex*) [7] if $A \sigma I_\sigma(A)$ (resp., $A \sigma I_\sigma(G_\sigma(A))$).

Remark 2.16. For a function $f : X \to Y$ on two preconvexity spaces (X, σ) and (Y, μ), from Theorem 2.4 and Theorem 2.10, the following things are obtained:

(1) If f is strongly c-convex, then for every convex set A of X, $f(A)$ is semi-preconvex (β-preconvex) set B in Y.

(2) If f is strongly c-concave, then for every convex set B of Y, $f^{-1}(B)$ is semi-preconvex (β-preconvex) set B in Y.

(3) If f is strongly c-convex, then for every co$r$-preconvex set B of Y, $f^{-1}(B)$ is convex.

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