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SOME PROPERTIES OF TOEPLITZ OPERATORS WITH SYMBOL μ

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Abstract. For a complex regular Borel measure μ on Ω which is a subset of $\mathbb{C}^k$, where k is a positive integer we define the Toeplitz operator T_μ on a reproducing analytic space which comtains polynomials. Using every symmetric polynomial is a polynomial of elementary polynomials, we show that if T_μ has finite rank then μ is a finite linear combination of point masses.

1. Introduction

Let Ω be a subset of $\mathbb{C}^k$ and let μ be a complex regular Borel measure on Ω. Suppose H is a seperable reproducing analytic space on Ω which contains polynomials.
If $T_\mu f(z)=\int_\Omega f(w)\overline{K_z(w)}d\mu(w)$ has finite rank, where K_z is the reproducing kernel of H, then μ must be singular with supp$\mu=\{z_1,\cdots,z_M\}$, that is, μ is a finite linear combination of point masses. Toeplitz operators are an important role in the physics and engeneering area. First Toeplitz operators were defined on the Hardy space H^2 by $T_\varphi f=P(\varphi f)$, where φ is in $L^\infty(\partial\mathbb{D})$ and P is the Szegö projection. Similarly Toeplitz operators on the Bergman space L^2_a are defined by $T_\varphi(f)=P(uf)$, where P is the Bergman projection from $L^2(dA)$ to L^2_a([3]).

Since H^∞ is dense in L^2_a, we can densely define Toeplitz operators with symbols that are measures. Moreover, we can extend the notion of Toeplitz operators to those with symbol measures ([1],[2],[5]). Luecking's paper ([1]) is devoted to characterization of a complex regular Borel measure μ on the unit disk whenever the Toeplitz operator T_μ has finite rank.

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In this paper, we introduce the symmetrization and the antisymmetrization of functions defined on a subset of $\mathbb{C}^k$ and we prove that every symmetric polynomial is a polynomial of elementary polynomials.

Section 3 is deal with the following theorem.

THEOREM 1.1. *Suppose μ is a complex regular Borel measure on $\Omega \subset \mathbb{C}^k$ and H is a separable reproducing analytic space on Ω which contains all polynomials. Then T_μ has finite rank if and only if suppμ is a finite set.*

2. Polynomials

Let k and N be fixed positive integers and let Ω be a subset of $\mathbb{C}^k$. In the following, we will assume that $f : \Omega^N \to \mathbb{C}$ is a function of N-variables. For $z \in \Omega$, let z=$(z^1, \cdots, z^k)$ and let e_0=1, $e_1(z_1, \cdots, z_N) = z_1+\cdots+z_N = (z_1^1+z_2^1+\cdots+z_N^1, \cdots, z_1^k+z_2^k+\cdots+z_N^k)$, $e_2(z_1, \cdots, z_N) = \sum_{i<j} z_i z_j = (\sum_{i<j} z_i^1 z_j^1, \sum_{i<j} z_i^2 z_j^2, \cdots, \sum_{i<j} z_i^k z_j^k)$, $e_3(z_1, \cdots, z_N)$=$(\sum_{i<j<l} z_i^1 z_j^1 z_l^1, \cdots, \sum_{i<j<l} z_i^k z_j^k z_l^k)$, $\cdots$ and $e_N(z_1, z_2, \cdots, z_N) = (z_1^1 \cdots z_N^1, z_1^2 \cdots z_N^2, \cdots, z_1^k \cdots z_N^k)$. Then $\prod_{j=1}^{N}(t - z_j) = t^N - e_1 t^{N-1} + e_2 t^{N-2} + \cdots + (-1)^N e_N$ $= (t^N - (z_1^1 + \cdots + z_N^1)t^{N-1} + \cdots + (-1)^N z_1^1 \cdots z_N^1, \cdots, t^N - (z_1^k + \cdots + z_N^k)t^{N-1} + \cdots + (-1)^N z_1^k \cdots z_N^k)$ and e_i's are elementary polynomials. For example, let $f(z_1, z_2) = z_1^{(2,2,\cdots,2)} + z_2^{(2,\cdots,2)}$. Then $f(z_1, z_2)(=z_1^2 + z_2^2)$ $= (z_1 + z_2)^2 - 2z_1 z_2 = e_1^2 - 2e_2$ and hence f is a polynomial of elementary polynomials. Let S_N denote the set of all bijection from N to N, where $N = \{1, 2, \cdots, N\}$. For $\sigma \in S_N$, we define ε_σ=+1 for an even permutation and -1 for an odd permutation.

DEFINITION 2.1. Suppose p is a polynomial function on Ω^N.
Then we say that
(1) p is symmetric if for any $\sigma \in S_N$, $\sigma p = p$, where $\sigma p(z_1, \cdots, z_N) = p(z_{\sigma(1)}, \cdots, z_{\sigma(N)}) = p((z^1_{\sigma(1)}, \cdots, z^1_{\sigma(N)}), \cdots, (z^1_{\sigma(N)}, \cdots, z^k_{\sigma(N)}))$.
(2) p is antisymmetric if for each $\sigma \in S_N$, $\sigma p = \varepsilon_\sigma p$.

We note that e_1 and e_2 are symmetric polynomials. Suppose f is a polynomial function of $z_1, z_2, \cdots, z_N$. We define the symmetrization and the antisymmetrization of f, that is, $Sf(z_1,\cdots,z_N)=\dfrac{1}{|S_N|}\sum_{\sigma\in S_N}\sigma f(z_1,\cdots,z_N)$ and $Af(z_1,\cdots,z_N)=\dfrac{1}{|S_N|}\sum_{\sigma\in S_N}\varepsilon_\sigma\sigma f(z_1,\cdots,z_N)$. Then f is a symmetric polynimial if and only if $Sf=f$ and f is antisymmetric if and only if $Af=f$. Moreover, $ASf=0=SAf$.

Proposition 2.2. *Let $f(z)=z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_N^{\alpha_n}$, where each multi-index α_i is a k-tuple of nonnegative integers $(\alpha_i^1,\cdots,\alpha_i^k)$.*
(1) If $\alpha_s=\alpha_t$ for some $s\neq t$ then $Af=0$.
(2) (a) If σ is an even permutation then $A\sigma f=f$.
(b) If σ is an odd permutation then $A\sigma f=-f$.

Proof. (1) Let $\sigma=(s,t)$. Then $\sigma f=f$ and hence $\sum_{\sigma\in S_N}\varepsilon_\sigma\sigma f=0$. Thus $Af=0$.
(2) If follows from the fact that for any $\tau\in S_N$,

$$\varepsilon_\tau\tau\sigma f(z_1,\cdots,z_N)=\begin{cases}\varepsilon_\gamma\gamma f(z_1,\cdots,z_N) & \text{if } \sigma \text{ is even and } \gamma=\tau\sigma\\ -\varepsilon_\gamma\gamma f(z_1,\cdots,z_N) & \text{if } \sigma \text{ is odd and } \gamma=\tau\sigma.\end{cases}$$

□

Corollary 2.3. *If $\sigma f=g$ for some $\sigma\in S_N$ then*

$$Ag=\begin{cases}Af & ,\ \sigma \text{ is even}\\ -Af & ,\ \sigma \text{ is odd.}\end{cases}$$

Let $V=\begin{vmatrix}1 & z_1 & z_1^2 & \dots & z_1^{N-1}\\ 1 & z_2 & z_2^2 & \dots & z_2^{N-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & z_N & z_N^2 & \dots & z_N^{N-1}\end{vmatrix}$. Then V is the Vandermonde determinant and by induction, $V=\prod_{i<j}(z_j-z_i)=\prod_{i<j}\prod_{l=1}^{k}(z_j^l-z_i^l)$. For $J=(\alpha_1,\alpha_2,\dots,\alpha_N)$, where whenever $s<t$, $\alpha_s^i<\alpha_t^i$ for all $i=1,2,\dots,k$, define $V_J=\begin{vmatrix}z_1^{\alpha_1} & z_1^{\alpha_2} & \dots & z_1^{\alpha_N}\\ z_2^{\alpha_1} & z_2^{\alpha_2} & \dots & z_2^{\alpha_N}\\ \vdots & \vdots & \ddots & \vdots\\ z_N^{\alpha_1} & z_N^{\alpha_2} & \dots & z_N^{\alpha_N}\end{vmatrix}$. Let $J_1=(0,1,2,\cdots,N-1)$. Then $V_{J_1}=V$ and V is the minimal-degree polynomial vanishing on $\bigcup_{i\neq j}\{(z_1,\cdots,z_N):z_i^s=z_j^s \text{ for some } s\in\{1,2,\cdots,k\}\}$. Let $p(z)=a_0+a_1z+\cdots+$

a_nz^n. Then $p(z)-p(z_0)=(z-z_0)(a_1+a_1(z+z_0)+\cdots+a_n(z^{n-1}+z^{n-2}z_0+\cdots+z_0^{n-1}))=((z^1-z_0^1)(a_1+a_2(z^1+z_0^1)+\cdots+a_n((z^1)^{n-1}+(z^1)^{n-2}z_0^1+\cdots+(z_0^1)^{n-1})),\cdots,(z^k-z_0^k)(a_1+a_2(z^k+z_0^k)+\cdots+a_n((z^k)^{n-1}+(z^k)^{n-2}z_0^k+\cdots+(z_0^k)^{n-1}))$. Since the degree of $a_1+a_2(z+z_0)+\cdots+a_n(z^{n-1}+\cdots+z_0^{n-1})$ is $n-1$, $p(z)-p(z_0)=(z-z_0)Q(z)$ for some $Q(z)$ with $\deg(Q(z))=n-1$ and hence if a polynomial p vanishes at $a\in\mathbb{C}^k$ then $z-a$ divides p.

LEMMA 2.4. *Suppose f is an antisymmetric polynomial. Then there is a symmetric polynomial g such that $f=Vg$.*

Proof. Since f is antisymmetric, $f(z_1,z_2,z_3,\cdots,z_N)=-f(z_2,z_1,z_3,\cdots,z_N)$ and hence $f(a,a,\ z_3,\cdots,z_N)=0$ for all $a\in\mathbb{C}$. Let $p(z)=f(z,a,z_3,\cdots,z_N)$. Since $z-a$ divides $p(z)$, z_2-z_1 divides $f(z_1,z_2,z_3,\cdots,z_N)$. Since $z_2-z_1=\prod_{s=1}^{k}(z_2^s-z_1^s)$, V divides f. Let $g(z_1,z_2,\cdots,z_N)=\frac{f(z_1,\cdots,z_N)}{V}$. Take any permutation σ in S_N. Since $\sigma g=\frac{\sigma f}{\sigma V}=\frac{\varepsilon_\sigma f}{\varepsilon_\sigma V}=\frac{f}{V}=g$, g is symmetric. This completes the proof. □

Let $J=(\alpha_1,\alpha_2,\cdots,\alpha_N)$, where for $s<t$, $\alpha_s^i<\alpha_t^i$ for all $i=1,2,\cdots,k$. Let $g(z_1,z_2,\cdots,z_N)=z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_N^{\alpha_N}$ be a monomial. We note that the range of A is the vector space of all antisymmetric polynomials and hence the images of all monomials span the range of A. By Lemma 2.4, $A(g(z_1,\cdots,z_N))=\frac{V_J}{N!}$. This implies that each antisymmetric polynomial is a linear combination of $V_J's$, that is, for any antisymmetric polynomial f, $f=\sum_J C_JV_J$.

LEMMA 2.5. *Each symmetric polynomial $f(z_1,z_2,\cdots,z_N)$ can be written a polynomial of elementary polynomials $e_1,\cdots,e_N$.*

Proof. We note that the statement is trivially true for $N=1$. Let $\mathbb{C}[z_1,z_2,\cdots,z_N]$ denote the set of all polynomials of N-variables. We define $Q:\mathbb{C}[z_1,z_2,\cdots,z_N]\to\mathbb{C}[z_1,z_2,\cdots,z_N]$ by $Q(p(z_1,z_2,\cdots,z_N))=p(z_1,\cdots,z_{N-1},0)$. Suppose f is a symmetric polynomial of $z_1,z_2,\cdots,z_N$. Put $g(z_1,z_2,\cdots,z_{N-1})=Q(f(z_1,z_2,\cdots,z_N))$. Then $Q(f(z_1,z_2,\cdots,z_N))$ is also a symmetric polynomial of $z_1,z_2,\cdots,z_{N-1}$. By induction hypothesis, there exists a polynomial $p(z_1,z_2,\cdots,z_{N-1})$ such that $g(z_1,z_2,\cdots,z_{N-1})=p(e_1(z_1,\cdots,z_{N-1}),e_2(z_1,\cdots,z_{N-1}),\cdots,e_{N-1}(z_1,$

$\cdots, z_{N-1})$). Define $F(z_1, z_2, \cdots, z_N) = p(e_1(z_1, \cdots, z_{N-1}), \cdots, e_{N-1}(z_1, \cdots, z_{N-1})) = g(z_1, z_2, \cdots, z_{N-1}) = Q(f(z_1, z_2, \cdots, z_N))$, $Q(f(z_1, z_2, \cdots, z_N) - F(z_1, z_2, \cdots, z_N)) = 0$. Put $G(z_1, z_2, \cdots, z_N) = f(z_1, z_2, \cdots, z_N) - F(z_1, z_2, \cdots, z_N)$. Since $Q(G(z_1, z_2, \cdots, z_N)) = G(z_1, z_2, \cdots, z_{N-1}, 0) = 0$, z_N divides G. Since G is a symmetric polynomial, $z_1 z_2 \cdots z_N (= e_N(z_1, z_2, \cdots, z_N))$ divies G and hence $f(z_1, z_2, \cdots, z_N) - p(e_1(z_1, z_2, \cdots, z_N), \cdots, e_{N-1}(z_1, z_2, \cdots, z_N)) = e_N(z_1, z_2, \cdots, z_N) f_1(z_1, z_2, \cdots, z_N)$ for some polynomial f_1. Since $\deg f_1 \leq \deg f - N$, by induction on the degree of f, we obtain $f_1(z_1, z_2, \cdots, z_N) = p_1(e_1(z_1, z_2, \cdots, z_N), \cdots, e_N(z_1, z_2, \cdots, z_N))$ for some polynomial $p_1(z_1, z_2, \cdots, z_N)$. Since $p(e_1(z_1, z_2, \cdots, z_N), \cdots, e_{N-1}(z_1, z_2, \cdots, z_N)) + e_N(z_1, z_2, \cdots, z_N) p_1(e_0, e_1(z_1, z_2, \cdots, z_N), \cdots, e_N(z_1, z_2, \cdots, z_N))$ is a polynomial of elementary polynomials, we get the result. □

Suppose $(z_1, z_2, \cdots, z_N)$ and $(w_1, w_2, \cdots, w_N)$ are in Ω^N. Then $(e_1(z_1, z_2, \cdots, z_N), \cdots, e_N(z_1, z_2, \cdots, z_N)) = (e_1(w_1, w_2, \cdots, w_N), \cdots, e_N(w_1, w_2, \cdots, w_N))$ if and only if $\prod_{i=1}^{N}(\lambda - z_j) = \prod_{i=1}^{N}(\lambda - w_j)$ for all $\lambda \in \mathbb{C}$ if and only if $\lambda^N - e_1(z_1, z_2, \cdots, z_N)\lambda^{N-1} + \cdots + (-1)^N e_N(z_1, z_2, \cdots, z_N) = \lambda^N - e_1(w_1, w_2, \cdots, w_N)\lambda^{N-1} + \cdots + (-1)^N e_N(w_1, w_2, \cdots, w_N)$ for all $\lambda \in \mathbb{C}$.
Since S_N acts on $(\mathbb{C}^k)^N$ as $\sigma(z_1, z_2, \cdots, z_N) = (z_{\sigma(1)}, z_{\sigma(2)}, \cdots, z_{\sigma(N)})$ for all $\sigma \in S_N$, it induces an equvalence relation on $(\mathbb{C}^k)^N$, that is, $(z_1, z_2, \cdots, z_N) \sim (w_1, w_2, \cdots, w_N)$ if and only if
$\sigma(z_1, z_2, \cdots, z_N) = (w_1, w_2, \cdots, w_N)$ for some $\sigma \in S_N$ if and only if
$(e_1(z_1, z_2, \cdots, z_N), \cdots, e_N(z_1, z_2, \cdots, z_N)) = (e_1(w_1, w_2, \cdots, w_N), \cdots, e_N(w_1, w_2, \cdots, w_N))$. Thus $\{e_1, e_2, \cdots, e_N\}$ does not seperate Ω^N. Since $\overline{\Omega}^N \big/ \sim$ is a compact Hausdorff space, span$\{f(e_1, \cdots, e_N)\overline{g(e_1, \cdots, e_N)} : f, g \in \mathbb{C}[z_1, z_2, \cdots, z_N]\}$ seperates points of $\overline{\Omega}^N \big/ \sim$. Let $E = \text{span}\{f(e_1, e_2, \cdots, e_N)\overline{g(e_1, e_2, \cdots, e_N)} : f, g \in \mathbb{C}[z_1, z_2, \cdots, z_N]\}$.
By Stone-Weierstrass theorem, E is dense in $C[e_1, e_2, \cdots, e_N]$, where $C[e_1, \cdots, e_N]$ is the set of continuous functions.

3. Toeplitz operators

Suppose H is a separable reproducing analytic space on Ω which contains polynomials. Let $K_z(w)$ be a reproducing kernel of H, that is, for each $f \in H$, $\int_\Omega f(w)\overline{K_z(w)}dV(w) = f(z)$, where dV is the Lebesque volume measure. For an orthnormal basis $\{e_n(z)\}$ of H and $f \in H$, $f(z) = \sum_{n=1}^{\infty} < f, e_n > e_n(z)$ and hence $K_z(w) = \sum_{n=1}^{\infty} \overline{e_n(z)} e_n(w)$. Thus $\overline{K_z(w)} = K_w(z)$. Let $\mathbb{C}[z]$ denote the set of polynomials. Given a complex reqular Borel measure μ on Ω, we define a Toeplitz operator T_μ with symbol μ by $T_\mu f(z) = \int_\Omega f(w)\overline{K_z(w)}d\mu(w)$, $f \in \mathbb{C}[z]$. Suppose f and g are in $\mathbb{C}[z]$. Then

$$\begin{aligned} < T_\mu f, g > &= \int_\Omega T_\mu f(z)\overline{g(z)}dV(z) \\ &= \int_\Omega \int_\Omega f(w)\overline{K_z(w)}d\mu(w)\overline{g(z)}dV(z) \\ &= \int_\Omega f(w)\overline{\int_\Omega g(z)\overline{K_w(z)}dA_z}d\mu(w) \\ &= \int_\Omega f(w)\overline{g(w)}d\mu(w). \end{aligned}$$

Since the closure of span $\{w^k, \overline{w}^n\}_{k,n\geq 0} = C(\overline{\mathbb{D}})$, $T_\mu = 0$ if and only if $\mu = 0$.

LEMMA 3.1. *Suppose T_μ has finte rank $N-1$. If $f \in \mathbb{C}[z_1, z_2, \cdots, z_N]$ and g is an antisymmetric polynomial then* $\int_{\Omega^N} f(z_1, z_2, \cdots, z_N) \times$ $\times \overline{g(z_1, z_2, \cdots, z_N)} d\mu(z_1)d\mu(z_2)\cdots d\mu(z_N) = 0.$

Proof. Suppose RangeT_μ = span$\{F_1, F_2, \cdots, F_{N-1}\}$. Then for any $f_j \in \mathbb{C}[z]$, there is $(c_{j1}, c_{j2}, \cdots, c_{j(N-1)})$ such that $T_\mu f_j = \sum_{i=1}^{N-1} c_{ji}F_i$.

Suppose $0 = c_1 T_\mu f_1 + \cdots + c_N T_\mu f_N = \sum_{j=1}^{N} c_j (\sum_{i=1}^{N-1} c_{ji}F_i) = \sum_{i=1}^{N-1} (\sum_{j=1}^{N} c_j c_{ji})F_i$.

Since $\{F_1, F_2, \cdots, F_{N-1}\}$ is linearly independent, $\sum_{j=1}^{N} c_j c_{ji} = 0$ for $i = 1, 2, \cdots, N-1$. Since the number of equation is less than the number of unknown, there exists $(c_1, c_2, \cdots, c_N) \neq (0, 0, \cdots, 0)$ such that $c_1 T_\mu f_1 + c_2 T_\mu f_2 + \cdots + c_N T_\mu f_N = 0$. Pick up other N functions

$g_1, g_2, \cdots, g_N$ in $\mathbb{C}[z]$. Since $\sum_{j=1}^{N} c_j T_\mu f_j = 0$, $0 = <\sum_{j=1}^{N} c_j T_\mu f_j, g_i>$ for $i = 1, 2, \cdots, N$ and hence we get a system of linear equations :

$$\left[\begin{array}{llll} c_1 < T_\mu f_1, g_1 > & +c_2 < T_\mu f_2, g_1 > & + \cdots & +c_N < T_\mu f_N, g_1 > = 0 \\ c_1 < T_\mu f_1, g_2 > & +c_2 < T_\mu f_2, g_2 > & + \cdots & +c_N < T_\mu f_N, g_2 > = 0 \\ \quad \vdots & & & \\ c_1 < T_\mu f_1, g_N > & +c_2 < T_\mu f_2, g_N > & + \cdots & +c_N < T_\mu f_N, g_N > = 0 \end{array}\right.$$

Let $A = \begin{vmatrix} < T_\mu f_1, g_1 > & < T_\mu f_2, g_1 > & \cdots & < T_\mu f_N, g_1 > \\ \vdots & & & \\ < T_\mu f_1, g_N > & < T_\mu f_2, g_N > & \cdots & < T_\mu f_N, g_N > \end{vmatrix}$ and let A_{ij} be the cofactor of A. Since the system has a non-trivial solution, $0 = A$

$= < T_\mu f_1, g_1 > A_{11} + < T_\mu f_1, g_2 > A_{21} + \cdots + < T_\mu f_1, g_N > A_{N1}$

$= \Big(\int_\Omega f_1(z_1)\overline{g_1(z_1)} d\mu\Big) A_{11} + \cdots + \Big(\int_\Omega f_1(z_1)\overline{g_N(z_1)} d\mu\Big) A_{N1}$

$= \int_\Omega f_1(z_1)[\overline{g_1(z_1)} A_{11} + \overline{g_1(z_1)} A_{21} + \cdots + \overline{g_N(z_1)} A_{N1}] d\mu(z_1)$

$$= \int_\Omega f_1(z_1) \begin{vmatrix} \overline{g_1(z_1)} & < T_\mu f_2, g_1 > & \cdots & < T_\mu f_N, g_1 > \\ \overline{g_2(z_1)} & < T_\mu f_2, g_2 > & \cdots & < T_\mu f_N, g_2 > \\ \vdots & & & \\ \overline{g_N(z_1)} & < T_\mu f_2, g_N > & \cdots & < T_\mu f_N, g_N > \end{vmatrix} d\mu(z_1)$$

$$= \int_\Omega \int_\Omega f_1(z_1) \cdots f_N(z_N) \begin{vmatrix} \overline{g_1(z_1)} & \overline{g_1(z_2)} & \cdots & \overline{g_1(z_N)} \\ \overline{g_2(z_1)} & \overline{g_2(z_2)} & \cdots & \overline{g_2(z_N)} \\ \vdots & & & \\ \overline{g_N(z_1)} & \overline{g_N(z_2)} & \cdots & \overline{g_N(z_N)} \end{vmatrix} d\mu(z_1) \cdots d\mu(z_N),$$

where the sixth equality comes from the induction.

$$\text{I.e., } 0 = \int_\Omega \int_\Omega \prod_{j=1}^{N} f_j(z_j) \overline{\begin{vmatrix} g_1(z_1) & g_1(z_2) & \cdots & g_1(z_N) \\ g_2(z_1) & g_2(z_2) & \cdots & g_2(z_N) \\ \vdots & & & \\ g_N(z_1) & g_N(z_2) & \cdots & g_N(z_N) \end{vmatrix}} d\mu(z_1) \cdots d\mu(z_N).$$

For $J = (\alpha_1, \alpha_2, \cdots, \alpha_N)$, where whenever $s < t$, $\alpha_s^i < \alpha_t^i$ for all $i = 1, 2, \cdots, k$, let $g_i(z_j) = z_j^{\alpha_i}$. Then $0 = \int_{\Omega^N} f(z)\overline{V_J(z)} d\mu^N(z)$. Take any antysymmetric polynomial g. Since each antisymmetric polynomial is a linear combination of $V_J's$, $g = \sum C_J V_J$ for some C_J and hence $\int_{\Omega^N} f(z)\overline{g(z)}d\mu^N(z) = \sum C_J \int_{\Omega^N} f(z)\overline{V_J}d\mu^N(z) = 0.$ □

Let $E = \text{span}\{f(e_1, e_2, \cdots, e_N)\overline{g(e_1, e_2, \cdots, e_N)}\} : f, g \in \mathbb{C}[z_1, z_2, \cdots, z_N]$ and let C denote the continuity of the function. Then $E \subset C[e_1, e_2, \cdots, e_N]$ and E is dense in $C[e_1, e_2. \cdots, e_N]$. Since each symmetric polynomial can be written a polynomial of elementary polynomials, for every $F : \Omega^N \to C(\overline{\Omega}^N)$,

$$0 = \int_\Omega \cdots \int_\Omega F(e_1, \cdots e_N)|V(z_1, \cdots, z_N)|^2)d\mu(z_1)\cdots d\mu(z_N).$$

Take any continuous function f in $C(\overline{\Omega^N})$. Let $Sf(z_1, \cdots, z_N)$ be the symmetrization of f. Put $F(z_1, z_2, \cdots, z_N) = Sf(z_1, z_2, \cdots, z_N)$. Then

$$0 = \int_\Omega \cdots \int_\Omega Sf(z_1, \cdots, z_N)|V(z_1, \cdots, z_N)|^2 d\mu(z_1)\cdots d\mu(z_N)$$

$$= \frac{1}{|S_N|}\sum_{\sigma\in S_N}\int_\Omega \cdots \int_\Omega f(z_{\sigma(1)}, \cdots, z_{\sigma(N)})|V(z_1, \cdots, z_N)|^2 d\mu(z_1)\cdots d\mu(z_N).$$

Since $|V(z_1, \cdots, z_N)|^2$ and $d\mu^N$ are both invariant under permutations of the coordinates, $|V(z_1, \cdots, z_N)|^2 d\mu^N = 0$. Thus μ is supported on the set where V vanishes. Therefore we have the following :

THEOREM 3.2. *Let μ be a complex regular Borel measure on Ω which is a subset of $\mathbb{C}^k$ and H a separable reproducing analytic space on Ω which contains all polynomials. Then T_μ has finite rank if and only if μ is supported on the finite set, that is, μ is a finite linear combination of point masses.*

Suppose μ is a complex regular Borel measure on the unit disk $\mathbb{D}$. For $\alpha > -1$, the weighted Bergman space A^p_α consists of the analytic functions in $L^p(\mathbb{D}, dA_\alpha)$, where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z) = \frac{1}{\pi}(\alpha+1)(1-|z|^2)^\alpha dxdy$. Then $K^\alpha_z(w) = \frac{1}{(1-\overline{z}w)^{2+\alpha}}$ is a reproducing kernel of A^2_α. Then $< f, K^\alpha_z > = \int_\mathbb{D} f(w)\overline{K^\alpha_z(w)}dA_\alpha$ for all $f \in A^2_\alpha$, A^2_α is a separable reproducing analytic space and contains all polynomials. If μ is absolutely continuous with respect to dA_α then $d\mu = \varphi dA_\alpha$ for some $\varphi \in L^1(\mathbb{D}, dA_\alpha)$. If T_μ has finite rank then $\{z \in \mathbb{D} : \varphi(z) \neq 0\}$ is a finite set. Since $A^p_\alpha \subset L^p(\mathbb{D}, d\mu)$, μ is a Carleson measure on the weighted Bergman space A^p_α and hence T_μ is a bounded linear operator.

In fact, the measure μ is the zero measure and whenever ν is a complex regular Borel measure on the unit disk $\mathbb{D}$ and T_ν has finite rank, ν is a finite linear combination of point masses.

References

[1] D. H. Luecking, *Finite rank Toeplitz operators on the Bergman Space*, Proc. Amer. Math. Soc. **136** (2008), no. 5, 1717-1723.

[2] D. H. Luecking, *Trace and criteria for Toeplitz operators*, J. Fanct. Anal. **73** (1987), 345-368.

[3] G. McDonald and C. Sundberg, *Toeplitz operators on the disc*, Indiana Univ. Math. J. **28** (1979), no. 4, 595-611.

[4] H. L. Royden, *Real Analysis*, The Macmillan company, New York, 1998.

[5] K. Zhu, *Operator theory in Function Spaces, 2nd ed.*, Amer. Math. Soc. Providence, RI, 2005.

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