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LOGARITHMIC CAPACITY UNDER CONFORMAL MAPPINGS OF THE UNIT DISC

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Abstract. If $P(f,r)$ is the set of endpoints of radii which have length greater than or equal to $r > 0$ under a conformal mapping f of the unit disc. Then for large r, the logarithmic capacity of $P(f,r)$, $\frac{1}{2\sqrt{r}} \leq cap(P(f,r)) \leq \frac{k}{\sqrt{r}}$. Where k is the positive constant.

1. Modulus and logarithmic capacity

The theory of modulus has been successfully applied to analytic functions of a complex variable, and it has found application in the study of conformal mappings.

Throughout this note, $\Omega = \{z\}$ will denote the complex plane, D is a domain in Ω. And U is the unit disc in Ω. A curve $\gamma : I \to \Omega$ is a continuous mapping of an interval I. If we speak of a curve in D, then we allow the endpoints of the curve to lie on ∂D. A curve γ in D connects two sets $A, B \subseteq \bar{D}$, if γ has one endpoint in A and one in B. We denote by $len(\gamma)$ the euclidean length of γ.

Definition 1.1. ([1]) The modulus $mod(\Gamma)$ of a family Γ of locally rectifiable curves(simply, curves or arcs) in a domain D is defined as

$$mod(\Gamma) = inf_\rho \int\int_D \rho(z)^2 dm_2(z).$$

Where m_2 is two-dimensional Lebesque measure and infimum is taken over all non-negative Borel measurable functions ρ that satisfy

$$\int_\gamma \rho(z)|dz| \geq 1$$

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for all $\gamma \in \Gamma$, where $|dz|$ means integration with respect to euclidean arc-length. We shall call a function $\rho(z)$ in D *admissible* in association with Γ. Obviously $0 \leq mod(\Gamma) \leq \infty$.

PROPOSITION 1.2. ([9]) If $\Gamma_1 = \{\gamma\}$ is a curve family in some domain D_1, $f : D_1 \to D_2$ is a conformal mapping and Γ_2 is the curve family in D_2 consisting of the curves $f \circ \gamma$, then

$$mod(\Gamma_1) = mod(\Gamma_2).$$

EXAMPLE 1.3. Let T be a Jordan domain with three distinguished boundary points and Γ the family of all curves in T which touch all three sides, then

$$mod(\Gamma) = \frac{1}{\sqrt{3}}.$$

In fact, by Proposition 1.2, we begin by mapping conformally on an equilateral triangle with side 1. The minimum length of $\gamma \in \Gamma$ is that of the altitude: $\sqrt{3}/2$. We set $\rho = 2/\sqrt{3}$. Then ρ is admissible in association with Γ and it follows that $mod(\Gamma) = (2/\sqrt{3})^2(\sqrt{3}/4) = 1/\sqrt{3}$.

EXAMPLE 1.4. Let R be a rectangle of sides a and b, Γ the family of all curves in R which join the two sides of length a. Then

$$mod(\Gamma) = \frac{a}{b}.$$

In fact, since the minimum length of $\gamma \in \Gamma$ is b, we set $\rho = 1/b$. Then ρ is admissible in association with Γ, and we obtain $mod(\Gamma) = a/b$.

DEFINITION 1.5. ([2]) Let E be a bounded Borel set in Ω, μ a positive mass-distribution on E with total mass unity. Then

$$\mathbb{U}^{\mu}(z) = \int_E \log \left| \frac{1}{z - \zeta} \right| \, d\mu(\zeta)$$

is called a *logarithmic potential* of μ on E, where

$$\mathbb{V}_{\mu}(E) = \sup_{z \in E} \mathbb{U}^{\mu}(z), \quad \mathbb{V} = \inf_{\mu} \mathbb{V}_{\mu}(E).$$

We define the *logarithmic capacity*(simply, *capacity*), $cap(E)$ of E by

$$cap(E) = exp(-\mathbb{V}).$$

Obviously $0 \leq cap(E) < \infty$.

EXAMPLE 1.6. For the Cantor ternary set $E\{2/3\}$, $cap(E) \geq 1/18$.

Proposition 1.7. ([10]) The capacity of a countable set is also zero, and the union of a countable set of sets of capacity zero is of capacity zero.

The following statements which relates the modulus and capacity is needed in the proof of the theorem.

Example 1.8. ([10]) Let E be a compact set in U and Γ be the familly of all curves which join $\{z \,|\, |z| = 1\}$ to E. Then

$$mod(\Gamma) = 0 \quad \text{if and only if} \quad cap(E) = 0.$$

Theorem 1.9. ([9]) Let E be a Borel subset of ∂U and $\Gamma(E, \alpha)$ the family of all curves γ in $D = \{z \in U \,|\, \alpha < |z| < 1\}$ that connect $\{z \,|\, |z| = \alpha\}$ and E.

$$cap(E) \leq \frac{1+\alpha}{\sqrt{\alpha}}(-1)^{\frac{1}{\Gamma(E,\alpha)}},$$

where $\alpha > 0$ is a sufficiently small constant.

2. Capacity under some conformal mappings

Now we are ready to state our result. The following theorem states that if $f : U \to \Omega$ is conformal, then the set of radii whose images under f have infinite length has vanishing capacity.

Theorem 2.1. *Let $f : U \to \Omega$ be a conformal mapping with $f'(0) = 1$. If $P(f, r)$ is the set of all $p \in \partial U$ with*

$$len(f([0, p))) \geq r > 0,$$

then

$$cap(P(f, r)) \leq \frac{k}{\sqrt{r}},$$

where k is the positive constant. And for large r, there exist functions f, such that

$$cap(P(f, r)) \geq \frac{1}{2\sqrt{r}}.$$

For our proof of the theorem 2.1, we will need the followings.

Theorem 2.2. ([9]) Let $f : U \to \Omega$ be a conformal mapping, γ a curve in U with endpoints 0, $p \in \partial U$ and $[0, p)$ the radius of U with endpoint p. Then

$$len(f([0, p))) \leq c \; len(f \circ \gamma),$$

where c is a positive constant.

The following lemma states a modulus estimate. It shows us the usefullness of the method of modulus.

LEMMA 2.3. *Let D be a domain and Γ a family of curves in D which have one endpoint in a compact set $F \subseteq \bar{D}$. Suppose F is contained in a disc of diameter $\eta > 0$ centered at the origin. If $L \geq \eta$ and $len(\gamma) \geq L$ for all $\gamma \in \Gamma$, then*

$$mod(\Gamma) \leq \frac{2\pi}{\log(1 + L/\eta)}.$$

Proof. In addition to our assumptions on M we may assume that there exists at least one rectifiable curve in D which connects a point in D to a point in M. For otherwise it is easy to see that

$$mod(\Gamma) = 0.$$

(Consider test functions ρ which are equal to $\alpha > 0$ on $B \cap D$ where B is some open disc containing M and 0 elsewhere. Let α tend to 0.)

For $w \in D$ define

$$l(w) = inf_\gamma \, len(\gamma),$$

where the infimum is taken over all curves in D connecting w and M. The additional assumption on M implies that

$$l(w) < \infty$$

for all $w \in D$. The function l is continuous on D and satisfies

$$l(w) \geq |w| - \frac{\eta}{2}$$

for $w \in D$. Moreover, if $\gamma : [0, t_0] \to \Omega$ is a curve in D parameterized with respect to arc-length and if $\gamma(0) \in M$, then

$$l(\gamma(t)) \leq t$$

for $t \in (0, t_0]$.

Define $\rho : D \to [0, \infty)$ by

$$\rho(w) = \begin{cases} \frac{1}{(\log(1+L/\eta))(\eta + l(w))} & \text{if } l(w) \leq L, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, the function ρ is Borel measurable and we claim that

$$\int_\gamma \rho(w) |dw| \geq 1$$

for all $\gamma \in \Gamma$. Hence forth, $\log(1 + L/\eta)$ will simply denote δ.

To see this let $\gamma \in \Gamma$ be arbitrary. We may assume that $\gamma : I \to \Omega$ has an arc-length parametrization with $I = [0, len(\gamma)]$ and that $\gamma(0) \in M$. We have $l(\gamma(s)) \leq s$ for all $s \in I - \{0\}$. By assumption $len(\gamma) \geq L$ and so

$$\begin{aligned}\int_\gamma \rho(w)|dw| &\geq \frac{1}{\delta}\int_0^L \frac{ds}{\eta + l(\gamma(s))} \\ &\geq \frac{1}{\delta}\int_0^L \frac{ds}{\eta + s} \\ &= 1.\end{aligned}$$

Therefore, if $L \geq \eta$

$$\begin{aligned}\text{mod }(\Gamma) &\leq \int\int_D \rho(w)^2 dm_2(w) \\ &= \frac{1}{\delta^2}\int\int_{\{w\in D: l(w)\leq L\}} \frac{dm_2(w)}{(\eta + l(w))^2} \\ &\leq \frac{1}{\delta^2}\int\int_{\{w\in\Omega: |w|\leq L+\eta/2\}} \frac{dm_2(w)}{(\eta/2 + |w|)^2} \\ &= \frac{2\pi}{\delta} + 2\pi\frac{\log 2 - 1 + \eta/(2L+2\eta)}{\delta^2} \\ &\leq \frac{2\pi}{\delta} \\ &= \frac{2\pi}{\log(1+L/\eta)}.\end{aligned}$$

This completes the proof of the lemma. □

3. Proof of the theorem 2.1

The idea of the proof is essentially the same as in [9]. A limiting argument is employed in Pfluger's theorem which is related to the concept of reduced extremal distance([1], [8]). The new ingredients in our proof are the more refined modulus estimate of the lemma and the use of the Gehring-Hayman theorem. The proof will show that for the constant k in the theorem we can take

$$k = \sqrt{2c}$$

where c is the constant in the Gehring-Hayman theorem([7]).

We use the notation of the theorem and may assume $f(0) = 0$. Let $\alpha \in (0,1)$ be arbitrary. Let $\Gamma_1(\alpha)$ be the family of all curves in $\{z \in U \,|\, \alpha < |z| < 1\}$ connecting $\{z \in U \,|\, |z| = \alpha\}$ and $P(f,r)$. We leave it to the reader to show that the set $P(f,r)$ is a countable intersection of open subsets of ∂U. Hence it is a Borel set.

Suppose $\gamma \in \Gamma_1(\alpha)$ and let $z_0 \in U$, $|z_0| = \alpha$, and $p \in P(f,r)$ be the endpoints of γ. Let $[0, z_0]$ be the line segment with endpoints 0 and z_0. If we join $[0, z_0]$ and γ, then we get a curve $\tilde{\gamma}$ in U connecting 0 and p. By the Gehring-Hayman theorem and by definition of $P(f,r)$

$$\begin{aligned} len(f \circ \tilde{\gamma}) &\geq \frac{1}{c}\, len(f([0,p])) \\ &\geq \frac{r}{c}. \end{aligned}$$

By Koebe's distortion theorem([9]),

$$|f'(z)| \leq 1 + 5\alpha$$

if $|z| \leq \alpha$ and $\alpha > 0$ is sufficiently small. It follows that for small α

$$\begin{aligned} len(f \circ \gamma) &\geq \frac{r}{c} - (\alpha + 5\alpha^2) \\ &= L. \end{aligned}$$

We now apply the lemma for the region

$$D = f(U - \{z \in U \,|\, |z| \leq \alpha\}),$$

the compact set

$$M = f(\{z \in U \,|\, |z| = \alpha\}) \subseteq \bar{D}$$

and the curve family

$$\Gamma_2(\alpha) = \{f \circ \gamma \,|\, \gamma \in \Gamma_1(\alpha)\}.$$

By Koebe's distortion theorem M is contained in a disc centered at the origin of diameter

$$\eta = 2\alpha(1 + 3\alpha)$$

for small $\alpha > 0$. It follows that for small $\alpha > 0$

$$\begin{aligned} mod(\Gamma_1(\alpha)) &= mod(\Gamma_2(\alpha)) \\ &\leq \frac{2\pi}{\log\left(\frac{r/c + \alpha + \alpha^2}{2\alpha(1+3\alpha)}\right)}. \end{aligned}$$

Hence Pfluger's theorem implies

$$\begin{aligned} cap(P(f,r)) &\leq \liminf_{\alpha\to o} \frac{(1+\alpha)\sqrt{2+6\alpha}}{\sqrt{r/c+\alpha+\alpha^2}} \\ &= \frac{\sqrt{2c}}{\sqrt{r}} \\ &= \frac{k}{\sqrt{r}}. \end{aligned}$$

The first part of the theorem follows.

For the second part consider the Koebe function

$$f(z) = \frac{z}{(1-z)^2},\ z \in \Omega - \{1\}.$$

If $r > \frac{1}{4}$ there exists $\varphi \in (0,\pi)$ such that

$$R = \frac{1}{4\sin^2(\varphi/2)}.$$

Since

$$len(f([0,p))) \geq |f(p)|$$

for $p \in \partial U$, we have

$$A = \{e^{i\beta}\,|\,\beta \in [-\varphi,\varphi]\} \subseteq P(f,r).$$

Since the capacity of the circular arc A is

$$cap(A) = \sin\frac{\varphi}{2}$$

([9]) we obtain

$$cap(P(f,r)) \geq \frac{1}{2\sqrt{r}}.$$

This completes the proof of the theorem.

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