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SOLVABILITY OF A THIRD ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATION

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Abstract. This work deals with the existence of uncountably many bounded positive solutions for the third order nonlinear neutral delay differential equation

$$\frac{d^3}{dt^3}[x(t)+p(t)x(t-\tau)]+f(t,x(t-\tau_1),\dots,x(t-\tau_k))=0,\quad t\geq t_0,$$

where $\tau>0$, $\tau_i\in\mathbb{R}^+$ for $i\in\{1,2,\dots,k\}$, $p\in C([t_0,+\infty),\mathbb{R}^+)$ and $f\in C([t_0,+\infty)\times\mathbb{R}^k,\mathbb{R})$.

1. Introduction

The oscillation, nonoscillation and asymptotic behavior of first and second order neutral delay differential equations have been investigated by several authors, see, for example, [1]-[8] and the references cited therein.

Grammatikopoulos, Grove and Ladas [1] and Ladas and Sficas [4] studied the asymptotic properties of the first order neutral delay differential equation

$$\frac{d}{dt}[x(t)+px(t-\tau)]+qx(t-\delta)=0,\quad t\geq t_0, \tag{1.1}$$

where $q,\tau,\delta\in\mathbb{R}^+$ and $p\in\mathbb{R}$. Yu, Chen and Zhang [7] discussed the existence of a positive solution and an unbounded positive solution for the first order neutral delay differential equation

$$\frac{d}{dt}[x(t)+p(t)x(t-\tau)]+q(t)x(t-\delta)=0,\quad t\geq t_0, \tag{1.2}$$

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where $\tau, \delta \in \mathbb{R}^+$ and $p, q \in C([t_0, +\infty), \mathbb{R}^+)$. Öcalan [5] and Zhang, Feng, Yan and Song [8] investigated the existence of a positive solution for the below first order neutral delay differential equation with positive and negative coefficients

$$\begin{aligned} \frac{d}{dt}[x(t) + p(t)x(t-\tau)] &+ q_1(t)x(t-\delta_1) \\ &- q_2(t)x(t-\delta_2) = 0, \quad t \geq t_0, \end{aligned} \tag{1.3}$$

where $\tau > 0$, $\delta_1, \delta_2 \in \mathbb{R}^+$, $p \in C([t_0, +\infty), \mathbb{R})$, $q_1, q_2 \in C([t_0, +\infty), \mathbb{R}^+)$. Kulenović and Hadžiomerspahić [3] studied the second order linear neutral delay differential equation with positive and negative coefficients

$$\frac{d^2}{dt^2}[x(t) + px(t-\tau)] + q_1(t)x(t-\delta_1) - q_2(t)x(t-\delta_2) = 0, \quad t \geq t_0, \tag{1.4}$$

where $\tau > 0$, $\delta_1, \delta_2 \in \mathbb{R}^+$, $p \in \mathbb{R}$ and $q_1, q_2 \in C([t_0, +\infty), \mathbb{R}^+)$, and established the existence of a nonoscillatory solution for Eq.(1.4) by means of the Banach fixed point theorem.

The purpose of this paper is to study the third order nonlinear neutral delay differential equation

$$\begin{aligned} \frac{d^3}{dt^3}&[x(t) + p(t)x(t-\tau)] \\ &+ f(t, x(t-\tau_1), \ldots, x(t-\tau_k)) = 0, \quad t \geq t_0, \end{aligned} \tag{1.5}$$

where $\tau > 0$, $\tau_i \in \mathbb{R}^+$ for $i \in \{1, 2, \ldots, k\}$, $p \in C([t_0, +\infty), \mathbb{R})$ and $f \in C([t_0, +\infty) \times \mathbb{R}^k, \mathbb{R})$. By using the Banach fixed point theorem, we get two sufficient conditions for the existence of uncountably many bounded positive solutions of Eq.(1.5). Two nontrivial examples are considered to illustrate the results in this paper.

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\alpha = t_0 - \max\{\tau, \tau_i : i \in \{1, 2, \ldots, k\}\}$ and $C([\alpha, +\infty), \mathbb{R})$ stands for the set of all continuous and bounded functions on $[\alpha, +\infty)$ with norm $\|x\| = \sup_{t \geq \alpha} |x(t)|$ for each $x \in C([\alpha, +\infty), \mathbb{R})$ and

$$A(N, M) = \{x \in X : N \leq x(t) \leq M, t \in [\alpha, +\infty)\}, \quad M > N.$$

Obviously, $A(N,M)$ is a nonempty bounded closed subset of $C([\alpha,+\infty),\mathbb{R})$. By a solution of Eq.(1.5), we mean a function $x \in C([\alpha,+\infty),\mathbb{R})$ for some $T \geq t_0$, such that $x(t)+p(t)x(t-\tau)$ is thrice continuously differentiable on $[T,+\infty)$ and Eq.(1.5) holds for $t \geq T$.

2. Main results

Now we prove the existence of uncountably many bounded positive solutions for Eq.(1.5) by using the Banach fixed point theorem.

THEOREM 2.1. *Assume that there exist constants p^*, M, N and T_0 and functions $q, r \in C([t_0,+\infty),\mathbb{R}^+)$ satisfying*

$$\begin{aligned} &0 \leq p(t) \leq p^* < 1, \\ &\quad \forall t \geq T_0 > |t_0| + \max\{\tau, \tau_i : i \in \{1,2,\ldots,k\}\}; \end{aligned} \tag{2.1}$$

$$M > \frac{N}{1-p^*} > 0; \tag{2.2}$$

$$\begin{aligned} &|f(t,u_1,u_2,\ldots,u_k)| \leq q(t), \\ &\quad \forall (t,u_i) \in [t_0,+\infty) \times [N,M],\ i \in \{1,2,\ldots,k\}; \end{aligned} \tag{2.3}$$

$$\begin{aligned} &|f(t,u_1,u_2,\ldots,u_k) - f(t,\overline{u}_1,\overline{u}_2,\ldots,\overline{u}_k)| \\ &\quad \leq r(t)\max\{|u_i - \overline{u}_i| : 1 \leq i \leq k\}, \\ &\qquad \forall (t,u_i,\overline{u}_i) \in [t_0,+\infty) \times [N,M]^2,\ i \in \{1,2,\ldots,k\}; \end{aligned} \tag{2.4}$$

$$\int_{t_0}^{+\infty} s^2 \max\{q(s), r(s)\} ds < +\infty. \tag{2.5}$$

Then Eq.(1.5) has uncountably many bounded positive solutions in $A(N,M)$.

Proof. Let $L \in (p^*M+N, M)$. It follows from (2.1), (2.2) and (2.5) that there exist $\theta \in (0,1)$ and $T > T_0$ sufficiently large satisfying

$$\theta = p^* + \frac{1}{2}\int_T^{+\infty} s^2 r(s) ds, \tag{2.6}$$

$$\frac{1}{2}\int_T^{+\infty} s^2 q(s) ds \leq \min\{M-L, L-p^*M-N\}. \tag{2.7}$$

Define a mapping S_L: $A(N,M) \longrightarrow C([\alpha,+\infty),\mathbb{R})$ by

$$(2.8) \qquad (S_Lx)(t) = \begin{cases} L - p(t)x(t-\tau) \\ \quad +\dfrac{1}{2}\displaystyle\int_t^{+\infty}(s-t)^2 f(s,x(s-\tau_1), \\ \qquad \ldots, x(s-\tau_k))ds, \quad t \geq T,\ x \in A(N,M), \\ (S_Lx)(T), \quad \alpha \leq t < T,\ x \in A(N,M). \end{cases}$$

In terms of (2.1), (2.3), (2.7) and (2.8), we get that for every $x \in A(N,M)$ and $t \geq T$

$$\begin{aligned} (S_Lx)(t) &= L - p(t)x(t-\tau) \\ &\quad + \frac{1}{2}\int_t^{+\infty}(s-t)^2 f(s,x(s-\tau_1),\ldots,x(s-\tau_k))ds \\ &\leq L + \frac{1}{2}\int_t^{+\infty}(s-t)^2|f(s,x(s-\tau_1),\ldots,x(s-\tau_k))|ds \\ &\leq L + \frac{1}{2}\int_t^{+\infty} s^2 q(s)ds \\ &\leq L + \min\{M-L, L-p^*M-N\} \\ &\leq M \end{aligned}$$

and

$$\begin{aligned} (S_Lx)(t) &\geq L - p^*M - \frac{1}{2}\int_t^{+\infty}(s-t)^2|f(s,x(s-\tau_1),\ldots,x(s-\tau_k))|ds \\ &\geq L - p^*M - \frac{1}{2}\int_t^{+\infty} s^2 q(s)ds \\ &\geq L - p^*M - \min\{M-L, L-p^*M-N\} \\ &\geq N, \end{aligned}$$

which imply that $S_L(A(N,M)) \subseteq A(N,M)$. In view of (2.1), (2.4), (2.7) and (2.8), we deduce that for every $x_1, x_2 \in A(N,M)$ and $t \geq T$

$$\begin{aligned} &|(S_Lx_1)(t) - (S_Lx_2)(t)| \\ &= \bigg| -p(t)x_1(t-\tau) + \frac{1}{2}\int_t^{+\infty}(s-t)^2 f(s,x_1(s-\tau_1),\ldots,x_1(s-\tau_k))ds \\ &\qquad + p(t)x_2(t-\tau) - \frac{1}{2}\int_t^{+\infty}(s-t)^2 f(s,x_2(s-\tau_1),\ldots,x_2(s-\tau_k))ds \bigg| \end{aligned}$$

$$
\begin{aligned}
&\le p(t)|x_1(t-\tau)-x_2(t-\tau)| \\
&\quad + \frac{1}{2}\int_t^{+\infty}(s-t)^2|f(s,x_1(s-\tau_1),\dots,x_1(s-\tau_k)) \\
&\qquad - f(s,x_2(s-\tau_1),\dots,x_2(s-\tau_k))|ds \\
&\le p^*\|x_1-x_2\| \\
&\quad + \frac{1}{2}\int_T^{+\infty} s^2 r(s)\max\{|x_1(s-\tau_i)-x_2(s-\tau_i)| : 1\le i\le k\}ds \\
&\le p^*\|x_1-x_2\| + \left(\frac{1}{2}\int_T^{+\infty} s^2 r(s)ds\right)\|x_1-x_2\| \\
&\le \theta\|x_1-x_2\|,
\end{aligned}
$$

which gives that

$$
\|S_L x_1 - S_L x_2\| \le \theta\|x_1-x_2\|, \quad \forall x_1,x_2\in A(N,M), \tag{2.9}
$$

which yields that S_L is a contraction mapping and it has a unique fixed point $x\in A(N,M)$, which is a bounded positive solution of Eq.(1.5).

Now we show that Eq.(1.5) has uncountably many bounded positive solutions in $A(N,M)$. Let $L_1,L_2\in(p^*M+N,M)$ with $L_1\neq L_2$. For $i\in\{1,2\}$, as in the above proof, we can conclude that there exist constants $\theta_i\in(0,1), T_i>T_0$ and a mapping $S_{L_i}: A(N,M)\to C\big([\alpha,+\infty),\mathbb{R}\big)$ satisfying (2.6)-(2.8), where θ,T,L and S_L are replaced by θ_i,T_i,L_i and S_{L_i}, respectively, and S_{L_i} has a unique fixed point $z_k\in A(N,M)$. It is clear that z_1 and z_2 are bounded positive solutions of Eq.(1.5) in $A(N,M)$. That is,

$$
\begin{aligned}
z_1(t) &= L_1 - p(t)z_1(t-\tau) \\
&\qquad + \frac{1}{2}\int_t^{+\infty}(s-t)^2 f(s,z_1(s-\tau_1),\dots,z_1(s-\tau_k))ds
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
z_2(t) &= L_2 - p(t)z_2(t-\tau) \\
&\qquad + \frac{1}{2}\int_t^{+\infty}(s-t)^2 f(s,z_2(s-\tau_1),\dots,z_2(s-\tau_k))ds.
\end{aligned}
\tag{2.11}
$$

Next we only need to show that $z_1\neq z_2$. It follows from (2.1), (2.4), (2.10) and (2.11) that for any $t \ge \max\{T_1, T_2\}$

$$
\begin{aligned}
&|z_1(t) - z_2(t)| \\
&= \Big| L_1 - L_2 - p(t)z_1(t-\tau) + p(t)z_2(t-\tau) \\
&\qquad + \frac{1}{2}\int_t^{+\infty} (s-t)^2 f(s, z_1(s-\tau_1), \dots, z_1(s-\tau_k))ds \\
&\qquad - \frac{1}{2}\int_t^{+\infty} (s-t)^2 f(s, z_2(s-\tau_1), \dots, z_2(s-\tau_k))ds \Big| \\
&\ge |L_1 - L_2| - p^*\|z_1 - z_2\| \\
&\quad - \frac{1}{2}\int_t^{+\infty} s^2 \big| f(s, z_1(s-\tau_1), \dots, z_1(s-\tau_k)) \\
&\qquad - f(s, z_2(s-\tau_1), \dots, z_2(s-\tau_k)) \big| ds \\
&\ge |L_1 - L_2| - p^*\|z_1 - z_2\| - \left(\int_{\max\{T_1,T_2\}}^{+\infty} s^2 r(s)ds \right) \|z_1 - z_2\| \\
&\ge |L_1 - L_2| - \max\{\theta_1, \theta_2\}\|z_1 - z_2\|,
\end{aligned}
$$

which implies that

$$
\|z_1 - z_2\| \ge \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0,
$$

that is, $z_1 \ne z_2$. This completes the proof. □

THEOREM 2.2. *Assume that there exist constants* p^*, M, N *and* T_0 *and functions* $q, r \in C([t_0, +\infty), \mathbb{R}^+)$ *satisfying* (2.2)-(2.5) *and*

$$
\begin{aligned}
&-1 < -p^* \le p(t) \le 0, \\
&\quad \forall t \ge T_0 > |t_0| + \max\{\tau, \tau_i : i \in \{1, 2, \dots, k\}\}.
\end{aligned}
\tag{2.12}
$$

Then Eq.(1.5) *has uncountably many bounded positive solutions in* $A(N, M)$.

Proof. Let $L \in (N, M(1 - p^*))$. It follows from (2.2), (2.5) and (2.12) that there exist $\theta \in (0, 1)$ and $T > T_0$ sufficiently large satisfying

$$
\frac{1}{2}\int_T^{+\infty} s^2 q(s)ds \le \min\{M(1-p^*) - L, L - N\}. \tag{2.13}
$$

Define a mapping $S_L : A(N, M) \longrightarrow C([\alpha, +\infty), \mathbb{R})$ by (2.8).

Using (2.3), (2.4), (2.8), (2.12) and (2.13), we get that for every $x, y \in A(N, M)$ and $t \geq T$

$$\begin{aligned}(S_L x)(t) &= L - p(t)x(t-\tau) \\ &\quad + \frac{1}{2}\int_t^{+\infty} (s-t)^2 f(s, x(s-\tau_1), \ldots, x(s-\tau_k))ds \\ &\leq L + p^* M + \frac{1}{2}\int_t^{+\infty} s^2 |f(s, x(s-\tau_1), \ldots, x(s-\tau_k))|ds \\ &\leq L + p^* M + \frac{1}{2}\int_t^{+\infty} s^2 q(s)ds \\ &\leq L + \min\{M(1-p^*) - L, L - N\} \\ &\leq M,\end{aligned}$$

$$\begin{aligned}(S_L x)(t) &\geq L - \frac{1}{2}\int_t^{+\infty} (s-t)^2 |f(s, x(s-\tau_1), \ldots, x(s-\tau_k))|ds \\ &\geq L - \frac{1}{2}\int_t^{+\infty} s^2 q(s)ds \\ &\geq L - \min\{M(1-p^*) - L, L - N\} \\ &\geq N\end{aligned}$$

and

$$\begin{aligned}&|(S_L x)(t) - (S_L y)(t)| \\ &= \Big| - p(t)x(t-\tau) + \frac{1}{2}\int_t^{+\infty} (s-t)^2 f(s, x(s-\tau_1), \ldots, x(s-\tau_k))ds \\ &\qquad + p(t)y(t-\tau) - \frac{1}{2}\int_t^{+\infty} (s-t)^2 f(s, y(s-\tau_1), \ldots, y(s-\tau_k))ds\Big| \\ &\leq |p(t)||x(t-\tau) - y(t-\tau)| \\ &\quad + \frac{1}{2}\int_t^{+\infty} (s-t)^2 |f(s, x(s-\tau_1), \ldots, x(s-\tau_k)) \\ &\qquad - f(s, y(s-\tau_1), \ldots, y(s-\tau_k))|ds \\ &\leq p^* \|x - y\| + \left(\frac{1}{2}\int_T^{+\infty} s^2 r(s)ds\right)\|x - y\| \\ &\leq \theta \|x - y\|,\end{aligned}$$

which imply that $S_L(A(N, M)) \subseteq A(N, M)$ and (2.9) holds. Thus S_L has a unique fixed point $x \in A(N, M)$, which is a bounded positive solution of

Eq.(1.5). The rest of the proof is similar to that of Theorem 2.1, and is omitted. This completes the proof. □

3. Examples

Finally we construct two examples to show applications of the results in section 2.

EXAMPLE 3.1. Consider the following third order nonlinear neutral delay differential equation

$$
\begin{aligned}
&\frac{d^3}{dt^3}\left[x(t) + \frac{t^2 \sin^2(t^5-1)}{1+3t^2} x(t-\tau)\right] \\
&\quad + \frac{\sqrt{1+2t}\,x^2(t-5)x^3(t-9)}{t^4 + (1+2t^2+3t^3)\ln(1+t^2)} = 0, \quad t \geq 1,
\end{aligned} \tag{3.1}
$$

where $\tau > 0$ is a constant. Let $t_0 = 1$, $k = 2$, $M = 4$, $N = 2$, $p^* = \frac{1}{3}, T_0 = 2 + \max\{9, \tau\}$ and

$$
\begin{aligned}
p(t) &= \frac{t^2 \sin^2(t^5-1)}{1+3t^2}, \quad \tau_1 = 5, \quad \tau_2 = 9, \\
f(t,u,v) &= \frac{\sqrt{1+2t}\,u^2 v^3}{t^4 + (1+2t^2+3t^3)\ln(1+t^2)}, \quad q(t) = \frac{M^5\sqrt{1+2t}}{t^4}, \\
r(t) &= \frac{5M^4\sqrt{1+2t}}{t^4}, \quad \forall (t,u,v) \in [t_0, +\infty) \times \mathbb{R}^2.
\end{aligned}
$$

It is easy to see that (2.1)-(2.5) are fulfilled. It follows from Theorem 2.1 that Eq.(3.1) has uncountably many bounded positive solutions in $A(N, M)$.

EXAMPLE 3.2. Consider the following third order nonlinear neutral delay differential equation

$$
\begin{aligned}
&\frac{d^3}{dt^3}\left[x(t) - \frac{t^2 \ln(1+t^4)}{1+2t^2\ln(1+t^4)} x(t-\tau)\right] \\
&\quad + \frac{tx^7(t-2) - \ln\left(1+t^2x^2(t-3)\right)}{(1+t)^4 + tx^2(t-49)} = 0, \quad t \geq 0,
\end{aligned} \tag{3.2}
$$

where $\tau > 0$ is a constant. Let $t_0 = 0$, $k = 3$, $M = 7$, $N = 3$, $p^* = \frac{1}{2}$,

$T_0 = 1 + \max\{49, \tau\}$ and

$$p(t) = -\frac{t^2 \ln(1+t^4)}{1+2t^2\ln(1+t^4)}, \quad \tau_1 = 2, \quad \tau_2 = 3, \quad \tau_3 = 49,$$
$$f(t,u,v,w) = \frac{tu^7 - \ln\left(1+t^2v^2\right)}{(1+t)^4 + tw^2}, \quad q(t) = \frac{tM^7 + \ln(1+M^2t^2)}{(1+t)^4 + tN^2},$$
$$r(t) = \frac{Q}{[(1+t)^4 + tN^2]^2}, \quad \forall (t,u,v,w) \in [t_0, +\infty) \times \mathbb{R}^3,$$

where $Q = Mt[9M^7t + 2M^2t^2 + 7M^5(1+t)^4 + 2t(1+t)^4 + 2\ln(1+M^2t^2)]$. It is easy to see that (2.2)-(2.5) and (2.12) are fulfilled. It follows from Theorem 2.2 that Eq.(3.2) has uncountably many bounded positive solutions in $A(N,M)$.

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