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REMARK FOR CALCULATING THE BEST SIMULTANEOUS APPROXIMATIONS

Hyang Joo Rhee*

Abstract. This paper is concerned with algorithm for calculating one-sided best simultaneous approximation, in the case of continuous functions. we will apply any subgradient algorithm. In other words, we consider the algotithms from a mathematical, rather then computational, point of view.

1. Introduction

We assume that W is a normed linear space and K is a nonempty subset of W. For any subset F of W, we define

$$d(F, K) := \inf_{k \in K} \sup_{f \in F} ||f - k||$$

and the elements in K which attain the above infimum are called a best simultaneous approximation for F from K.

Lemma 1.1. [6] *Suppose that K is a nonempty closed convex subset of a finite-dimensional subspace of a normed linear space W. For any compact subset $F \subset W$, there exists a best simultaneous approximation for F from K.*

Throughout this article, we denote that X is a compact subset of $\mathbb{R}^m$ satisfying $X = \overline{\text{int} X}$, S is an n-dimensional subspace of $C(X)$ with the $L_1(X, \mu)$-norm where μ is an admissible measure on X.

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Suppose that we are given $l-$tuple $F = \{f_1, \cdots, f_\ell\}$ in $C(X)$ with

$$S(F) = \bigcap_{i=1}^{\ell} \{s \in S | \ s \leq f_i\}$$

is non-empty. We define

$$d(F, S(F)) := \inf_{s \in S(F)} \sup_{f \in F} ||f - s||_1$$

and the elements in $S(F)$ which attain the above infimum are called a one-sided best simultaneous approximation for F from $S(F)$. The case by $\ell = 1$, we called a one-sided best approximation for f from $S(f)$ [5].

Finding a one-sided best simultaneous approximation for F from $S(F)$ is equivalent to finding a $s \in S(F)$ satisfying

$$\sup\{\int_X s \ d\mu | \ s \in S(F)\}.$$

Since $S(F)$ is closed and convex, we have that $S(F) \neq \phi$ for all $l-$tuple $F = \{f_1, \cdots, f_\ell\}$ in $C(X)$ if and only if S contains a strictly positive function. By lemma 1.0.1., if $S(F)$ is nonempty, then there exists a one-sided best simultaneous approximation for F from $S(F)$. We choose and fix a basis $s_1, \cdots, s_n$ for S where s_1 is strictly positive and

$$\int_X s_1 d\mu = 1,$$

while

$$\int_X s_i d\mu = 0, \quad i = 2, \cdots, n.$$

Thus our problem can be reformulated as

$$\max\{a_1 \ | \ \sum_{i=1}^{n} a_i s_i \leq f_j, \ j = 1, \cdots, \ell\}.$$

For each $\hat{a} = (a_2, \cdots, a_n) \in \mathbb{R}^{n-1}$, define

$$h_j(\hat{a}) = \min\{\frac{f_j(x) - \sum_{i=2}^{n} a_i s_i(x)}{s_1(x)} : \ x \in X\}$$

and $H(\hat{a}) = \min_{1 \leq j \leq \ell} h_j(\hat{a})$.

Remark 1.2. Since s_1 is strictly positive,

$$\begin{aligned} \max\{a_1 \ | \ \sum_{i=1}^{n} a_i s_i \leq f_j, \ j = 1, \cdots, \ell\} &= \max_{\hat{a} \in \mathbb{R}^{n-1}} \min_{1 \leq j \leq \ell} h_j(\hat{a}) \\ &= \max_{\hat{a} \in \mathbb{R}^{n-1}} H(\hat{a}). \end{aligned}$$

2. A subgadient to H at $\hat{a}$

In this section, we shall consider vector in $\mathbb{R}^{n-1}$ as being indexed 2 to n. This algorithm is based on ideas from gradients and subgradients.

Corollary 2.1. *The function H is a continuous concave function on $\mathbb{R}^{n-1}$.*

Proof. For each h_j $(1 \le j \le \ell)$, the continuity and concavity may be proved as follows. The continuity is obvious. Let $\hat{a}, \hat{b} \in \mathbb{R}^{n-1}$. By definition

$$f_j \ge h_j(\hat{a})s_1 + \sum_{i=2}^{n} a_i s_i$$

$$f_j \ge h_j(\hat{b})s_1 + \sum_{i=2}^{n} b_i s_i$$

on all of X. Thus, for any $\lambda \in [0,1]$,

$$f_j \ge (\lambda h_j(\hat{a}) + (1-\lambda)h_j(\hat{b}))s_1 + \sum_{i=2}^{n}(\lambda a_i + (1-\lambda)b_i)s_i$$

on X. Which implies that

$$h_j(\lambda\hat{a} + (1-\lambda)\hat{b}) \ge \lambda h_j(\hat{a}) + (1-\lambda)h_j(\hat{b}).$$

Thus h_j is concave. By definition, $H(\hat{a}) = \min_{1\le j\le \ell} h_j(\hat{a})$, the function H is a continuous concave function on a one-sided best simultaneous approximation for F from $S(F)$. □

We also have by definition that H is finite on $\mathbb{R}^{n-1}$. We claim that

$$\lim_{||\hat{a}||\to\infty} H(\hat{a}) = -\infty$$

where $||\cdot||$ is any norm on $\mathbb{R}^{n-1}$. To see this, set $T = \text{span}\,\{s_2, \cdots, s_n\}$. Since T is a finite dimensional subspace of $C(X)$, and $\int_X t d\mu = 0$ for all $t \in T$, we necessarily have

$$\lim_{||\hat{a}||\to\infty} \max\{\sum_{i=2}^{n} a_i s_i(x) : x \in X\} = \infty.$$

Thus

$$\lim_{||\hat{a}||\to\infty} \min\{\frac{f_j(x) - \sum_{i=2}^{n} a_i s_i(x)}{s_1(x)} : x \in X\} = -\infty,$$

i.e., $\lim_{||\hat{a}||\to\infty} h_j(\hat{a}) = -\infty$, so $\lim_{||\hat{a}||\to\infty} H(\hat{a}) = -\infty$.

Maximizing H over $\mathbb{R}^{n-1}$ is therefore a problem of maximizing a concave function.

DEFINITION 2.2. Let H be as above and $\hat{a} \in \mathbb{R}^{n-1}$. A vector $g \in \mathbb{R}^{n-1}$ is said to be a subgradient to H at $\hat{a}$ if

$$H(\hat{b}) - H(\hat{a}) \leq (g, \hat{b} - \hat{a})$$

for all $\hat{b} \in \mathbb{R}^{n-1}$. We let $G(\hat{a})$ denote the set of subgradients to H at $\hat{a}$.

DEFINITION 2.3. Let H be as above and if $G(\hat{a})$ is a singleton, then this singleton is called the gradient to H at $\hat{a}$.

By definition, a gradient to H exists at $\hat{a}$ if and only if there is a unique supporting hyperplane to H at $\hat{a}$. Thus we can take a remark that a^* is a maximum point of H if and only if $0 \in G(a^*)$.

Since $G(\hat{a})$ is a compact convex set, it is uniquely determined by its extreme points. These extreme points are related to one-sided directional derivatives as follows.

PROPOSITION 2.4. [4] *Let H be as above and $\hat{a} \in \mathbb{R}^{n-1}$. For each $d \in \mathbb{R}^{n-1}$*

$$\lim_{t \to 0^+} \frac{H(\hat{a} + td) - H(\hat{a})}{t} = H'_d(\hat{a})$$

exists. Furthermore,

$$H'_d(\hat{a}) = \min\{(g, d) : g \in G(\hat{a})\}.$$

As a consequence of proposition, the above definition of a gradient implies the existence of the partial derivatives to H at $\hat{a}$.

For any $\hat{a} \in \mathbb{R}^{n-1}$, set

$$Z(\hat{a}) = \{x : (f_{j_0} - H(\hat{a})s_1 - \sum_{i=2}^{n} a_i s_i)(x) = 0 \ \text{ where } \ h_{j_0}(\hat{a}) = H(\hat{a})\}.$$

By definition $Z(\hat{a}) \neq \phi$ for each $\hat{a} \in \mathbb{R}^{n-1}$. For each $x \in Z(\hat{a})$, set

$$g^x = (-s_2(x)/s_1(x), \cdots, -s_n(x)/s_1(x)).$$

Let $\tilde{G}(\hat{a})$ denote the convex hull of the set of vectors $\{g^x : x \in Z(\hat{a})\}$. Then the set $\tilde{G}(\hat{a})$ is closed since $Z(\hat{a})$ are closed.

THEOREM 2.5. *The set $\tilde{G}(\hat{a})$ is the set of subgradients to H at $\hat{a}$.*

Proof. For each $x \in Z(\hat{a})$, there exists $j \in \{1, \cdots, \ell\}$ such that

$$f_j(x) = H(\hat{a})s_1(x) + \sum_{i=2}^{n} a_i s_i(x)$$

$$= h_j(\hat{a})s_1(x) + \sum_{i=2}^{n} a_i s_i(x).$$

By definition, that is $f_j \geq h_j(\hat{b})s_1 + \sum_{i=2}^{n} b_i s_i$, for each $\hat{b} \in \mathbb{R}^{n-1}$,

$$f_j(x) \geq h_j(\hat{b})s_1(x) + \sum_{i=2}^{n} b_i s_i(x)$$

$$\geq H(\hat{b})s_1(x) + \sum_{i=2}^{n} b_i s_i(x).$$

Since $s_1(x) \geq 0$,

$$H(\hat{b}) - H(\hat{a}) \leq \sum_{i=2}^{n} a_i \frac{s_i(x)}{s_1(x)} - \sum_{i=2}^{n} b_i \frac{s_i(x)}{s_1(x)}$$

$$= \sum_{i=2}^{n} (a_i - b_i) \frac{s_i(x)}{s_1(x)}$$

$$= (g^x, \hat{b} - \hat{a}).$$

So g^x is a subgradient to H at $\hat{a}$, that is, $\tilde{G}(\hat{a}) \subset G(\hat{a})$.

It remains to prove that all subgradients to H at $\hat{a}$ are in $\tilde{G}(\hat{a})$. Suppose that $\tilde{G}(\hat{a}) \neq G(\hat{a})$. Then $\tilde{G}(\hat{a}) \subset G(\hat{a})$, and $\tilde{G}(\hat{a}), G(\hat{a})$ are both convex and compact, there exists a $g^* \in G(\hat{a})$ and $d \in \mathbb{R}^{n-1}$ for which

$$(g, d) > (g^*, d)$$

for all $g \in \tilde{G}(\hat{a})$. Thus

$$\min\{(g, d) : g \in \tilde{G}(\hat{a})\} > \min\{(g, d) : g \in G(\hat{a})\} = H'_d(\hat{a}).$$

If the strictly inequality will be equality, we have proved our result.

It suffices to prove that, for each $d \in \mathbb{R}^{n-1}$, there exists a $g \in \tilde{G}(\hat{a})$ for which

$$H'_d(\hat{a}) = \lim_{t \to 0^+} \frac{H(\hat{a} + td) - H(\hat{a})}{t} = (g, d).$$

We therefore calculate $H'_d(\hat{a})$. Set

$$g(x) = \frac{f_{j_0}(x) - \sum_{i=2}^{n} a_i s_i(x)}{s_1(x)} - H(\hat{a})$$

and $v_i(x) = \frac{s_i(x)}{s_1(x)}$, $i = 2, \cdots, n$, where $h_{j_0}(\hat{a}) = H(\hat{a})$. Note that $\min\{g(x) : x \in X\} = 0$.

Now, for each $d \in \mathbb{R}^{n-1}$,

$$\frac{H(\hat{a}+td)-H(\hat{a})}{t} = \frac{1}{t}[h_{j_0}(\hat{a}+td)-h_{j_0}(\hat{a})]$$

$$= \frac{1}{t}[\min_{x\in X}\frac{f_{j_0}(x)-\sum_{i=2}^{n}(a_i+td_i)s_i(x)}{s_1(x)} - \min_{x\in X}\frac{f_{j_0}(x)-\sum_{i=2}^{n}a_is_i(x)}{s_1(x)}]$$

$$= \frac{1}{t}[\min_{x\in X}\frac{f_{j_0}(x)-\sum_{i=2}^{n}a_is_i(x)-t\sum_{i=2}^{n}d_is_i(x)}{s_1(x)} - H(\hat{a})]$$

$$= \frac{1}{t}\min_{x\in X}[\frac{f_{j_0}(x)-\sum_{i=2}^{n}a_is_i(x)}{s_1(x)} - t\sum_{i=2}^{n}d_iv_i(x) - H(\hat{a})]$$

$$= \frac{1}{t}\min_{x\in X}\{g(x)-t\sum_{i=2}^{n}d_iv_i(x)\}.$$

For each $x \in Z(\hat{a})$,

$$\frac{1}{t}\{g(x)-t\sum_{i=2}^{n}d_iv_i(x)\} = -\sum_{i=2}^{n}d_iv_i(x).$$

Thus

$$H'_d(\hat{a}) = \lim_{t\to 0^+}\frac{H(\hat{a}+td)-H(\hat{a})}{t} = \lim_{t\to 0^+}\frac{1}{t}\min_{x\in X}\{g(x)-t\sum_{i=2}^{n}d_iv_i(x)\}$$

$$\le \min_{x\in Z(\hat{a})} -\sum_{i=2}^{n}d_iv_i(x)$$

$$= \min_{x\in Z(\hat{a})}(g^x, d).$$

If equality holds, that is, for each $d \in \mathbb{R}^{n-1}$, there exists a $g \in \tilde{G}(\hat{a})$ for which

$$H'_d(\hat{a}) = \min\{(g^x, d) : g^x \in G(\hat{a})\}$$
$$= \min\{(g^x, d) : g^x \in \tilde{G}(\hat{a})\},$$

then by proposition 2.0.6., the proof is complete, that is, $\tilde{G}(\hat{a}) = G(\hat{a})$. We assume that equality does not hold. Set

$$c^* = \min_{x\in Z(\hat{a})} -\sum_{i=2}^{n}d_iv_i(x) = -\max_{x\in Z(\hat{a})}\sum_{i=2}^{n}d_iv_i(x).$$

Assume that, there exists a $\delta > 0$ such that

$$H'_d(a) \le c^* - \delta.$$

Let $t_k \to 0$ and $x_k \in X$ satisfy

$$H'_d(\hat{a}) = \lim_{k\to\infty} \frac{1}{t_k}\{g(x_k) - t_k \sum_{i=2}^{n} d_i v_i(x_k)\} \le c^* - \delta.$$

Because the limit exists, it follows that

$$\lim_{k\to\infty} g(x_k) = 0.$$

Since X is compact, there exists a subsequence of the $\{x_k\}$, again denoted by $\{x_k\}$, converging to some x^*. Since g is continuous and therefore $g(x^*) = 0$, i.e., $x^* \in Z(\hat{a})$.

The function $\sum_{i=2}^{n} d_i v_i$ is continuous and there exists a K_1 such that for all $k \ge K_1$

$$|\sum_{i=2}^{n} d_i v_i(x_k) - \sum_{i=2}^{n} d_i v_i(x^*)| < \frac{\delta}{2}.$$

Thus for $k \ge K_1$

$$c^* + \sum_{i=2}^{n} d_i v_i(x_k) = -\max_{x\in Z(\hat{a})} \sum_{i=2}^{n} d_i v_i(x) + \sum_{i=2}^{n} d_i v_i(x_k)$$

$$\le -\sum_{i=2}^{n} d_i v_i(x^*) + \sum_{i=2}^{n} d_i v_i(x_k) < \frac{\delta}{2}.$$

There exists a K_2 such that for all $k \ge K_2$

$$g(x_k) \le t_k\{c^* - \frac{\delta}{2} + \sum_{i=2}^{n} d_i v_i(x_k)\}.$$

Thus, for all $k \ge \max\{K_1, K_2\}$,

$$g(x_k) \le t_k\{c^* - \frac{\delta}{2} + \sum_{i=2}^{n} d_i v_i(x_k)\} < 0.$$

Hence $g(x) \ge 0$ for all $x \in X$. This contradiction proves the theorem. □

As we have shown, the subgradients to H at $\hat{a}$ are easily determined theoretically. We can apply any subgradient algorithm. Suffice it to say that convergence is generally very slow, at least in theory. This paper is concerned with algorithms for calculating best one-sided simultaneous approximations. And this algorithms will expand the algorithms for calculating best two-sided simultaneous approximations.

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*
Department of Mathematics
Duksung Women's University
Seoul 132-714, Republic of Korea
E-mail: rhj@duksung.ac.kr