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REMARKS CONCERNING SOME GENERALIZED CESÀRO OPERATORS ON ℓ^2

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In memory of Harvey Lynn Elder (1934-2009)

Abstract. Here we see that the p-Cesàro operators, the generalized Cesàro operators of order one, the discrete generalized Cesàro operators, and their adjoints are all posinormal operators on ℓ^2, but many of these operators are not dominant, not normaloid, and not spectraloid. The question of dominance for C_k, the generalized Cesàro operators of order one, remains unsettled when $\frac{1}{2} \leq k < 1$, and that points to some general questions regarding terraced matrices.

Sufficient conditions are given for a terraced matrix to be normaloid. Necessary conditions are given for terraced matrices to be dominant, spectraloid, and normaloid. A very brief new proof is given of the well-known result that C_k is hyponormal when $k \geq 1$.

1. Introduction

A lower triangular infinite matrix $M = M\ (\{a_n\},\{c_n\})$, acting through multiplication to give a bounded linear operator on ℓ^2, is *factorable* if its nonzero entries m_{ij} satisfy $m_{ij} = a_i c_j$ where a_i depends only on i and c_j depends only on j. A factorable matrix is *terraced* (see [5, 10]) if $c_j = 1$ for all j. For the operator M on ℓ^2 to be *posinormal*, there must exist a positive operator P on ℓ^2 satisfying $MM^* = M^*PM$. These operators were introduced and studied in [11], where it was observed that the set of all posinormal operators on any Hilbert space is an enormous collection that includes every invertible operator and all the hyponormal operators. A *dominant* operator M is one for which $Ran(M - \lambda) \subset Ran(M - \lambda)^*$ for all λ in the spectrum of M (see [15]), and a *hyponormal* operator M

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satisfies $\langle (M^*M - MM^*)f, f\rangle \geq 0$ for all f in ℓ^2. Hyponormal operators are necessarily dominant; it was shown in [11, Proposition 3.5] that M is dominant if and only if $M - \lambda$ is posinormal for all complex λ, so it follows that

$$\{\text{posinormal operators}\} \supset \{\text{dominant operators}\} \\ \supset \{\text{hyponormal operators}\}.$$

Here we will explore how these properties apply to three special types of factorable matrices that arise as generalized Cesàro matrices. This will lead us to a more general situation involving some terraced matrices and also to sufficient conditions for a terraced matrix to be normaloid, as well as necessary conditions for terraced matrices to be dominant, spectraloid, and normaloid.

Before proceeding, we recall that the Cesàro matrix C is the terraced matrix that arises when $a_i = \frac{1}{i+1}$ for all i. In [**1**] it is shown that C is bounded, noncompact, and hyponormal; its norm is $||C|| = 2$, and its spectrum is

$$\sigma(C) = \{\lambda : |\lambda - 1| \leq 1\}.$$

Since C is hyponormal, it is necessarily dominant and posinormal. C^* is shown to be posinormal in [11, Theorem 5.3], and it is a consequence of [13, Theorem 3.3] that C^* cannot be dominant.

2. On posinormality and dominance

First we will consider the p-Cesàro matrices and the generalized Cesàro matrices of order one, and these examples will point us toward some questions concerning terraced matrices. Throughout this section we will make repeated use of the following results found in [11, Theorem 2.1 and Corollary 2.3].

PROPOSITION 2.1. *For a bounded linear operator M on ℓ^2, the following statements are equivalent:*
(1) M is posinormal;
(2) $RanM \subseteq RanM^*$;
(3) $MM^* \leq \gamma^2 M^*M$ for some $\gamma \geq 0$; and
(4) There exists a bounded operator T on ℓ^2 such that $M = M^*T$.

PROPOSITION 2.2. *If M is posinormal, then $KerM \subseteq KerM^*$.*

For fixed $p \geq 1$, the p-Cesàro matrix M_p is the terraced matrix associated with the sequence defined by $a_i = \frac{1}{(i+1)^p}$ for all i. Note that

$M_1 = C$. In [9] it is shown that when $p > 1$ these operators are bounded, compact, and not hyponormal; they have spectrum

$$\sigma(M_p) = \{\frac{1}{(n+1)^p} : n \geq 0\} \cup \{0\}.$$

THEOREM 2.3. *(1) Both M_p and M_p^* are posinormal for $p > 1$.*
(2) Both M_p and M_p^* are not dominant for $p > 1$.

Proof. (1) From [11, Theorem 2.2] we know that M_p is posinormal for all $p > 1$. To begin our proof that M_p^* is posinormal, we define $T = [t_{mn}]$ by

$$t_{mn} = \begin{cases} \frac{(n+1)^p - n^p}{(m+1)^p} & if \quad n \leq m; \\ -1 & if \quad n = m+1; \\ 0 & if \quad n > m+1. \end{cases}$$

Let U denote the unilateral shift. Then $T + U^*$ is a lower triangular matrix whose entries are all nonnegative and dominated by the corresponding entries of pC, a fact that relies heavily on the inequality $1 - \frac{n^p}{(n+1)^p} \leq \frac{p}{n+1}$ for all n and for all $p > 1$ (see [4, Theorem 42, 2.15.3, page 40]). It follows that $T + U^*$ is a bounded operator, and hence T is also bounded. A routine computation shows that $M_p = TM_p^*$ and hence $M_p^* = M_p T^*$, so M_p^* is posinormal for all $p > 1$.
(2) To see that M_p cannot be dominant, we consider f defined as follows: $f(0) = 1$ and $f(n) = \prod_{j=1}^{n} \frac{j^p}{(j+1)^p - 1}$ for $n \geq 1$. In [9] it is verified that $f \in \ell^2$ is an eigenvector for M_p associated with eigenvalue 1, so $f \in Ker(M_p - I)$; but $f \notin Ker(M_p^* - I)$ since $((M_p^* - I)f)(0) > 0$. It follows that $M_p - I$ cannot be posinormal, and therefore M_p is not dominant. Similarly, M_p^* is not dominant since $g :\equiv < 1, 0, 0, 0, ... >^T \in Ker(M_p^* - I)$ but $g \notin Ker(M_p - I)$. □

For fixed $k > 0$, the generalized Cesàro matrices of order one are the terraced matrices C_k that occur when $a_i = \frac{1}{k+i}$ for all i. From [11, Theorem 5.1] we know that these operators have spectrum

$$\sigma(C_k) = \{\lambda : |\lambda - 1| \leq 1\} \cup \{\frac{1}{k}\}$$

and are, consequently, not compact for $k > 0$.

THEOREM 2.4. *(1) Both C_k and C_k^* are posinormal for $k > 0$.*
(2) (a) C_k is hyponormal and hence dominant if $k \geq 1$, but C_k is not dominant for $0 < k < \frac{1}{2}$; and (b) C_k^* is not dominant for $k > 0$.

Proof. (1) See [11, Theorems 5.2 and 5.3] for the proof that C_k is posinormal and the proof that C_k^* is posinormal for $k > 0$.
(2) (a) If Q denotes the diagonal matrix with diagonal $\{k, 1, 1, 1, ...\}$ and P denotes the diagonal matrix with diagonal $\{\frac{k+i}{k+i+1} : i = 0, 1, 2, 3, ...\}$, it can be verified that

$$C_k Q C_k^* = \begin{pmatrix} \frac{1}{k} & \frac{1}{k+1} & \frac{1}{k+2} & \cdots \\ \frac{1}{k+1} & \frac{1}{k+1} & \frac{1}{k+2} & \cdots \\ \frac{1}{k+2} & \frac{1}{k+2} & \frac{1}{k+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = C_k^* P C_k$$

for all $k > 0$. Therefore

$$\begin{aligned}\langle (C_k^* C_k - C_k C_k^*) f, f\rangle &= \langle (C_k^* C_k - C_k^* P C_k + C_k Q C_k^* - C_k C_k^*) f, f\rangle \\ &= \langle (I - P) C_k f, C_k f\rangle + \langle (Q - I) C_k^* f, C_k^* f\rangle \geq 0\end{aligned}$$

for all f in ℓ^2 when $k \geq 1$, so C_k is hyponormal and therefore also dominant for those values of k. To see that C_k cannot be dominant for $0 < k < \frac{1}{2}$, we consider f defined as follows: $f(0) = 1$ and $f(n) = \frac{\prod_{j=0}^{n-1}(k+j)}{n!}$ for $n \geq 1$. Using Raabe's test (see [6, page 396]), it can be verified that $f \in \ell^2$ is an eigenvector for C_k associated with eigenvalue $\frac{1}{k}$ for $0 < k < \frac{1}{2}$, so $f \in Ker(C_k - \frac{1}{k})$; but $f \notin Ker(C_k^* - \frac{1}{k})$ since $((C_k^* - \frac{1}{k})f)(0) > 0$. Therefore $C_k - \frac{1}{k}$ is not posinormal for $0 < k < \frac{1}{2}$, and hence C_k cannot be dominant for those values of k. (b) It is a consequence of [13, Theorem 3.3] that C_k^* is not dominant for $k > 0$. □

See [11, 14] for earlier proofs of the hyponormality of C_k when $k \geq 1$.

Corollary 2.5. *$\frac{1}{k}||C_k f||^2 \geq ||C_k^* f||^2$ for all f in ℓ^2 when $0 < k \leq 1$.*

Proof. We make use of the relationship $C_k Q C_k^* = C_k^* P C_k$ from the proof of the theorem. For all f in ℓ^2 we have

$$\begin{aligned}\langle (\tfrac{1}{k} C_k^* C_k - C_k C_k^*) f, f\rangle &= \langle (\tfrac{1}{k} C_k^* C_k - \tfrac{1}{k} C_k^* P C_k + \tfrac{1}{k} C_k Q C_k^* - C_k C_k^*) f, f\rangle \\ &= \langle \tfrac{1}{k}(I - P) C_k f, C_k f\rangle + \langle (\tfrac{1}{k} Q - I) C_k^* f, C_k^* f\rangle \geq 0\end{aligned}$$

when $0 < k \leq 1$, and this gives the result. □

We note that part of the second sentence in the statement of [11, Theorem 5.1], concerning the point spectrum of C_k, should be corrected to read as follows:

$$\pi_0(C_k) = \phi \textit{ unless } k < \tfrac{1}{2}, \textit{ in which case } \pi_0(C_k) = \{\tfrac{1}{k}\}.$$

We note also that the question of dominance has not been settled here for C_k when $\frac{1}{2} \leq k < 1$.

When $k > 1$, the situation for C_k can be regarded as a special case of the following theorem, whose proof occurs in [13, Theorem 2.2].

THEOREM 2.6. *Assume* $M(a) :\equiv M(a,1)$ *is a terraced matrix associated with a sequence* $a = \{a_n\}$ *satisfying the following conditions:*
(1) $\{a_n\}$ *is a strictly decreasing sequence that converges to* 0*;*
(2) $\{(n+1)a_n\}$ *is a strictly increasing sequence that converges to* $L < +\infty$*; and*
(3) $\frac{1}{a_{n+1}} \geq \frac{1}{2}(\frac{1}{a_n} + \frac{1}{a_{n+2}})$ *for all* n*.*
Then $M(a)$ *is hyponormal.*

Since $M(a)$ is hyponormal, it will necessarily be dominant and posinormal. We note that C_k does not satisfy condition (2) when $0 < k < 1$. Consider what happens when that condition is changed to the following:

$(2')$ $\{na_n\}$ *is a strictly increasing sequence that converges to* $L < +\infty$.

This new condition is satisfied by C_k for all $k > 0$. We observe that $(2')$ will allow $a_0 > L$, whereas the original condition (2) did not. Since $a_n \leq L$ for $n \geq 1$, we conclude from [10, Theorems 2.4 and 2.5] that

$$\sigma(M(a)) = \{\lambda : |\lambda - L| \leq L\} \cup \{a_0\}$$

and that a_0 is an eigenvalue for $M(a)$ when $a_0 > 2L$, just as $\frac{1}{k}$ was an eigenvalue for C_k when $0 < k < \frac{1}{2}$; so, when $a_0 > 2L$, an argument similar to what we used in Theorem 2.4 for C_k can be used to show that $M(a)$ fails to be dominant, and therefore also fails to be hyponormal, although it is posinormal (see [13, Corollary 2.2]). We note further that $M(a)$ also fails to be hyponormal when $a_0 = 2L$ since $||M(a)|| > a_0 = 2L = r(M(a))$, where $r(M(a))$ is the spectral radius of $M(a)$. The question of dominance for $M(a)$ remains unsettled for the cases when $L < a_0 \leq 2L$ and $a_1 < a_0 \leq L$.

The next theorem gives a necessary condition for a terraced matrix to be dominant. The proof is omitted since the key ideas have already been presented here.

THEOREM 2.7. *Assume that* $a = \{a_n\}$ *is a strictly decreasing sequence that converges to* 0 *and that* $\{(n+1)a_n\}$ *converges to* L $(0 < L < +\infty)$*. In order for the terraced matrix* $M(a)$ *to be dominant, it is necessary that* $a_0 \leq 2L$*; moreover, if* $a_0 = 2L$*, then* a_0 *must not be an eigenvalue of* $M(a)$*.*

We note that it is known that a_0 will not be an eigenvalue of $M(a)$ when $a_0 < 2L$ (see [10, Theorem 2.4]).

The previous examples have been terraced matrices, but our next ones are not. For fixed $\alpha \in (0,1]$, the discrete generalized Cesàro matrices A_α (see [7,8]) are the factorable matrices that occur when $a_i = \frac{\alpha^i}{i+1}$ and

$c_j = \frac{1}{\alpha^j}$ for all i,j. Note that $A_1 = C$. In [7] it is shown that when $0 < \alpha < 1$ these operators are bounded, compact, and not hyponormal; they have spectrum

$$\sigma(A_\alpha) = \{\frac{1}{n+1} : n \geq 0\} \cup \{0\}.$$

Theorem 2.8. *(1) Both A_α and A_α^* are posinormal for $0 < \alpha < 1$. (2) Both A_α and A_α^* are not dominant for $0 < \alpha < 1$.*

Proof. (1) From [11] we know that A_α is posinormal when $\alpha \in (0,1)$. To aid in the proof that A_α^* is posinormal, we define $T = [t_{mn}]$ by

$$t_{mn} = \begin{cases} \frac{n+1}{m+1}\alpha^{m-n}(1 - \frac{n}{n+1}\alpha^2) & if \quad n \leq m; \\ -\alpha & if \quad n = m+1; \\ 0 & if \quad n > m+1. \end{cases}$$

T can be shown to be bounded by an argument similar to that presented in the proof of [11, Theorem 4.1]. Then it is straightforward to verify that $A_\alpha = TA_\alpha{}^*$ and hence $A_\alpha{}^* = A_\alpha T^*$, so $A_\alpha{}^*$ is posinormal for $0 < \alpha < 1$.

(2) It is routine to verify that $f :\equiv < 1, \alpha, \alpha^2, \alpha^3, ... >^T \in \ell^2$ is an eigenvector associated with eigenvalue 1 for A_α $(0 < \alpha < 1)$ and hence $f \in Ker(A_\alpha - I)$, but $f \notin Ker(A_\alpha^* - I)$. Thus $A_\alpha - I$ is not posinormal, so A_α cannot be dominant. Similarly, A_α^* is not dominant since $g :\equiv < 1, 0, 0, 0, ... >^T \in Ker(A_\alpha^* - I)$ but $g \notin Ker(A_\alpha - I)$. □

3. Conditions for terraced matrices to be spectraloid and normaloid

Following Halmos, we recall that the *numerical range* $W(M)$ of the operator M is the set $\{< Mf, f >: ||f|| = 1\}$, and the *numerical radius* $\omega(M)$ is the number $sup\{|\lambda| : \lambda \in W(M)\}$. M is *spectraloid* if $\omega(M) = r(M)$, and M is *normaloid* if $\omega(M) = ||M||$. The inequality $r(M) \leq \omega(M) \leq ||M||$ holds for all M. From [3, Problem 218 (b)] we know that every normaloid operator is spectraloid. All hyponormal operators are normaloid.

The first theorem of this section gives sufficient conditions for a terraced matrix to be normaloid.

Theorem 3.1. *Assume that $a = \{a_n\}$ is a decreasing sequence that converges to 0 and that $\{(n+1)a_n\}$ is an increasing sequence that converges to L with $0 < L < +\infty$. Then $M(a)$ is normaloid (and hence spectraloid).*

Proof. If D is the diagonal matrix with diagonal $\{(n+1)a_n\}$, then $M(a) = DC$, so $||M(a)|| \leq ||D||||C|| = 2L$; and

$$\sigma(M(a)) = \{\lambda : |\lambda - L| \leq L\},$$

so $r(M(a)) = 2L$. Therefore $2L = r(M(a)) \leq \omega(M(a)) \leq ||M(a)|| = 2L$, so $r(M(a)) = \omega(M(a)) = ||M(a)||$, which means that $M(a)$ is both normaloid and spectraloid. □

Example 3.2. The sequence defined by $a_n = sin(\frac{1}{n+1})$ for all n satisfies the conditions of Theorem 3.1, so the associated terraced matrix $M(a)$ is normaloid. The same conclusion applies when $a_n = ln(1+\frac{1}{n+1})$ or $a_n = arctan(\frac{1}{n+1})$ for all n. It has not yet been determined whether or not these operators are hyponormal, although they are known to be posinormal (see [13, Corollary 2.2]).

Example 3.3. The terraced matrix associated with the sequence defined by $a_0 = 0.51$ and $a_n = \frac{1}{n+1}$ for all $n \geq 1$ is normaloid but not hyponormal (see [10, Example 3]). We note that this example does not satisfy condition (3) of Theorem 2.6.

Now we turn our attention to necessary conditions for a terraced matrix to be normaloid and spectraloid.

Theorem 3.4. *Assume that $a = \{a_n\}$ is a decreasing sequence that converges to 0 and that $\{(n+1)a_n\}$ converges to $L < +\infty$. (a) If $a_0 + a_1 + \sqrt{(a_0 - a_1)^2 + a_1^2} > 4L$, then the terraced matrix $M(a)$ is not spectraloid (and hence not normaloid). (b) If $\sum_{k=0}^{\infty} a_k^2 > 4L^2$, then $M(a)$ is not normaloid.*

Proof. (a) It is easy to see that if T is the matrix $[t_{mn}]$ with $t_{00} = a_0$, $t_{10} = t_{11} = a_1$, and $t_{mn} = 0$ for all other values of m, n, then $W(T) \subset W(M(a))$. It follows from [2] (see also [3, page 113]) that $W(T)$ is the closed elliptical disk bounded by the curve

$$\frac{(x - \frac{a_0+a_1}{2})^2}{(a_0 - a_1)^2 + a_1^2} + \frac{y^2}{a_1^2} = \frac{1}{4};$$

since the major axis has length $\sqrt{(a_0 - a_1)^2 + a_1^2}$, it follows that $\omega(M(a)) \geq \omega(T) = \frac{a_0+a_1}{2} + \frac{\sqrt{(a_0-a_1)^2+a_1^2}}{2} > a_0$. When $L > 0$,

$$\sigma(M(a)) = \{\lambda : |\lambda - L| \leq L\} \cup \{a_n : n = 0, 1, 2, ...\},$$

so $r(M(a)) = sup\{2L, a_0\}$, and the inequality $\omega(M(a)) > r(M(a))$ follows from our hypothesis. When $L = 0$, $M(a)$ is compact and $r(M(a)) = a_0$, so $\omega(M(a)) > r(M(a))$. (b) Since $||M(a)||^2 \geq \sum_{k=0}^{\infty} a_k^2$

and $r(M(a)) = sup\{a_0, 2L\}$, it follows from our hypothesis that $||M(a)|| > r(M(a))$ and hence $M(a)$ cannot be normaloid. □

Example 3.5. If $a_n = \frac{n+2}{(n+1)^2}$ for all n, then $0 < a_n \leq \frac{2}{n+1}$ for all n, so $M(a)$ is bounded and $||M(a)|| \leq 2||C|| = 4$. One easily verifies that $\{a_n\}$ is strictly decreasing and $L = 1$. Since $\sum_{k=0}^{\infty} a_k^2 > a_0^2 = 4 = 4L^2$, $M(a)$ is not normaloid (and hence also not hyponormal). Since $a_0 + a_1 + \sqrt{(a_0 - a_1)^2 + a_1^2} = \frac{11+\sqrt{34}}{4} > 4 = 4L$, $M(a)$ is not spectraloid. We note that $\{a_n\}$ satisfies the hypothesis of [13, Theorem 2.1], so $M(a)$ is posinormal.

In contrast, we point out that if U is the unilateral shift and $M :\equiv M(a)$ is the terraced matrix from Example 3.5, then U^*MU is the terraced matrix associated with sequence $a_n = \frac{n+3}{(n+2)^2}$, which is hyponormal (see [13, Example 2.2]). This example illustrates that even though M fails to be normaloid, spectraloid, and hyponormal, these failures may not be inherited by U^*MU (see [12]).

Corollary 3.6. *Assume that* $a = \{a_n\}$ *is a strictly decreasing sequence that converges to* 0 *and that* $\{(n+1)a_n\}$ *converges to* L *with* $0 < L < +\infty$. *(a) For* $M(a)$ *to be spectraloid, it is necessary that*

$$a_0 + a_1 + \sqrt{(a_0 - a_1)^2 + a_1^2} \leq 4L.$$

(b) For $M(a)$ *to be normaloid, it is necessary that* $\sum_{k=0}^{\infty} a_k^2 \leq 4L^2$.

Corollary 3.7. *If* $0 < k < z \approx 0.521$, *where* z *is the only positive zero of* $y = 16x^4 + 16x^3 - 5x^2 - 4x$, *then* C_k *is not spectraloid (and hence not normaloid).*

Corollary 3.8. *If* $0 < k < (5 - \frac{\pi^2}{6})^{-1/2} \approx 0.5459$, *then* C_k *is not normaloid.*

With a little patience, the estimate in the preceding corollary can be improved somewhat. For example, when $0 < k \leq 0.566$, it can be verified that $\sum_{n=0}^{\infty} \frac{1}{(k+n)^2} > \sum_{n=0}^{9} \frac{1}{(k+n)^2} + \frac{\pi^2}{6} - \sum_{n=1}^{10} \frac{1}{n^2} > 4$, so C_k is not normaloid for those values of k.

Corollary 3.9. *If* $p > 1$, *then* M_p *is not spectraloid and not normaloid.*

We note that the result of Corollary 3.9 appears in [9], albeit with a factor of $\frac{1}{2}$ missing from the supporting calculation there. We also note that Theorem 3.4 does not cover A_α for $0 < \alpha < 1$, since these are not terraced matrices. However, it was shown in [8] that A_α is not spectraloid and not normaloid for these values of α.

The following theorem summarizes what we know about C_k in terms of the topics of this section.

THEOREM 3.10. *C_k is spectraloid and normaloid for $k \geq 1$, but C_k is not spectraloid for $0 < k < z \approx 0.521$, and C_k is not normaloid for $0 < k \leq 0.566$.*

Proof. The first assertion follows from the fact that C_k is hyponormal for $k \geq 1$, and the subsequent assertions are justified by Corollary 3.7 and the comment following Corollary 3.8. □

4. Conclusion

In closing, we are reminded that the question of dominance for C_k is unresolved for $\frac{1}{2} \leq k < 1$, although it is known that these operators are not hyponormal. Similarly, the more general question of dominance for the terraced matrix $M(a)$ when $L < a_0 \leq 2L$ or $a_1 < a_0 \leq L$ for $0 < L < +\infty$ has not yet been settled. In view of the results of the preceding section, it is worth noting that dominant operators are not necessarily normaloid. For now, the question of whether or not there exists a dominant terraced matrix that is not hyponormal remains open.

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