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L^p ESTIMATES FOR SCHRÖDINGER TYPE OPERATORS ON THE HEISENBERG GROUP

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Abstract. We investigate the Schrödinger type operator $H_2 = (-\Delta_{\mathbb{H}^n})^2 + V^2$ on the Heisenberg group $\mathbb{H}^n$, where $\Delta_{\mathbb{H}^n}$ is the sublaplacian and the nonnegative potential V belongs to the reverse Hölder class B_q for $q \geq \frac{Q}{2}$, where Q is the homogeneous dimension of $\mathbb{H}^n$. We shall establish the estimates of the fundamental solution for the operator H_2 and obtain the L^p estimates for the operator $\nabla^4_{\mathbb{H}^n} H_2^{-1}$, where $\nabla_{\mathbb{H}^n}$ is the gradient operator on $\mathbb{H}^n$.

1. Introduction

The Schrödinger operators on the Euclidean spaces $\mathbb{R}^n$ with nonnegative potentials that belong to reverse Hölder class have been investigated by a number of authors (cf. [4], [14], [11]). The extension to more general setting had been given by Lu [10] and Li [8]. They had obtained the L^p estimates for the Schrödinger operator $-\sum_{i=1}^k X_i^2 + V$ on the space of homogeneous type and the nilpotent Lie group respectively, where $X_1, \ldots, X_k$ denote the vector fields satisfying extra conditions. When V is a non-negative polynomial, the Schrödinger type operator $(-\Delta)^2 + V^2$ had been studied by Zhong in [14] and he proved the L^p estimates related to this operator. Recently, Sugano [13] has generalized Zhong's results in [14] and has obtained the L^p estimate of some operators related to Schrödinger type operator with the potential V belonging to reverse Hölder class. In the present work, we want to extend Sugano's results to the setting of the Heisenberg group.

We recall that the Heisenberg group $\mathbb{H}^n$ is a Lie group with the underlying manifold $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and the multiplication

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y')).$$

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A basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$ is given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t},\ X_{n+j} \doteq Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},\ j = 1, 2, \ldots, n,\ \ T = \frac{\partial}{\partial t}.$$

All non-trivial commutation relations are given by $[X_j, X_{n+j}] = -4T$, $j = 1, \ldots, n$. Then the sublaplacian $\Delta_{\mathbb{H}^n}$ is defined by

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n} X_j^2$$

and the gradient operator $\nabla_{\mathbb{H}^n}$ is defined by

$$\nabla_{\mathbb{H}^n} = (X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}).$$

We adopt the following notation for higher order gradient operator. For $I = (i_1, \ldots, i_{2n}) \in \mathbb{N}^{2n}$, set

$$|\nabla^j_{\mathbb{H}^n} f| = \left(\sum_{|I|=j} |X^{|I|} f|^2 \right)^{\frac{1}{2}},$$

where $X^j = X_1^{i_1} \cdots X_{2n}^{2n}$, $j = |I| = i_1 + \cdots + i_{2n}$.

The dilations on $\mathbb{H}^n$ have the form

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t),\ \lambda > 0.$$

The Haar measure on $\mathbb{H}^n$ is the usual Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. We denote the measure of any measurable set E by $|E|$. Then $|\delta_\lambda E| = \lambda^Q |E|$. $Q = 2n + 2$ is called the homogeneous dimension of $\mathbb{H}^n$.

We define a homogeneous norm function on $\mathbb{H}^n$ by

$$|g| = \left((|x|^2 + |y|^2)^2 + |t|^2 \right)^{\frac{1}{4}}, \quad g = (x, y, t) \in \mathbb{H}^n.$$

This norm satisfies the triangular inequality and deduces a left-invariant distance function $d(g, h) = |g^{-1}h|$. Then the ball of radius r centered at g is given by

$$B(g, r) = \{h \in \mathbb{H}^n :\ |g^{-1}h| < r\}.$$

The ball $B(g, r)$ is the left translation by g of $B(0, r)$ and we have

$$|B(g, r)| = \alpha_1 r^Q,$$

where

$$\alpha_1 = |B(0, 1)| = \frac{2\pi^{n+\frac{1}{2}} \Gamma(\frac{n}{2})}{(n+1)\Gamma(n)\Gamma(\frac{n+1}{2})}$$

(cf. [3]).

Now we introduce the reverse Hölder class B_q. A nonnegative locally L^q integrable function V on $\mathbb{H}^n$ is said to belong to B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(g)^q\,dg\right)^{\frac{1}{q}} \le C\left(\frac{1}{|B|}\int_B V(g)\,dg\right) \tag{1}$$

holds for every ball B in $\mathbb{H}^n$.

It is important that the B_q class has a property of "self improvement" (cf. [6]); that is, if $V \in B_q$, then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$.

Assume that $V \in B_q$ for some $q > \frac{Q}{2}$. The definition of the auxiliary function $m(g,V)$ is given as follows.

Definition 1. For $g \in \mathbb{H}^n$, the function $m(g,V)$ is defined by

$$\frac{1}{m(g,V)} = \sup_{r>0}\left\{r : \frac{1}{r^{Q-2}}\int_{B(g,r)} V(h)\,dh \le 1\right\}.$$

Some basic properties of $m(g,V)$ on the Heisenberg group will be given in Section 2.

In this paper we consider the Schrödinger and Schrödinger type operators $H_1 = -\Delta_{\mathbb{H}^n} + V$ and $H_2 = (-\Delta_{\mathbb{H}^n})^2 + V^2$ where the nonnegative potentials V belong to the reverse Hölder class B_q for $q \ge \frac{Q}{2}$. We state our main results in the following theorems.

Theorem 1. *Suppose $V \in B_q$ for some $q \ge \frac{Q}{2}$ and there exists a constant $C > 0$ such that $V(g) \le Cm(g,V)^2$. Let $j = 0,1,2,3$. Then, for $1 < p \le \infty$,*

$$\| V^{2-\frac{j}{2}}\nabla^j_{\mathbb{H}^n} H_2^{-1} f \|_{L^p(\mathbb{H}^n)} \le C_j \| f \|_{L^p(\mathbb{H}^n)}, \tag{2}$$

where C_j depends only on j,p,q and the constant in (1)*.*

Corollary 1. *Suppose $V \in B_q$ for some $q \ge \frac{Q}{2}$ and there exists a constant $C > 0$ such that $V(g) \le Cm(g,V)^2$. Then there exists a positive constant C' such that, for $1 < p < \infty$,*

$$\| \nabla^4_{\mathbb{H}^n} H_2^{-1} f \|_{L^p(\mathbb{H}^n)} \le C' \| f \|_{L^p(\mathbb{H}^n)}\ . \tag{3}$$

To prove Theorem 1 and Corollary 1 we need the estimates and derivative estimates of the fundamental solution for H_2 which are given in the following propositions.

Proposition 1. *Assume $V \in B_{\frac{Q}{2}}$. For any positive integer N, there exists a constant $C_N > 0$ such that*

$$0 \le \Gamma_{H_2}(g,h) \le \frac{C_N}{(1+m(g,V)|g^{-1}h|)^N}\frac{1}{|g^{-1}h|^{Q-4}}.$$

Proposition 2. *Let $j = 1, 2, 3$. Suppose $V \in B_{\frac{Q}{2}}$ and there exists a constant $C > 0$ such that $V(g) \leq Cm(g, V)^2$. Then for any positive integer N, there exists a positive constant C_N such that*

$$|\nabla^j_{\mathbb{H}^n} \Gamma_{H_2}(g, h)| \leq \frac{C_N}{\left(1 + |g^{-1}h| m(g, V)\right)^N} \frac{1}{|g^{-1}h|^{Q-4+j}}. \tag{4}$$

This paper is organized as follows. In Section 2 we recall some properties of the auxiliary function $m(g, V)$ on the Heisenberg group. In Section 3 we establish the estimates and derivative estimates of the fundamental solution for the operator H_2. In Section 4 we give the proofs of our main results.

Similar to case of $\mathbb{R}^n$, the proof of the estimate of the fundamental solution for the operator H_2 depends on the estimates of the fundamental solution for the operator $-\Delta_{\mathbb{H}^n}$ and the heat kernel $e^{-s(-\Delta_{\mathbb{H}^n})}(g, h)$ on the Heisenberg group. However, the proof of Lemma 12 which is a key lemma in Section 3 is more complicate than the case of $\mathbb{R}^n$. We use a fact deduced from Lemma 3.3 in [8] (or Lemma 10 in this paper) and the idea of proof of Lemma 3.4 in [8] to prove Lemma 12 in the context of Heisenberg group.

In addition, the difference between our proof of some results for Schrödinger type operators on the Heisenberg group and that of Schrödinger operators in [8] lies in the estimates and derivative estimates of the fundamental solution of the operator $(-\Delta_{\mathbb{H}^n})^2 + V^2$. The difficulties are due to that the Schrödinger type operator has second power of $-\Delta_{\mathbb{H}^n}$, which increase the complexity of the estimates.

Throughout the paper, unless otherwise indicated, we will use C to denote the positive constant, which may be different and depends on the constant in (1) and the homogeneous dimension Q. By $A \sim B$, we mean that there exist $C > 0$ and $c > 0$ such that $c \leq \frac{A}{B} \leq C$.

2. The auxiliary function $m(g, V)$ on the Heisenberg group

In the section we outline some lemmas of the auxiliary function $m(g, V)$ on the Heisenberg group, which have been proved in [8]. We always assume that $V \in B_q$ for some $q > \frac{Q}{2}$ throughout this section.

Lemma 1. *If $V \in B_q$, $q > 1$, then $d\mu = V(g)\, dg$ is a doubling measure. That is, there exists a constant $C_0 > 0$ such that*

$$\int_{B(h,2r)} V(g)\, dg \leq C_0 \int_{B(h,r)} V(g)\, dg$$

for all balls $B(h, r)$ in $\mathbb{H}^n$.

Lemma 2. *There exists $C > 0$ such that, for $0 < r < R < \infty$,*

$$\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)\, dh \leq C\left(\frac{R}{r}\right)^{\frac{Q}{q}-2} \frac{1}{R^{Q-2}} \int_{B(g,R)} V(h)\, dh.$$

The auxiliary function $m(g,V)$ on the Euclidean space was introduced by Shen in [12]. After that Li [8] defined it on the nilpotent Lie group.

It is easy to see that $0 < m(g,V) < \infty$ for every $g \in \mathbb{H}^n$ and if $r = \frac{1}{m(g,V)}$, then

$$\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)\, dh = 1.$$

Moreover, by Lemma 2, if $\frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)\, dh \sim 1$, then $r \sim \frac{1}{m(g,V)}$.

Lemma 3. *There exist $C > 0$, $c > 0$ and $k_0 > 0$ such that, for any g, h in $\mathbb{H}^n$,*

(a) $m(h,V) \sim m(g,V)$ if $|g^{-1}h| \le \frac{C}{m(g,V)}$,

(b) $m(h,V) \le C\big(1 + |g^{-1}h|\, m(g,V)\big)^{k_0} m(g,V)$,

(c) $m(h,V) \ge \frac{c m(g,V)}{\big(1+|g^{-1}h|\, m(g,V)\big)^{\frac{k_0}{k_0+1}}}$.

As a consequence of Lemma 3, we have:

Corollary 2. *There exist $C > 0$, $c > 0$ and $k_0 > 0$ such that, for any $g, h \in \mathbb{H}^n$,*

$$c\{1 + |g^{-1}h|\, m(g,V)\}^{\frac{1}{k_0+1}} \le 1 + |g^{-1}h|\, m(h,V) \le C\{1 + |g^{-1}h|\, m(g,V)\}^{k_0+1}.$$

Lemma 4. *There exists $l_1 > 0$ such that*

$$\int_{B(g,R)} \frac{V(h)}{|g^{-1}h|^{Q-2}}\, dh \le \frac{C}{R^{Q-2}} \int_{B(g,R)} V(h)\, dh \le C\bigg(1 + Rm(g,V)\bigg)^{l_1}.$$

The proofs of the above Lemma 1–Lemma 4 are similar to the case of Euclidean space. See Section 2 in [8] for the details of proof of these lemmas.

Remark 1. If V is a nonnegative polynomial, then it is easy to see that there exists a constant $C > 0$ such that $V(g) \le Cm(g,V)^2$.

3. The estimates of fundamental solutions for H_2

This section is divided into two subsections to devote to estimates of the fundamental solution for the Schrödinger type operator $H_2 = (-\Delta_{\mathbb{H}^n})^2 + V^2$ on $\mathbb{H}^n$, that is, we will prove Proposition 1 and Proposition 2 respectively.

Assume that $V \in B_q$ for some $q \ge \frac{Q}{2}$ throughout this section.

3.1. Proof of Proposition 1

In this subsection we will prove Proposition 1, which follows easily from the Lemma 5.

Lemma 5. *If $V \in B_{\frac{Q}{2}}$ and $(-\Delta_{\mathbb{H}^n})^2 u + V^2 u = 0$ in $B(g_0,R)$, then for any positive integer N, there exists a constant $C_N > 0$ such that*

$$\sup_{B(g_0,\frac{R}{2})} |u(g)| \le \frac{C_N}{\big(1 + R\, m(g_0,V)\big)^N} \sup_{B(g_0,R)} |u(g)|. \tag{5}$$

Assume Lemma 5 for the moment, we give:

Proof of Proposition 1. Let $e^{-tA}(g,h)$ be the kernel of e^{-tA}, where $A = -\Delta_{\mathbb{H}^n}$, $(-\Delta_{\mathbb{H}^n})^2$, or H_2. By Trotter's formula

$$e^{t(A+B)} = \lim_{n\to\infty} (e^{\frac{tA}{n}} e^{\frac{tB}{n}})^n,$$

we get

$$0 \le e^{-tH_2}(g,h) \le e^{-t(-\Delta_{\mathbb{H}^n})^2}(g,h), \ \forall\, t \in \mathbb{R}^+. \tag{6}$$

According to the integral representation $\frac{1}{A^2} = \int_0^\infty se^{-sA}ds$, (6) and the estimates of heat kernel $e^{-t(-\Delta_{\mathbb{H}^n})}(g,h)$ in [7], we have

$$0 \le \Gamma_{H_2}(g,h) \le \int_0^\infty se^{-s(-\Delta_{\mathbb{H}^n})}(g,h)ds \le \frac{C}{|g^{-1}h|^{Q-4}}.$$

Fix $g_0, h_0 \in \mathbb{H}^n$ and put $R = |g_0^{-1}h_0|$. Then $u(g) = \Gamma_{H_2}(g,h_0)$ is a solution of $(-\Delta_{\mathbb{H}^n})^2u + V^2u = 0$ on $B(g_0, \frac{R}{4})$. By using the estimate $|\Gamma_{H_2}(g,h)| \le \frac{C}{|g^{-1}h|^{Q-4}}$ and (5), we can get the desired estimate. □

To prove the key Lemma 5 we need some lemmas. Firstly we need to recall an estimate of the fundamental solution for $\Delta^2_{\mathbb{H}^n}$ (cf. [1] or [2]). Furthermore, we have the following lemma for the derivative of $\Psi(g)$, which will be used in the proof of Lemma 13.

Lemma 6. *There exists a positive constant C which depends on n and the bounds of ψ and its derivative such that*

$$|X^j\Psi(g)| \le \frac{C}{|g|^{Q-4+j}} \quad \textit{for } j \in \mathbb{N}. \tag{7}$$

Proof. It follows from Theorem A in [1] that the fundamental solution $\Psi(g)$ of $\Delta^2_{\mathbb{H}^n}$ defined by

$$\Psi(g) = \frac{2(n-1)!}{\big(\frac{(x^2+y^2)^2}{16} + t^2\big)^{\frac{Q-4}{4}}}\psi(\theta)$$

extends to a function on $\mathbb{H}^n$ which is smooth away from 0, where $\theta = \arctan\frac{-4t}{|x|^2+|y|^2}$ and $\psi(\theta)$ is a smooth function for $|\theta| \le \frac{\pi}{2}$.

It is easy to see that (7) holds true for $j = 0$. We need only to prove (7) holds for $j = 1$. Other cases can be proved by induction. When $j = 1$, $X^1 = X_l, l = 1, 2, \ldots, 2n$. Set $r = \big(\frac{(x^2+y^2)^2}{16} + t^2\big)^{\frac{1}{4}}$. For $l = 1, 2, \ldots, n$, we have

$$\begin{aligned}
X^1\Psi(g) &= X_l\Psi(g) \\
&= \frac{\partial\Psi(g)}{\partial x_l} + 2y_l\frac{\partial\Psi(g)}{\partial t} \\
&= \frac{\partial\Psi(g)}{\partial r}\frac{\partial r}{\partial x_l} + \frac{\partial\Psi(g)}{\partial\theta}\frac{\partial\theta}{\partial x_l} + 2y_l\frac{\partial\Psi(g)}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial\Psi(g)}{\partial\theta}\frac{\partial\theta}{\partial t} \\
&= 2(4-Q)(n-1)!\frac{(|x|^2+|y|^2)x_l}{r^Q}\psi(\theta) + (n-1)!\frac{tx_l}{r^Q}\psi'(\theta)
\end{aligned}$$

$$+ (4-Q)(n-1)!\frac{ty_l}{r^Q}\psi(\theta) - \frac{(n-1)!}{2}\frac{(|x|^2+|y|^2)y_l}{r^Q}\psi'(\theta).$$

Since $|t| \le r^2, |x_l| \le \frac{r}{2}, |y_l| \le \frac{r}{2}$, we can get

$$|X^1\Psi(g)| \le Cr^{3-Q} \le \frac{C}{|g|^{Q-3}},$$

where the constant C depends on n and the bounds of $\psi(\theta)$ and its derivative.

Similarly, for $l = 1, 2, \dots, n$, we have

$$|X^1\Psi(g)| = |X_{n+l}\Psi(g)| \le Cr^{3-Q} = \frac{C}{|g|^{Q-3}}. \qquad \square$$

In the next theorem we shall prove the L^p estimate of the operator

$$m(g, V)\nabla_{\mathbb{H}^n} H_1^{-1}$$

related to $H_1 = -\Delta_{\mathbb{H}^n} + V$ which will be used later. In order to obtain this result, we recall an estimate of the fundamental solution for the operator H_1, which has been obtained in Lemma 5.1 in [8] in the setting of nilpotent Lie group. Let $\Gamma(g, h, \lambda)$ denote the fundamental solution for the operator $-\Delta_{\mathbb{H}^n} + V + \lambda$, where $\lambda \ge 0$. Then we have:

Lemma 7 ([8, Lemma 5.1])**.** *Suppose $V \in B_{\frac{Q}{2}}$ and $\lambda \ge 0$. Let $l > 0$ be an integer. Then there exists $C_l > 0$ such that, for $g \ne h$,*

$$\begin{aligned}\left|\nabla_{\mathbb{H}^n,\, g}\Gamma(g,h,\lambda)\right| \quad &\le \quad \frac{C_l}{\left(1+|g^{-1}h|\,\lambda^{\frac12}\right)^l\left(1+|g^{-1}h|m(g,V)\right)^l}\frac{1}{|g^{-1}h|^{Q-2}} \\ &\quad \times\left\{\int_{B(h,\frac{|g^{-1}h|}{4}))}\frac{V(h')dh'}{|h^{-1}h'|^{Q-1}} + \frac{1}{|g^{-1}h|}\right\}.\end{aligned}$$

Theorem 2. *Suppose $V \in B_{q_0}$ for some $\frac{Q}{2} \le q_0 < Q$. Then, for $1 < p \le p_0$, there exists a constant $C > 0$ such that*

$$\| m(\cdot, V)\nabla_{\mathbb{H}^n} H_1^{-1} f \|_p \le C \| f \|_p, \ \forall f \in L^p(\mathbb{H}^n),$$

where $\frac{1}{p_0} = \frac{1}{q_0} - \frac{1}{Q}$.

Proof. We adopt the method similar to the proof of Lemma 2 in [13]. Suppose $V \in B_{q_0}$ for some $\frac{Q}{2} \le q_0 < Q$. Then by the self improvement of B_q we have $V \in B_{q_1}$ for some $q_1 > q_0$. Denote by $\Gamma_{H_1}(g, h)$ the fundamental solution for H_1 and let

$$Tf(g) = m(g, V)\int_{\mathbb{H}^n}\nabla_{\mathbb{H}^n,\, g}\Gamma_{H_1}(g, h)f(h)dh.$$

The adjoint of T^* is defined by

$$T^*f(g) = \int_{\mathbb{H}^n}\nabla_{\mathbb{H}^n,\, h}\Gamma_{H_1}(h, g)m(h, V)f(h)dh.$$

By duality, it suffices to show that

$$\| T^*f \|_p \le C \| f \|_p \quad \text{for } p_0' \le p < \infty, \tag{8}$$

where $\frac{1}{p_0}+\frac{1}{p_0'}=1$. we choose t and p_1 such that $\frac{1}{t}=\frac{1}{q_1}-\frac{1}{Q}$, $\frac{1}{p_1}=1-\frac{1}{q_1}+\frac{1}{Q}$. Therefore, $\frac{1}{t}+\frac{1}{p_1}=1$. Let $r=\frac{1}{m(g,V)}$. Hence, by Hölder inequality,

$$\begin{aligned}|T^*f(g)| &\le \sum_{j=-\infty}^{\infty}\int_{2^{j-1}r<|g^{-1}h|\le 2^j r} \mid \nabla_{\mathbb{H}^n,h}\Gamma_{H_1}(h,g)m(h,V)f(h)\mid dh\\ &\le \sum_{j=-\infty}^{\infty}\left(\int_{2^{j-1}r<|g^{-1}h|\le 2^j r}\mid \nabla_{\mathbb{H}^n,h}\Gamma_{H_1}(h,g)m(h,V)\mid^t dh\right)^{\frac{1}{t}}\left(\int_{|g^{-1}h|\le 2^j r}\mid f(h)\mid^{p_1} dh\right)^{\frac{1}{p_1}}.\end{aligned}$$

It follows from Lemma 3, Lemma 4, Lemma 7 and the theorem on fractional integral on the Heisenberg group that

$$\begin{aligned}&\left(\int_{2^{j-1}r<|g^{-1}h|\le 2^j r}\mid \nabla_{\mathbb{H}^n,h}\Gamma_{H_1}(h,g)m(h,V)\mid^t dh\right)^{\frac{1}{t}}\\ &\le Cm(g,V)\{1+2^j rm(g,V)\}^{k_0}\left(\int_{2^{j-1}r<|g^{-1}h|\le 2^j r}\mid \nabla_{\mathbb{H}^n,h}\Gamma_{H_1}(h,g)\mid^t dh\right)^{\frac{1}{t}}\\ &\le Cm(g,V)\{1+2^j rm(g,V)\}^{k_0}\frac{C_k}{\{1+2^j rm(g,V)\}^k}\frac{1}{(2^j r)^{Q-2}}\left\{\left(\int_{|g^{-1}h|\le 2^{j+1}r}V^{q_1}dh\right)^{\frac{1}{q_1}}+(2^j r)^{\frac{Q}{t}-1}\right\}\\ &\le CC_k m(g,V)\frac{1}{\{1+2^j\}^{k-k_0}}\frac{1}{(2^j r)^{Q-2}}\left\{\left(\int_{|g^{-1}h|\le 2^{j+1}r}V^{q_1}dh\right)^{\frac{1}{q_1}}+(2^j r)^{\frac{Q}{t}-1}\right\}\\ &\le CC_k m(g,V)\frac{2^j}{\{1+2^j\}^{k-k_0}}(2^j r)^{-\frac{Q}{p_1}}\left(\frac{1}{(2^j r)^{Q-2}}\int_{|g^{-1}h|\le 2^{j+1}r}V(h)dh+1\right)\\ &\le CC_k\frac{2^j}{\{1+2^j\}^{k-k_0}}(2^j r)^{1-\frac{Q}{p_1}}\left((1+2^r m(g,V))^{l_1}+1\right)\\ &\le CC_k\frac{2^j}{\{1+2^j\}^{k-k_0-l_1}}(2^j r)^{-\frac{Q}{p_1}},\end{aligned}$$

where we take k sufficiently large. Thus,

$$\begin{aligned}|T^*f(g)| &\le CC_k\sum_{j=-\infty}^{\infty}\frac{2^j}{\{1+2^j\}^{k-k_0-l_1}}\left((2^j r)^{-Q}\int_{|g^{-1}h|\le 2^j r}\mid f(h)\mid^{p_1} dh\right)^{\frac{1}{p_1}}\\ &\le C\{M(|f|^{p_1})(g)\}^{\frac{1}{p_1}},\end{aligned}$$

where M is the Hardy-Littlewood maximal operator. Then we have

$$\parallel T^*f\parallel_p\le C\parallel f\parallel_p \quad \text{for } p_1<p\le\infty.$$

Then (8) follows since $p_0'>p_1$. □

The following lemma gives the L^p estimate of $m^2(g,V)(-\Delta_{\mathbb{H}^n}+V)^{-1}$ which is exactly Corollary 3.7 in [8].

Lemma 8 ([8, Corollary 3.7])**.** *Assume $V \in B_q, q > \frac{Q}{2}$. For $1 \le p \le \infty$, there exists a constant $C_p > 0$ such that*

$$\| m^2(g, V)(-\Delta_{\mathbb{H}^n} + V)^{-1} f \|_p \le C_p \| f \|_p, \ \forall f \in L^p(\mathbb{H}^n).$$

The following lemma will be used in the proof of Lemma 5.

Lemma 9. *Let $u \in C_0^\infty(\mathbb{H}^n)$ and $V \in B_{\frac{Q}{2}}$. Then there exists a positive constant C such that*

$$\begin{aligned} &\int_{\mathbb{H}^n} |u(g)|^2 m(g,V)^4 \, dg + \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u(g)|^2 m(g,V)^2 \, dg \\ &\le C \left(\int_{\mathbb{H}^n} |\Delta_{\mathbb{H}^n} u(g)|^2 \, dg + \int_{\mathbb{H}^n} |u(g)|^2 V^2(g) \, dg \right). \end{aligned}$$

Proof. At first by Lemma 8 for the case of $p = 2$ we have

$$\begin{aligned} \int_{\mathbb{H}^n} |u(g)|^2 m(g,V)^4 \, dg &= \int_{\mathbb{H}^n} \left| m(g,V)^2 (-\Delta_{\mathbb{H}^n} + V)^{-1} (-\Delta_{\mathbb{H}^n} + V) u(g) \right|^2 dg \\ &\le C \int_{\mathbb{H}^n} \left| (-\Delta_{\mathbb{H}^n} + V) u(g) \right|^2 dg. \end{aligned}$$

Secondly, we use Theorem 2 for the case of $p = 2$ to obtain

$$\begin{aligned} &\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u(g)|^2 m(g,V)^2 \, dg \\ &= \int_{\mathbb{H}^n} \left| m(g,V) \nabla_{\mathbb{H}^n} (-\Delta_{\mathbb{H}^n} + V)^{-1} (-\Delta_{\mathbb{H}^n} + V) u(g) \right|^2 dg \\ &\le C \int_{\mathbb{H}^n} \left| (-\Delta_{\mathbb{H}^n} + V) u(g) \right|^2 dg. \end{aligned}$$

Then

$$\begin{aligned} &\int_{\mathbb{H}^n} |u(g)|^2 m(g,V)^4 \, dg + \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u(g)|^2 m(g,V)^2 \, dg \\ &\le C \int_{\mathbb{H}^n} \left| (-\Delta_{\mathbb{H}^n} + V) u(g) \right|^2 dg \\ &\le C \left(\int_{\mathbb{H}^n} |\Delta_{\mathbb{H}^n} u(g)|^2 \, dg + \int_{\mathbb{H}^n} |u(g)|^2 V^2(g) \, dg \right), \end{aligned}$$

where the last inequality used the fact $(a + b)^2 \le 2(a^2 + b^2)$. □

Let $U \subseteq \mathbb{H}^n$ be an open set. Denote

$$S(U) = \left\{ u \in C^\infty(U) : \| u \|_{S(U)} = \left[\int_U (|\nabla_{\mathbb{H}^n} u|^2 + |u|^2) dg \right]^{\frac{1}{2}} < \infty \right\}$$

and

$$S_{loc}(U) = \{ u \in L^2_{loc}(U) : \phi u \in S(U), \forall \phi \in C_0^\infty(U) \}.$$

We call $u \in S_{loc}(U)$ is a weak solution of $-\Delta_{\mathbb{H}^n} u + Vu = 0$ on U if, for any $\phi \in C_0^\infty(U)$,

$$\int_U [\nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} \phi + Vu\phi] dg = 0$$

holds, where $\nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} \phi = \sum_{i=1}^{2n} (X_i u)(X_i \phi)$.

Lemma 10 ([8, Lemma 3.3]). *Suppose $u \in S_{loc}(B(g_0, 4C^2R))$ is a weak solution of $-\Delta_{\mathbb{H}^n} u + (V + \lambda)u = 0$ on $B(g_0, 4C^2R)$, where the constant $C \geq 1$ and $\lambda \geq 0$. Then there exists a constant $C' > 0$ such that*

$$\int_{B(g_0,R)} [|\nabla_{\mathbb{H}^n} u(g)|^2 + (V(g) + \lambda)|u(g)|^2] dg \leq \frac{C'}{R^2} \int_{B(g_0,2C^2R)} |u(g)|^2 dg.$$

Next we will prove the Caccioppoli's type inequality on the Heisenberg group.

Lemma 11. *Suppose $(-\Delta_{\mathbb{H}^n})^2 u + V^2 u = 0$ in $B(g_0, R)$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\int_{B(g_0,\frac{R}{2})} |\Delta_{\mathbb{H}^n} u(g)|^2 \, dg + \int_{B(g_0,\frac{R}{2})} |u(g)|^2 V^2(g) \, dg \\ &\leq \frac{C}{R^2} \int_{B(g_0,R)} |\nabla_{\mathbb{H}^n} u(g)|^2 \, dg + \frac{C}{R^4} \int_{B(g_0,R)} |u(g)|^2 \, dg. \end{aligned}$$

Proof. Let $\phi \in C_c^\infty(B(g_0, R))$ such that $\phi \equiv 1$ on $B(g_0, \frac{R}{2})$, $0 < \phi \leq 1$, $|\nabla_{\mathbb{H}^n} \phi| \leq CR^{-1}$ and $|\Delta_{\mathbb{H}^n} \phi| \leq CR^{-2}$. Multiplying the equation $(-\Delta_{\mathbb{H}^n})^2 u + V^2 u = 0$ by $\phi^4 \bar{u}$ and integrating over $B(g_0, R)$, we have

$$\int_{B(g_0,R)} \Delta_{\mathbb{H}^n}(\phi^4(g)\bar{u}(g)) \Delta_{\mathbb{H}^n} u(g) \, dg + \int_{B(g_0,R)} \phi^4(g)|u(g)|^2 V^2(g) \, dg = 0.$$

It is reduced to that

$$\begin{aligned} &\int_{B(g_0,R)} \phi^4(g)|\Delta_{\mathbb{H}^n} u(g)|^2 \, dg + \int_{B(g_0,R)} \phi^4(g)|u(g)|^2 V^2(g) \, dg \\ = &- 2\int_{B(g_0,R)} \nabla_{\mathbb{H}^n} \phi^4(g) \cdot \nabla_{\mathbb{H}^n} u(g) \Delta_{\mathbb{H}^n} u(g) \, dg \\ &- \int_{B(g_0,R)} \Delta_{\mathbb{H}^n}(\phi^4)(g) u(g) \Delta_{\mathbb{H}^n} u(g) \, dg \\ = &- \int_{B(g_0,R)} \big((12|\nabla_{\mathbb{H}^n} \phi|^2 + 4\phi \Delta_{\mathbb{H}^n} \phi) u(g)\big)(\phi^2 \Delta_{\mathbb{H}^n} u(g)) \, dg \\ &- 2\int_{B(g_0,R)} (4\phi \nabla_{\mathbb{H}^n} \phi \cdot \nabla_{\mathbb{H}^n} u(g))(\phi^2 \Delta_{\mathbb{H}^n} u(g)) \, dg \\ \leq &\frac{C}{R^4} \int_{B(g_0,R)} |u(g)|^2 dg + \frac{1}{4} \int_{\mathbb{H}^n} \phi^4 |\Delta_{\mathbb{H}^n} u|^2 dg + \frac{C}{R^2} \int_{B(g_0,R)} |\nabla_{\mathbb{H}^n} u(g)|^2 dg \\ &+ \frac{1}{4} \int_{\mathbb{H}^n} \phi^4 |\Delta_{\mathbb{H}^n} u|^2 dg \end{aligned}$$

$$\leq \frac{C}{R^4}\int_{B(g_0,R)}|u(g)|^2dg+\frac{C}{R^2}\int_{B(g_0,R)}|\nabla_{\mathbb{H}^n}u(g)|^2dg+\frac{1}{2}\int_{\mathbb{H}^n}\phi^4|\Delta_{\mathbb{H}^n}u|^2dg,$$

where we used $|ab|\leq a^2+\frac{1}{2}b^2$.

It follows that

$$\int_{B(g_0,\frac{R}{2})}|\Delta_{\mathbb{H}^n}u(g)|^2\,dg+\int_{B(g_0,\frac{R}{2})}|u(g)|^2V^2(g)\,dg$$
$$\leq \frac{C}{R^2}\int_{B(g_0,R)}|\nabla_{\mathbb{H}^n}u(g)|^2\,dg+\frac{C}{R^4}\int_{B(g_0,R)}|u(g)|^2\,dg.$$

This completes the proof of the lemma. □

The following lemma gives an estimate of the solution for the equation $(-\Delta_{\mathbb{H}^n})^2u+V^2u=0$.

Lemma 12. *Suppose* $(-\Delta_{\mathbb{H}^n})^2u+V^2u=0, u\geq 0$ *in* $B(g_0,R)$. *Then*

$$\sup_{B(g_0,\frac{R}{2})}|u(g)|\leq C\left(\frac{1}{R^Q}\int_{B(g_0,8R)}|u(g)|^2\,dg\right)^{\frac{1}{2}}+CR\left(\frac{1}{R^Q}\int_{B(g_0,8R)}|\nabla_{\mathbb{H}^n}u(g)|^2\,dg\right)^{\frac{1}{2}}.$$

Proof. We will adopt the idea of the proof different from the case on Euclidean spaces in [14].

Let $\phi\in C_c^\infty(B(g_0,2R))$ such that $\phi\equiv 1$ on $B(g_0,R)$, $0<\phi\leq 1$. Also, ϕ satisfies $|\nabla^j_{\mathbb{H}^n}\phi|\leq CR^{-j}$ for $j=1,2,3,4$.

Note that

$$\begin{aligned}[(-\Delta_{\mathbb{H}^n})^2+V^2](u\phi) &= 2\nabla_{\mathbb{H}^n}\Delta_{\mathbb{H}^n}u\cdot\nabla_{\mathbb{H}^n}\phi+2\Delta_{\mathbb{H}^n}u\Delta_{\mathbb{H}^n}\phi\\ &\quad+2\nabla_{\mathbb{H}^n}u\cdot\nabla_{\mathbb{H}^n}\Delta_{\mathbb{H}^n}\phi+u(\Delta_{\mathbb{H}^n})^2\phi\\ &\quad+2\sum_{j=1}^{2n}\Delta_{\mathbb{H}^n}X_juX_j\phi+4\sum_{j=1}^{2n}\nabla_{\mathbb{H}^n}X_ju\cdot\nabla_{\mathbb{H}^n}X_j\phi\\ &\quad+2\sum_{j=1}^{2n}X_ju\Delta_{\mathbb{H}^n}X_j\phi.\end{aligned}$$

Then, for any $g\in B(g_0,\frac{R}{2})$,

$$\begin{aligned}|u(g)| &= \big|\big((-\Delta_{\mathbb{H}^n})^2+V^2\big)^{-1}(2\nabla_{\mathbb{H}^n}\Delta_{\mathbb{H}^n}u\cdot\nabla_{\mathbb{H}^n}\phi+2\Delta_{\mathbb{H}^n}u\Delta_{\mathbb{H}^n}\phi\\ &\quad+2\nabla_{\mathbb{H}^n}u\cdot\nabla_{\mathbb{H}^n}\Delta_{\mathbb{H}^n}\phi+u(\Delta_{\mathbb{H}^n})^2\phi+2\sum_{j=1}^{2n}\Delta_{\mathbb{H}^n}X_juX_j\phi\\ &\quad+4\sum_{j=1}^{2n}\nabla_{\mathbb{H}^n}X_ju\cdot\nabla_{\mathbb{H}^n}X_j\phi+2\sum_{j=1}^{2n}X_ju\Delta_{\mathbb{H}^n}X_j\phi)\big)(g)\big|\\ &\leq C\int_{\mathbb{H}^n}\frac{1}{|g^{-1}h|^{Q-4}}\big(|\nabla_{\mathbb{H}^n}\Delta_{\mathbb{H}^n}u|\cdot|\nabla_{\mathbb{H}^n}\phi|+|\Delta_{\mathbb{H}^n}u|\cdot|\Delta_{\mathbb{H}^n}\phi|\end{aligned}$$

$$
\begin{aligned}
&+ |\nabla_{\mathbb{H}^n} u| \cdot |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} \phi| + |u| \cdot |(\Delta_{\mathbb{H}^n})^2 \phi| + \sum_{j=1}^{2n} |\Delta_{\mathbb{H}^n} X_j u| \cdot |X_j \phi| \\
&+ \sum_{j=1}^{2n} |\nabla_{\mathbb{H}^n} X_j u| \cdot |\nabla_{\mathbb{H}^n} X_j \phi| + \sum_{j=1}^{2n} |X_j u| \cdot |\Delta_{\mathbb{H}^n} X_j \phi| \big) dh \\
\le\ & \frac{C}{R^{Q-4}} \int_{B(g_0,2R)\setminus B(g_0,R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u| \cdot |\nabla_{\mathbb{H}^n} \phi| dh \\
&+ \frac{C}{R^{Q-4}} \int_{B(g_0,2R)\setminus B(g_0,R)} |\Delta_{\mathbb{H}^n} u| \cdot |\Delta_{\mathbb{H}^n} \phi| dh \\
&+ \frac{C}{R^{Q-4}} \int_{B(g_0,2R)\setminus B(g_0,R)} |\nabla_{\mathbb{H}^n} u| \cdot |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} \phi| dh \\
&+ \frac{C}{R^{Q-4}} \int_{B(g_0,2R)\setminus B(g_0,R)} |u| \cdot |(\Delta_{\mathbb{H}^n})^2 \phi| dh \\
&+ \frac{C}{R^{Q-4}} \int_{B(g_0,2R)\setminus B(g_0,R)} \sum_{j=1}^{2n} |\Delta_{\mathbb{H}^n} X_j u| \cdot |X_j \phi| dh \\
&+ \frac{C}{R^{Q-4}} \int_{B(g_0,2R)\setminus B(g_0,R)} \sum_{j=1}^{2n} |\nabla_{\mathbb{H}^n} X_j u| \cdot |\nabla_{\mathbb{H}^n} X_j \phi| dh \\
&+ \frac{C}{R^{Q-4}} \int_{B(g_0,2R)\setminus B(g_0,R)} \sum_{j=1}^{2n} |X_j u| \cdot |\Delta_{\mathbb{H}^n} X_j \phi| dh \\
\doteq\ & J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7.
\end{aligned}
$$

In the following we shall consider the term $J_i,\ i = 1, \ldots, 7$, respectively.

If we make a little modification for the proof of Lemma 10 (or Lemma 3.3 [8]), we can obtain the following fact: Suppose $-\Delta_{\mathbb{H}^n} u \le 0$ in $B(g_0, R)$. Then there exists a constant $C > 0$ such that

$$
\int_{B(g_0,\frac{R}{2})} |\nabla_{\mathbb{H}^n} u(h)|^2\, dh \le \frac{C}{R^2} \int_{B(g_0,R)} |u(h)|^2\, dh. \tag{9}
$$

Since $-\Delta_{\mathbb{H}^n} u$ satisfies $-\Delta_{\mathbb{H}^n}(-\Delta_{\mathbb{H}^n} u) = -V^2 u \le 0$, it follows from (9) that

$$
\int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(h)|^2 dh \le \frac{C}{R^2} \int_{B(g_0,4R)} |\Delta_{\mathbb{H}^n} u(h)|^2 dh.
$$

Thus by using Lemma 11 and the above inequality we get

$$
\begin{aligned}
J_1 &\le \frac{C}{R^{Q-4}} \int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(h)| \cdot |\nabla_{\mathbb{H}^n} \phi(h)| dh \\
&\le \frac{C}{R^{Q-3}} \int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(h)| dh
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{C}{R^{\frac{Q}{2}-3}}\left(\int_{B(g_0,2R)}|\nabla_{\mathbb{H}^n}\Delta_{\mathbb{H}^n}u(h)|^2dh\right)^{\frac{1}{2}}\\
&\le \frac{C}{R^{\frac{Q}{2}-2}}\left(\int_{B(g_0,4R)}|\Delta_{\mathbb{H}^n}u(h)|^2dh\right)^{\frac{1}{2}}\\
&\le \frac{C}{R^{\frac{Q}{2}-2}}\left\{\left(\frac{1}{R^4}\int_{B(g_0,8R)}|u(h)|^2dh\right)^{\frac{1}{2}}+\left(\frac{1}{R^2}\int_{B(g_0,8R)}|\nabla_{\mathbb{H}^n}u(h)|^2dh\right)^{\frac{1}{2}}\right\}\\
&\le C\left\{\left(\frac{1}{|B(g_0,8R)|}\int_{B(g_0,8R)}|u(h)|^2dh\right)^{\frac{1}{2}}+R\left(\frac{1}{|B(g_0,8R)|}\int_{B(g_0,8R)}|\nabla_{\mathbb{H}^n}u(h)|^2dh\right)^{\frac{1}{2}}\right\}.
\end{aligned}
$$

Similarly,

$$
J_2\le C\left\{\left(\frac{1}{|B(g_0,8R)|}\int_{B(g_0,8R)}|u(h)|^2dh\right)^{\frac{1}{2}}+R\left(\frac{1}{|B(g_0,8R)|}\int_{B(g_0,8R)}|\nabla_{\mathbb{H}^n}u(h)|^2dh\right)^{\frac{1}{2}}\right\}.
$$

For J_3 and J_4 we have

$$
\begin{aligned}
J_3+J_4&\le \frac{C}{R^{\frac{Q}{2}-1}}\left(\int_{B(g_0,2R)}|\nabla_{\mathbb{H}^n}u(h)|^2dh\right)^{\frac{1}{2}}+\frac{C}{R^{\frac{Q}{2}}}\left(\int_{B(g_0,2R)}|u(h)|^2dh\right)^{\frac{1}{2}}\\
&\le C\left\{\left(\frac{1}{|B(g_0,8R)|}\int_{B(g_0,8R)}|u(h)|^2dh\right)^{\frac{1}{2}}+R\left(\frac{1}{|B(g_0,8R)|}\int_{B(g_0,8R)}|\nabla_{\mathbb{H}^n}u(h)|^2dh\right)^{\frac{1}{2}}\right\}.
\end{aligned}
$$

The estimate of J_7 is similar to that of J_3. Thus we omit the detail. For J_6, by Proposition 1 in [9] we have

$$
\begin{aligned}
J_6&\le \frac{C}{R^{Q-2}}\int_{B(g_0,2R)}|\nabla^2_{\mathbb{H}^n}u(h)|dh\\
&\le \frac{C}{R^{\frac{Q}{2}-2}}\left(\int_{B(g_0,2R)}|\nabla^2_{\mathbb{H}^n}u(h)|^2dh\right)^{\frac{1}{2}}\\
&\le \frac{C}{R^{\frac{Q}{2}-2}}\left(\int_{B(g_0,2R)}|\Delta_{\mathbb{H}^n}u(h)|^2dh\right)^{\frac{1}{2}}\\
&\le \frac{C}{R^{\frac{Q}{2}-2}}\left\{\left(\frac{1}{R^4}\int_{B(g_0,8R)}|u(h)|^2dh\right)^{\frac{1}{2}}+\left(\frac{1}{R^2}\int_{B(g_0,8R)}|\nabla_{\mathbb{H}^n}u(h)|^2dh\right)^{\frac{1}{2}}\right\}\\
&\le C\left\{\left(\frac{1}{|B(g_0,8R)|}\int_{B(g_0,8R)}|u(h)|^2dh\right)^{\frac{1}{2}}+R\left(\frac{1}{|B(g_0,8R)|}\int_{B(g_0,8R)}|\nabla_{\mathbb{H}^n}u(h)|^2dh\right)^{\frac{1}{2}}\right\},
\end{aligned}
$$

where we used Lemma 11 for the fourth inequality.

Finally, we consider the term J_5. By the commutation of left invariant vector fields it is easy to see that $X_j^2X_i=8X_jT+X_iX_j^2$ when $j=i+n$ and

$X_j^2 X_i = X_i X_j^2$ when $j \neq i+n$ for $1 \leq i,j \leq n$. Thus,

$$\begin{aligned}
J_5 &\leq \frac{C}{R^{Q-3}} \sum_{j=1}^{2n} \left(\int_{B(g_0,2R)} |X_j T u(h)| dh + \int_{B(g_0,2R)} |X_j \Delta_{\mathbb{H}^n} u(h)| dh \right) \\
&\leq \frac{C}{R^{Q-3}} \left(\int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} T u(h)| dh + \int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(h)| dh \right) \\
&\leq \frac{C}{R^{\frac{Q}{2}-3}} \left\{ \left(\int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} T u(h)|^2 dh \right)^{\frac{1}{2}} + \left(\int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(h)|^2 dh \right)^{\frac{1}{2}} \right\} \\
&\leq \frac{C}{R^{\frac{Q}{2}-3}} \left\{ \left(\int_{B(g_0,2R)} |(\Delta_{\mathbb{H}^n})^{\frac{1}{2}} T u(h)|^2 dh \right)^{\frac{1}{2}} + \left(\int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(h)|^2 dh \right)^{\frac{1}{2}} \right\} \\
&\leq \frac{C}{R^{\frac{Q}{2}-3}} \left\{ \left(\int_{B(g_0,2R)} |T (\Delta_{\mathbb{H}^n})^{\frac{1}{2}} u(h)|^2 dh \right)^{\frac{1}{2}} + \left(\int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(h)|^2 dh \right)^{\frac{1}{2}} \right\} \\
&\leq \frac{C}{R^{\frac{Q}{2}-3}} \left\{ \left(\int_{B(g_0,2R)} |\nabla^2_{\mathbb{H}^n} (\Delta_{\mathbb{H}^n})^{\frac{1}{2}} u(h)|^2 dh \right)^{\frac{1}{2}} + \left(\int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(h)|^2 dh \right)^{\frac{1}{2}} \right\} \\
&\leq \frac{C}{R^{\frac{Q}{2}-3}} \left\{ \left(\int_{B(g_0,2R)} |\Delta_{\mathbb{H}^n} (\Delta_{\mathbb{H}^n})^{\frac{1}{2}} u(h)|^2 dh \right)^{\frac{1}{2}} + \left(\int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(h)|^2 dh \right)^{\frac{1}{2}} \right\} \\
&\leq \frac{C}{R^{\frac{Q}{2}-3}} \left\{ \left(\int_{B(g_0,2R)} |(\Delta_{\mathbb{H}^n})^{\frac{1}{2}} \Delta_{\mathbb{H}^n} u(h)|^2 dh \right)^{\frac{1}{2}} + \left(\int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(h)|^2 dh \right)^{\frac{1}{2}} \right\} \\
&\leq \frac{2C}{R^{\frac{Q}{2}-3}} \left(\int_{B(g_0,2R)} |\nabla_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(h)|^2 dh \right)^{\frac{1}{2}}.
\end{aligned}$$

By using the estimate of J_1 we obtain

$$J_6 \leq C \left\{ \left(\frac{1}{|B(g_0,8R)|} \int_{B(g_0,8R)} |u(h)|^2 dh \right)^{\frac{1}{2}} + R \left(\frac{1}{|B(g_0,8R)|} \int_{B(g_0,8R)} |\nabla_{\mathbb{H}^n} u(h)|^2 dh \right)^{\frac{1}{2}} \right\}.$$

This completes the proof of the lemma. □

We need another lemma before we carry out the proof of Lemma 5.

Lemma 13. *Let $j = 1,2,3$. Suppose $V \in B_q$, $\frac{Q}{2} < q < \frac{2Q}{4-j}$, and $(-\Delta_{\mathbb{H}^n})^2 u + V^2 u = 0$ in $B(g_0,R)$. Then there exist constants $C > 0$ and $l_1 > 0$ such that*

$$\left(\int_{B(g_0,\frac{R}{2})} |\nabla^j_{\mathbb{H}^n} u(g)|^t \, dg \right)^{\frac{1}{t}} \leq C R^{\frac{2Q}{q}-4} \big(1 + R\, m(g_0,V)\big)^{l_1} \sup_{B(g_0,R)} |u(g)|,$$

where $\frac{1}{t} = \frac{2}{q} - \frac{4-j}{Q}$.

Proof. Let $\phi \in C_0^\infty(B(g_0,R))$ such that $\phi \equiv 1$ on $B(g_0,\frac{3R}{4})$, $0 < \phi \leq 1$. Also, ϕ satisfies $|\nabla^j_{\mathbb{H}^n} \phi| \leq C R^{-j}$ for $j = 1,2,3,4$. Denote by $\Gamma_0(g,h)$ the fundamental solution of $(-\Delta_{\mathbb{H}^n})^2$. Note that,

$$
\begin{aligned}
u(g)\phi(g) &= \int_{\mathbf{H}^n} \Gamma_0(g,h)(-\Delta_{\mathbb{H}^n})^2(u\phi)(h)\,dh \\
&= \int_{\mathbb{H}^n} \Gamma_0(g,h)\big(-V^2(h)u(h)\phi(h) + 4\Delta_{\mathbb{H}^n}(\nabla_{\mathbb{H}^n}u(h)\cdot\nabla_{\mathbb{H}^n}\phi(h)) \\
&\quad + 2\Delta_{\mathbb{H}^n}(u(h)\Delta_{\mathbb{H}^n}\phi(h)) - 4\nabla^2_{\mathbb{H}^n}u(h)\cdot\nabla^2_{\mathbb{H}^n}\phi(h) \\
&\quad - 4\nabla_{\mathbb{H}^n}u(h)\cdot\nabla_{\mathbb{H}^n}(\Delta_{\mathbb{H}^n}\phi(h)) - u(h)(\Delta^2_{\mathbb{H}^n}\phi(h))\big)\,dh, \qquad (10)
\end{aligned}
$$

where $\nabla^2_{\mathbb{H}^n}u(h)\cdot\nabla^2_{\mathbb{H}^n}\phi(h) = \sum_{i,j=1}^{2n} X_iX_ju(h)\cdot X_iX_j\phi(h)$.

Then by integration by parts and (7) in Lemma 6 we have, for $g \in B(g_0,\frac{R}{2})$,

$$
\begin{aligned}
|\nabla^j_{\mathbb{H}^n}u(g)| &\le C\int_{B(g_0,R)} \frac{V^2(h)|u(h)||\phi(h)|}{|g^{-1}h|^{Q-4+j}}\,dh + \frac{C}{R^{Q+j}}\int_{B(g_0,R)}|u(h)|\,dh \\
&\le C\sup_{B(g_0,R)}|u(g)|\left(\int_{B(g_0,R)}\frac{V^2(h)|\phi(h)|}{|g^{-1}h|^{Q-4+j}}\,dh + \frac{1}{R^j}\right).
\end{aligned}
$$

Then by the theorem on fractional integration on the Heisenberg group (cf. [5]), it follows that

$$
\begin{aligned}
\left(\int_{B(g_0,\frac{R}{2})}|\nabla^j_{\mathbb{H}^n}u(g)|^t\,dg\right)^{\frac{1}{t}} &\le C\sup_{B(g_0,R)}|u(g)|\left(\left(\int_{B(g_0,R)}V(g)^q\,dg\right)^{\frac{2}{q}} + R^{\frac{2Q}{q}-4}\right) \\
&\le CR^{\frac{2Q}{q}-4}\sup_{B(g_0,R)}|u(g)|\left(\frac{1}{R^{Q-2}}\int_{B(g_0,R)}V(g)\,dg + 1\right) \\
&\le CR^{\frac{2Q}{q}-4}\big(1+R\,m(g_0,V)\big)^{l_1}\sup_{B(g_0,R)}|u(g)|,
\end{aligned}
$$

where $\frac{1}{t} = \frac{2}{q} - \frac{4-j}{Q}$ and we have used (1) and Lemma 4. $\square$

Corollary 3. *Let $j = 1,2$. Suppose $V \in B_{\frac{Q}{2}}$ and $(-\Delta_{\mathbb{H}^n})^2u + V^2u = 0$ in $B(g_0,R)$. Then there exist constants $C > 0$ and $l_1 > 0$ such that*

$$
\left(\int_{B(g_0,\frac{R}{2})}|\nabla^j_{\mathbb{H}^n}u(g)|^2\,dg\right)^{\frac{1}{2}} \le \frac{C\big(1+R\,m(g_0,V)\big)^{l_1}}{R^j}\sup_{B(g_0,R)}|u(g)|. \qquad (11)
$$

Now we are in a position to give the proof of Lemma 5.

Proof of Lemma 5. Let $\phi \in C_0^\infty(B(g_0,\frac{R}{2}))$ such that $\phi \equiv 1$ on $B(g_0,\frac{R}{4})$, $0 < \phi \le 1$. Also, ϕ satisfies $|\nabla_{\mathbb{H}^n}\phi| \le CR^{-1}$ and $|\nabla^2_{\mathbb{H}^n}\phi| \le CR^{-2}$. Using Lemma 9 and Lemma 11 we have

$$
\begin{aligned}
&\int_{B(g_0,\frac{R}{4})}|u(g)|^2m(g,V)^4\,dg + \int_{B(g_0,\frac{R}{4})}|\nabla_{\mathbb{H}^n}u(g)|^2m(g,V)^2\,dg \\
\le\ &\frac{C}{R^4}\int_{B(g_0,R)}|u(g)|^2\,dg + \frac{C}{R^2}\int_{B(g_0,R)}|\nabla_{\mathbb{H}^n}u(g)|^2\,dg.
\end{aligned}
$$

From (c) in Lemma 3 it follows that

$$\begin{aligned}
&\int_{B(g_0,R)} |u(g)|^2\,dg \\
&\le \frac{C\{1+Rm(g_0,V)\}^{\frac{4k_0}{k_0+1}}}{R^4 m(g_0,V)^4} \int_{B(g_0,R)} |u(g)|^2\,dg + R^2 \int_{B(g_0,R)} |\nabla_{\mathbb{H}^n} u(g)|^2\,dg \\
&\le \frac{C}{\{1+Rm(g_0,V)\}^{\frac{4}{k_0+1}}} \int_{B(g_0,R)} |u(g)|^2\,dg + R^2 \int_{B(g_0,R)} |\nabla_{\mathbb{H}^n} u(g)|^2\,dg.
\end{aligned}$$

Then

$$\begin{aligned}
&\left(\frac{1}{|B(g_0,\frac{R}{4})|}\int_{B(g_0,R)} |u(g)|^2\,dg\right)^{\frac12} \\
&\le \frac{C}{\{1+Rm(g_0,V)\}^{\frac{2}{k_0+1}}}\left[\left(\frac{1}{|B(g_0,R)|}\int_{B(g_0,R)}|u(g)|^2dg\right)^{\frac12} + R\left(\frac{1}{|B(g_0,R)|}\int_{B(g_0,R)}|\nabla_{\mathbb{H}^n} u(g)|^2\,dg\right)^{\frac12}\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&R\left(\frac{1}{|B(g_0,\frac{R}{4})|}\int_{B(g_0,R)} |\nabla_{\mathbb{H}^n} u(g)|^2\,dg\right)^{\frac12} \\
&\le \frac{C}{\{1+Rm(g_0,V)\}^{\frac{2}{k_0+1}}}\left[\left(\frac{1}{|B(g_0,R)|}\int_{B(g_0,R)}|u(g)|^2dg\right)^{\frac12} + R\left(\frac{1}{|B(g_0,R)|}\int_{B(g_0,R)}|\nabla_{\mathbb{H}^n} u(g)|^2\,dg\right)^{\frac12}\right].
\end{aligned}$$

By repeating above argument, for any $N>0$ we have

(12)

$$\begin{aligned}
&\left(\frac{1}{|B(g_0,\frac{R}{4^N})|}\int_{B(g_0,\frac{R}{4^N})} |u(g)|^2dg\right)^{\frac12} + R\left(\frac{1}{|B(g_0,\frac{R}{4^N})|}\int_{B(g_0,\frac{R}{4^N})} |\nabla_{\mathbb{H}^n} u(g)|^2\,dg\right)^{\frac12} \\
&\le \frac{C_N}{\{1+Rm(g_0,V)\}^{\frac{N}{k_0+1}}}\left[\left(\frac{1}{|B(g_0,R)|}\int_{B(g_0,R)}|u(g)|^2dg\right)^{\frac12} + R\left(\frac{1}{|B(g_0,R)|}\int_{B(g_0,R)}|\nabla_{\mathbb{H}^n} u(g)|^2\,dg\right)^{\frac12}\right].
\end{aligned}$$

Then by using Lemma 12, (11) and (12) we arrive at the desired estimate. □

3.2. Proof of Proposition 2

In this subsection we investigate the proof of Proposition 2. We arrive at it by combining the following Lemma 14 and Lemma 5.

Lemma 14. *Let $j=1,2,3$. Suppose $V\in B_{\frac{Q}{2}}$ and there exists a constant $C>0$ such that $V(g)\le Cm(g,V)^2$. Assume also that $(-\Delta_{\mathbb{H}^n})^2u+V^2u=0$ in $B(g_0,R)$. Then there exist constants $C'>0$ and $l_0>0$ such that*

$$\sup_{g\in B(g_0,\frac{R}{2})} |\nabla^j_{\mathbb{H}^n} u(g)| \le C'\frac{\big(1+R\,m(g_0,V)\big)^{l_0}}{R^j}\sup_{g\in B(g_0,R)}|u(g)|. \tag{13}$$

Proof. Let $\phi \in C_0^\infty(B(g_0,R))$ such that $\phi \equiv 1$ on $B(g_0,\frac{3R}{4})$, $0<\phi\le 1$. Also, ϕ satisfies $|\nabla^j_{\mathbb{H}^n}\phi| \le CR^{-j}$ for $j=1,2,3,4$. Using (10) and (b) of Lemma 3 we get

$$\begin{aligned}
&|\nabla^j_{\mathbb{H}^n} u(g_0)| \\
&\le C\int_{B(g_0,R)} \frac{V^2(h)|u(h)|}{|g_0^{-1}h|^{Q-4+j}}\,dh + \frac{C}{R^{Q+j}}\int_{B(g_0,R)} |u(h)|\,dh \\
&\le C\sup_{B(g_0,R)}|u(g)|\left(\int_{B(g_0,R)} \frac{V^2(h)}{|g_0^{-1}h|^{Q-4+j}}\,dh + \frac{1}{R^j}\right) \\
&\le C\sup_{B(g_0,R)}|u(g)|\left(\left(\int_{B(g_0,R)} V^{\frac{Q}{2}}(h)dh\right)^{\frac{4}{Q}}\left(\int_{B(g_0,R)} |g_0^{-1}h|^{-(Q-4+j)p}dh\right)^{\frac1p} + \frac{1}{R^j}\right) \\
&\le C\sup_{B(g_0,R)}|u(g)|\left(\left(\frac{1}{|B(g_0,R)|}\int_{B(g_0,R)} V(h)dh\right)^2 R^4\left(\int_0^R r^{Q-(Q-4+j)p-1}dr\right)^{\frac1p} + \frac{1}{R^j}\right) \\
&\le C\sup_{B(g_0,R)}|u(g)|\left(\frac{\{1+Rm(g_0,V)\}^{4k_0}R^4m^4(g_0,V)}{R^j} + \frac{1}{R^j}\right) \\
&\le C'\frac{\{1+Rm(g_0,V)\}^{4k_0+4}}{R^j}\sup_{B(g_0,R)}|u(g)|,
\end{aligned}$$

where $\frac{4}{Q}+\frac{1}{p}=1$.

From the above inequality we conclude for all $g\in B(g_0,\frac{R}{2})$,

$$|\nabla^j_{\mathbb{H}^n} u(g)| \le C\frac{\{1+Rm(g,V)\}^{4k_0+4}}{R^j}\sup_{B(g,\frac{R}{4})}|u(h)|.$$

Using (b) of Lemma 3 we get

$$\sup_{B(g_0,\frac{R}{2})}|\nabla^j_{\mathbb{H}^n} u(g)| \le C\frac{\{1+Rm(g_0,V)\}^{(4k_0+4)^2}}{R^j}\sup_{B(g_0,R)}|u(g)|.$$

This completes the proof. □

4. Proof of the main results

Theorem 1 immediately follows from the following lemma.

Lemma 15. (1) *Suppose $V\in B_{\frac{Q}{2}}$. Then there exists a constant $C>0$ such that*

(14) $$|m(g,V)^4 H_2^{-1}f(g)| \le CM(|f|)(g) \quad \text{for any } f\in C_0^\infty(\mathbb{H}^n),$$

where M is the Hardy-Littlewood maximal operator.

(2) *Let $j=1,2,3$. Suppose $V\in B_{\frac{Q}{2}}$ and there exists a constant $C>0$ such that $V(g)\le Cm^2(g,V)$. Then there exists a constant $C'>0$ such that*

(15) $$|m(g,V)^{4-j}\nabla^j_{\mathbb{H}^n} H_2^{-1}f(g)| \le C'M(|f|)(g) \quad \text{for any } f\in C_0^\infty(\mathbb{H}^n),$$

where M is the Hardy-Littlewood maximal operator.

Proof. Let $r = \frac{1}{m(g,V)}$. It follows from Proposition 1 that

$$\begin{aligned}
|m(g,V)^4 H_2^{-1} f(g)| &\le C_N \int_{\mathbb{H}^n} \frac{m(g,V)^4 |f(h)|}{\{1+m(g,V)|g^{-1}h|\}^N |g^{-1}h|^{Q-4}} dh \\
&\le C_N \sum_{j=-\infty}^{\infty} \int_{2^{j-1}r < |g^{-1}h| \le 2^j r} \frac{|f(h)|}{r^4\{1+r^{-1}|g^{-1}h|\}^N |g^{-1}h|^{Q-4}} dh \\
&\le C_N \sum_{j=-\infty}^{\infty} \frac{2^{4(j-1)+Q}}{(1+2^j)^N} \frac{1}{(2^j r)^Q} \int_{|g^{-1}h| \le 2^j r} |f(h)| dh \\
&\le CC_N \sum_{j=-\infty}^{\infty} \frac{2^{4(j-1)+Q}}{(1+2^j)^N} M(|f|)(g) \\
&\le C_1 C_N M(|f|)(g),
\end{aligned}$$

where we take N sufficiently large.

Similarly, we can conclude the proof of (15) by using Proposition 2. □

Proof of Theorem 1. Since $V(g) \le Cm^2(g,V)$, we can obtain the estimate (2) by using (14), (15) and the fact that the Hardy-Littlewood maximal operator is bounded on $L^p(\mathbb{H}^n)$, $1 < p \le \infty$. □

Proof of Corollary 1. It follows from Theorem 4.4 of Chapter 2 in [2] that $\nabla^4_{\mathbb{H}^n}(\Delta^2_{\mathbb{H}^n})^{-1}$ is bounded on $L^p(\mathbb{H}^n)$, $1 < p < \infty$. Then

$$\begin{aligned}
\| \nabla^4_{\mathbb{H}^n} H_2^{-1} f \|_{L^p(\mathbb{H}^n)} &= \| \nabla^4_{\mathbb{H}^n}(\Delta^2_{\mathbb{H}^n})^{-1}(\Delta^2_{\mathbb{H}^n}) H_2^{-1} f \|_{L^p(\mathbb{H}^n)} \\
&\le C \| (\Delta^2_{\mathbb{H}^n} - V^2 + V^2) H_2^{-1} f \|_{L^p(\mathbb{H}^n)} \\
&\le C \| f \|_{L^p(\mathbb{H}^n)},
\end{aligned}$$

where the last inequality follows from Theorem 1 for $j = 0$. □

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