IDENTITIES ARISING FROM GAUSS SUMS FOR SYMPLECTIC AND ORTHOGONAL GROUPS

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Abstract. We express Gauss sums for symplectic and orthogonal groups over finite fields as averages of exponential sums over certain maximal tori. Together with our previous results, we obtain some interesting identities involving various classical Gauss and Kloosterman sums.

1. Introduction

In a series of papers [7, 8, 9, 10, 11, 12, 15], the second author considered Gauss sums for various finite classical groups, which are exponential sums of the form

$$\sum_{g \in G^F} \chi(\det g)\lambda((\operatorname{tr} g)^r),$$

where G is a classical group over a finite field, F is the Frobenius map, G^F denotes the set of fixed points of G under F, and χ (resp. λ) is a multiplicative (resp. nontrivial additive) character of the finite field. He obtained explicit expressions of these sums in terms of classical Gauss and Kloosterman sums and proved some applications. On the other hand, we proved the next formula in [3, Theorem 3.1], which follows from the representation theory of finite reductive groups: let ψ be a complex valued class function on $G(\overline{\mathbb{F}}_q)$ whose value depend only on semisimple part of each element. Then we have

$$(1) \qquad \sum_{g \in G^F} \psi(g) = \frac{|G^F|}{|W|} \sum_{w \in W} \frac{1}{|T^{wF}|} \sum_{a \in T^{wF}} \psi(a),$$

where T is an F-stable maximal torus of G, $W = N(T)/T$ is the Weyl group of G with respect to T, and $_wF$ is the twisted Frobenius map of T given by $_wF(t) = wF(t)w^{-1}$.

In this paper, we apply the above formula to calculate the Gauss sums for Sp_{2n}, GSp_{2n}, $\operatorname{SO}_{2n}^{\pm}$, and SO_{2n+1} (Theorems 3.1, 3.2, 3.5, 3.6, and 3.7). Combined with the second author's previous results cited above, these yield new

identities involving various classical Gauss and Kloosterman sums (Theorems 4.1, 4.3, and 4.5), which may be of interest in number theory and combinatorics.

Another interesting consequence is the complete relations among Gauss sums for $\mathrm{Sp}(2n, q)$, $\mathrm{SO}(2n+1, q)$, $\mathrm{SO}^+(2n, q)$, and $\mathrm{SO}^-(2n, q)$. Namely, we have the following identities (Theorem 4.8):

$$\sum_{g \in \mathrm{SO}(2n+1,q)} \lambda(\mathrm{tr}\, g) = \lambda(1) \sum_{g \in \mathrm{Sp}(2n,q)} \lambda(\mathrm{tr}\, g),$$

$$\sum_{g \in \mathrm{SO}^+(2n,q)} \lambda(\mathrm{tr}\, g) = q^{-n} \sum_{g \in \mathrm{Sp}(2n,q)} \lambda(\mathrm{tr}\, g),$$

$$\sum_{g \in \mathrm{SO}^-(2n,q)} \lambda(\mathrm{tr}\, g) = -q^{-n} \sum_{g \in \mathrm{Sp}(2n,q)} \lambda(\mathrm{tr}\, g).$$

The first of these was noted in [11, (5.12)]. However, as we confessed at that time, we didn't have any philosophical reason that this should be so. Now, that is an easy consequence of the main formula (1). The second identity was observed in [15], while the last one was obtained from the old and new expressions of Gauss sums for $\mathrm{SO}^+(2n, q)$ and $\mathrm{SO}^-(2n, q)$. This last identity would have been found earlier if we pushed ourselves a little further in getting an expression of Gauss sums for $\mathrm{SO}^-(2n, q)$, as in Proposition 4.9.

2. Preliminaries and notations

One is referred to [2] and [17] for some elementary facts below. Let $\mathbb{F}_q$ be the finite field with q elements. Then, for each $m \in \mathbb{Z}_{>0}$, $\mathbb{F}_{q^m}$ denotes the extension field of degree m of $\mathbb{F}_q$ in a fixed algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. For each $m \in \mathbb{Z}_{>0}$, C_{q^m} denotes the kernel of the norm map $N_{\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^m}} : \mathbb{F}_{q^{2m}}^\times \to \mathbb{F}_{q^m}^\times$, so that

$$(2) \qquad C_{q^m} = \{\alpha \in \mathbb{F}_{q^{2m}} | \alpha^{q^m+1} = 1\}.$$

Let λ be a nontrivial additive character of $\mathbb{F}_q$, χ a multiplicative character of $\mathbb{F}_q$, and let $\widetilde{\chi}$ be a multiplicative character of $\mathbb{F}_{q^2}$. For each $m \in \mathbb{Z}_{>0}$, we put

$$(3) \qquad \lambda^{(m)} = \lambda \circ \mathrm{tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}, \ \chi^{(m)} = \chi \circ N_{\mathbb{F}_{q^m}/\mathbb{F}_q}, \ \widetilde{\chi}^{(2m)} = \widetilde{\chi} \circ N_{\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^2}}.$$

To simplify notations, the lifting $\lambda^{(2)}$ of λ to $\mathbb{F}_{q^2}$ will be denoted by λ'. For $m \in \mathbb{Z}_{>0}$ and $\gamma \in \mathbb{F}_q^\times$, we put

$$(4) \qquad K(\lambda^{(m)}; \gamma) = \sum_{\alpha \in \mathbb{F}_{q^m}^\times} \lambda^{(m)}(\alpha + \gamma\alpha^{-1}),$$

$$(5) \qquad K(\lambda^{(m)}) = K(\lambda^{(m)}; 1) = \sum_{\alpha \in \mathbb{F}_{q^m}^\times} \lambda^{(m)}(\alpha + \alpha^{-1}),$$

$$(6) \qquad K(\lambda^{(2m)}, \widetilde{\chi}^{(2m)}) = \sum_{\alpha \in \mathbb{F}_{q^{2m}}^{\times}} \widetilde{\chi}^{(2m)}(\alpha) \lambda^{(2m)}(\alpha + \alpha^{-1}),$$

$$(7) \qquad J(\lambda^{(m)}; \gamma) = \sum_{\alpha \in N_{\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^m}}^{-1}(\gamma)} \lambda^{(m)}(\alpha + \gamma\alpha^{-1})$$

$$= \sum_{\alpha \in N_{\mathbb{F}_{q^{2m}}/\mathbb{F}_{q^m}}^{-1}(\gamma)} \lambda^{(2m)}(\alpha),$$

$$(8) \qquad J(\lambda^{(m)}) = J(\lambda^{(m)}; 1) = \sum_{\alpha \in C_{q^m}} \lambda^{(m)}(\alpha + \alpha^{-1})$$

$$= \sum_{\alpha \in C_{q^m}} \lambda^{(2m)}(\alpha),$$

$$(9) \qquad J(\lambda^{(m)}, \widetilde{\chi}^{(2m)}) = \sum_{\alpha \in C_{q^m}} \widetilde{\chi}^{(2m)}(\alpha) \lambda^{(m)}(\alpha + \alpha^{-1})$$

$$= \sum_{\alpha \in C_{q^m}} \widetilde{\chi}^{(2m)}(\alpha) \lambda^{(2m)}(\alpha),$$

$K(\lambda; \gamma)$, $K(\lambda', \widetilde{\chi})$, $J(\lambda; \gamma)$, and $J(\lambda, \widetilde{\chi})$ are respectively called a Kloosterman sum, a twisted Kloosterman sum, a unitary Kloosterman sum, and a twisted unitary Kloosterman sum. The ones in (4), (6), (7), (9) are liftings of those classical exponential sums. In addition, we need the Gauss sum $G(\chi, \lambda)$ and the hyperkloosterman sum $K_m(\lambda)$ which are respectively defined by

$$(10) \qquad G(\chi, \lambda) = \sum_{\alpha \in \mathbb{F}_q^{\times}} \chi(\alpha) \lambda(\alpha),$$

$$(11) \qquad K_m(\lambda; \gamma) = \sum_{\alpha_1, \ldots, \alpha_m \in \mathbb{F}_q^{\times}} \lambda(\alpha_1 + \cdots + \alpha_m + \gamma\alpha_1^{-1} \cdots \alpha_m^{-1}),$$

$$(12) \qquad K_m(\lambda) = K_m(\lambda; 1) = \sum_{\alpha_1, \ldots, \alpha_m \in \mathbb{F}_q^{\times}} \lambda(\alpha_1 + \cdots + \alpha_m + \alpha_1^{-1} \cdots \alpha_m^{-1}).$$

For $\alpha \in \overline{\mathbb{F}}_q^{\times}$, $m \in \mathbb{Z}_{>0}$, we will need the following diagonal matrices:

$$(13) \qquad t_m(\alpha) = \mathrm{diag}(\alpha, \alpha^q, \ldots, \alpha^{q^{m-1}}),$$

$$(14) \qquad t'_m(\alpha) = \mathrm{diag}(\alpha, \alpha^{-q}, \ldots, \alpha^{(-q)^{m-1}}).$$

We now recall some basic terminologies and notations about partitions from [18]. A partition of a positive integer n is a finite nondecreasing sequence $\mu_1, \ldots, \mu_s$ of positive integers with $\sum_{i=1}^{s} \mu_i = n$. The partition $(\mu_1, \ldots, \mu_s)$ is denoted by μ, and we write

$$(15) \qquad \mu \vdash n \quad \text{if } \mu \text{ is a partition of } n.$$

The μ_i's are the parts of μ, and

(16) $l(\mu) =$ the number of parts of μ,

(17) $|\mu| =$ the sum of parts of μ.

The multiplicity of i in $\mu \vdash n$ is

$$m_i = m_i(\mu) = |\{j \,|1 \leqslant j \leqslant s, \ \mu_j = i\}|,$$

in which case we write

$$\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n}).$$

So $\sum_{i=1}^n im_i = n$. The set of all partitions of n is denoted by $\mathcal{P}_n$ $(n \in \mathbb{Z}_{\geqslant 0})$, and the set of all partitions $\bigcup_{n \in \mathbb{Z}_{\geqslant 0}} \mathcal{P}_n$ by $\mathcal{P}$. Here $\mathcal{P}_0$ consists of a single element, the 'empty partition of 0', which we denote by $\mathbf{0}$. If $\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ is a partition of n, we write

(18) $z_\mu = 1^{m_1} 2^{m_2} \cdots n^{m_n} m_1! m_2! \cdots m_n!.$

For $\mu = \mathbf{0} \in \mathcal{P}_0$, we agree that

(19) $l(\mu) = 0, \ |\mu| = 0, \ z_\mu = 1.$

For integers n, r with $0 \leqslant r \leqslant n$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \prod_{j=0}^{r-1} \frac{q^{n-j} - 1}{q^{r-j} - 1}.$$

$[y]$ denotes the greatest integer $\leqslant y$ for a real number y.

3. Calculation of Gauss sums

In this section we apply the formula (1) to calculate the Gauss sums $\sum \lambda(\operatorname{tr} g)$ for symplectic and orthogonal groups. In doing so, since the summand of the summation over $w \in W$ in the right hand side of (1) depends only on the F-conjugacy class $\{v^{-1} w F(v) \mid v \in W\}$ of w in W (cf. the proof of [3, Theorem 3.1]),we will sum over the F-conjugacy classes in W with summands multiplied by their orders.

3.1. Symplectic groups

Let $G = \operatorname{Sp}(2n, \overline{\mathbb{F}}_q) = \{g \in \operatorname{GL}(2n, \overline{\mathbb{F}}_q) \mid {}^t g \widetilde{J} g = \widetilde{J}\}$, with $\widetilde{J} = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}$. F is the standard Frobenius map Fr_q on G, and $G^F = \operatorname{Sp}(2n, q)$, with

$$|\operatorname{Sp}(2n, q)| = q^{n^2} \prod_{j=1}^n (q^{2j} - 1).$$

$T = \{\text{diagonal matrices in } G\} = \{\operatorname{diag}(\alpha_1, \ldots, \alpha_n, \alpha_1^{-1}, \ldots, \alpha_n^{-1})\}$ is an F-stable maximal torus of G which is split over $\mathbb{F}_q$. $W = N(T)/T \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ is the Weyl group with respect to T and F-conjugacy classes in W are just

conjugacy classes in W, which are in bijective correspondence with the set of double partitions of n

$$(20) \qquad \mathcal{P}_n^{(2)} = \{ (\mu, \nu) \mid \mu, \nu \in \mathcal{P}, |\mu| + |\nu| = n \} \text{ (cf. (17))}.$$

Here in $\mathcal{P}_n^{(2)}$ the special pairs $(\mathbf{0}, \nu)$ with $\nu \in \mathcal{P}_n$ or $(\mu, \mathbf{0})$ with $\mu \in \mathcal{P}_n$ are included (cf. (19)).

If the conjugacy class $\mathrm{conj}(w)$ of w in W corresponds to the double partition $(\mu, \nu) = \big((\mu_1, \ldots, \mu_s), (\nu_1, \ldots, \nu_t) \big) \in \mathcal{P}_n^{(2)}$, then, in the notation of (13),

$$\begin{aligned}
T^{wF} = \Big\{ &\mathrm{diag}\big(t_{\mu_1}(\alpha_1), \ldots, t_{\mu_s}(\alpha_s), t_{\nu_1}(\beta_1), \ldots, t_{\nu_t}(\beta_t), \\
& t_{\mu_1}(\alpha_1^{-1}), \ldots, t_{\mu_s}(\alpha_s^{-1},), t_{\nu_1}(\beta_1^{-1}), \ldots, t_{\nu_t}(\beta_t^{-1}) \big) \Big| \\
& \alpha_1 \in \mathbb{F}_{q^{\mu_1}}^{\times}, \ldots, \alpha_s \in \mathbb{F}_{q^{\mu_s}}^{\times}, \beta_1 \in C_{q^{\nu_1}}, \ldots, \beta_t \in C_{q^{\nu_t}} \Big\} \\
\cong\ & \mathbb{F}_{q^{\mu_1}}^{\times} \times \cdots \times \mathbb{F}_{q^{\mu_s}}^{\times} \times C_{q^{\nu_1}} \times \cdots \times C_{q^{\nu_t}} \text{ (cf. (2))},
\end{aligned}$$

$$(21) \qquad |\mathrm{conj}(w)| = \frac{|W|}{2^{l(\mu)+l(\nu)} z_\mu z_\nu} = \frac{2^n n!}{2^{l(\mu)+l(\nu)} z_\mu z_\nu}.$$

Also, we have

$$\begin{aligned}
(22) \quad \sum_{a \in T^{wF}} \lambda(\mathrm{tr}\, a) &= \sum_{\alpha_1 \in \mathbb{F}_{q^{\mu_1}}^{\times}} \lambda^{(\mu_1)}(\alpha_1 + \alpha_1^{-1}) \cdots \sum_{\alpha_s \in \mathbb{F}_{q^{\mu_s}}^{\times}} \lambda^{(\mu_s)}(\alpha_s + \alpha_s^{-1}) \\
&\quad \times \sum_{\beta_1 \in C_{q^{\nu_1}}} \lambda^{(\nu_1)}(\beta_1 + \beta_1^{-1}) \cdots \sum_{\beta_t \in C_{q^{\nu_t}}} \lambda^{(\nu_t)}(\beta_t + \beta_t^{-1}) \\
&= K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) J(\lambda^{(\nu_1)}) \cdots J(\lambda^{(\nu_t)}) \\
&= (-1)^t K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})
\end{aligned}$$

which follows from (5), (8), and the following fact which is a special case of [5, (7.2.5)]: for a nontrivial additive character λ of $\mathbb{F}_q$, $\gamma \in \mathbb{F}_q^{\times}$, and $m \in \mathbb{Z}_{>0}$,

$$(23) \qquad J(\lambda^{(m)}; \gamma) = -K(\lambda^{(m)}; \gamma).$$

The next result follows now from (1) and the above.

Theorem 3.1.

$$\begin{aligned}
&\sum_{g \in \mathrm{Sp}(2n,q)} \lambda(\mathrm{tr}\, g) \\
&= q^{n^2} \prod_{j=1}^n (q^{2j} - 1) \\
&\quad \times \sum_{(\mu,\nu) \in \mathcal{P}_n^{(2)}} \frac{(-1)^{l(\nu)} K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{2^{l(\mu)+l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}
\end{aligned}$$

3.2. Symplectic similitude groups

Let $G = \mathrm{GSp}(2n, \overline{\mathbb{F}}_q) = \{g \in \mathrm{GL}(2n, \overline{\mathbb{F}}_q) \,|\, {}^t g \widetilde{J} g = \nu(g) \widetilde{J} \text{ for some } \nu(g) \in \overline{\mathbb{F}}_q^\times\}$, with $\widetilde{J} = \left[\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right]$. F is the standard Frobenius map Fr_q on G, so that

$$G^F = \mathrm{GSp}(2n, q) = \{g \in \mathrm{GL}(2n, q) \,|\, {}^t g \widetilde{J} g = \nu(g) \widetilde{J} \text{ for some } \nu(g) \in \mathbb{F}_q^\times\},$$

with

$$|\mathrm{GSp}(2n, q)| = (q-1) q^{n^2} \prod_{j=1}^n (q^{2j} - 1).$$

$T = \{\text{diagonal matrices in } G\} = \{\mathrm{diag}(\alpha_1, \ldots, \alpha_n, \gamma\alpha_1^{-1}, \ldots, \gamma\alpha_n^{-1})\}$ is an F-stable maximal torus of G which is split over $\mathbb{F}_q$. $W = N(T)/T \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ is the Weyl group with respect to T, and F-conjugacy classes in W are just conjugacy classes in W.

If $\mathrm{conj}(w)$ $(w \in W)$ corresponds to $(\mu, \nu) = \big((\mu_1, \ldots, \mu_s), (\nu_1, \ldots, \nu_t)\big) \in \mathcal{P}_n^{(2)}$ (cf. (20)), then, in the notation of (13),

$$\begin{aligned}
T^{wF} = \Big\{ &\mathrm{diag}\big(t_{\mu_1}(\alpha_1), \ldots, t_{\mu_s}(\alpha_s), t_{\nu_1}(\beta_1), \ldots, t_{\nu_t}(\beta_t), \\
&\gamma t_{\mu_1}(\alpha_1^{-1}), \ldots, \gamma t_{\mu_s}(\alpha_s^{-1}), \gamma t_{\nu_1}(\beta_1^{-1}), \ldots, \gamma t_{\nu_t}(\beta_t^{-1}))\big) \Big| \\
&\gamma \in \mathbb{F}_q^\times, \alpha_1 \in \mathbb{F}_{q^{\mu_1}}^\times, \ldots, \alpha_s \in \mathbb{F}_{q^{\mu_s}}^\times, \\
&\beta_1 \in N_{\mathbb{F}_q^{2\nu_1}/\mathbb{F}_q^{\nu_1}}^{-1}(\gamma), \ldots, \beta_t \in N_{\mathbb{F}_q^{2\nu_t}/\mathbb{F}_q^{\nu_t}}^{-1}(\gamma) \Big\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{a \in T^{wF}} \chi(\det a) \lambda(\mathrm{tr}\, a) \\
&= \sum_{\gamma \in \mathbb{F}_q^\times} \chi^n(\gamma) K(\lambda^{(\mu_1)}; \gamma) \cdots K(\lambda^{(\mu_s)}; \gamma) J(\lambda^{(\nu_1)}; \gamma) \cdots J(\lambda^{(\nu_t)}; \gamma) \\
&= (-1)^t \sum_{\gamma \in \mathbb{F}_q^\times} \chi^n(\gamma) K(\lambda^{(\mu_1)}; \gamma) \cdots K(\lambda^{(\mu_s)}; \gamma) K(\lambda^{(\nu_1)}; \gamma) \cdots K(\lambda^{(\nu_t)}; \gamma)
\end{aligned}$$

(cf. (4), (7), (23)).

The following theorem now follows from (1), (21) and the above.

Theorem 3.2.

$$\begin{aligned}
&\sum_{g \in \mathrm{GSp}(2n, q)} \chi(\det g) \lambda(\mathrm{tr}\, g) \\
&= q^{n^2} \prod_{j=1}^n (q^{2j} - 1) \\
&\quad \times \sum_{(\mu, \nu) \in \mathcal{P}_n^{(2)}} \frac{(-1)^{l(\nu)} \sum_{\gamma \in \mathbb{F}_q^\times} \chi^n(\gamma) K(\lambda^{(\mu_1)}; \gamma) \cdots K(\lambda^{(\mu_s)}; \gamma) K(\lambda^{(\nu_1)}; \gamma) \cdots K(\lambda^{(\nu_t)}; \gamma)}{2^{l(\mu)+l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}
\end{aligned}$$

Remark 3.3. Let χ^n be trivial. Then the sum in the numerator of the summand of RHS of the above equation is

$$\sum_{\gamma\in\mathbb{F}_q^\times} K(\lambda^{(\mu_1)};\gamma)\cdots K(\lambda^{(\mu_s)};\gamma)K(\lambda^{(\nu_1)};\gamma)\cdots K(\lambda^{(\nu_t)};\gamma),$$

which, according to [4, Theorem 1], is equal to

$$\frac{q^2}{q-1}|A_{(\mu,\nu)}| - \frac{(q^{\mu_1}-1)\cdots(q^{\mu_s}-1)(q^{\nu_1}-1)\cdots(q^{\nu_t}-1)}{q-1} + 2(-1)^{l(\mu)+l(\nu)-1}$$

with

$$A_{(\mu,\nu)} = \left\{ (\alpha_1,\ldots,\alpha_s,\beta_1,\ldots,\beta_t) \in \mathbb{F}_{q^{\mu_1}}^\times \times \cdots \times \mathbb{F}_{q^{\mu_s}}^\times \times \mathbb{F}_{q^{\nu_1}}^\times \times \cdots \mathbb{F}_{q^{\nu_t}}^\times \right.$$

$$\sum_{i=1}^s \mathrm{tr}_{\mathbb{F}_{q^{\mu_i}}/\mathbb{F}_q}(\alpha_i) + \sum_{j=1}^t \mathrm{tr}_{\mathbb{F}_{q^{\nu_j}}/\mathbb{F}_q}(\beta_j)$$

$$\left. = 0 = \sum_{i=1}^s \mathrm{tr}_{\mathbb{F}_{q^{\mu_i}}/\mathbb{F}_q}(\alpha_i^{-1}) + \sum_{j=1}^t \mathrm{tr}_{\mathbb{F}_{q^{\nu_j}}/\mathbb{F}_q}(\beta_j^{-1}) \right\}.$$

As $|A_{(\mu,\nu)}|$ is divisible by $q-1$, it is an integer, depending only on (μ,ν).

3.3. $\mathrm{SO}^+(2n,q)$

Let J^+ be the $2n \times 2n$ matrix with $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ along the diagonal blocks and zeros elsewhere, and let

$$G = \mathrm{SO}^+(2n,\overline{\mathbb{F}}_q) = \{\, g \in \mathrm{GL}(2n,\overline{\mathbb{F}}_q) \,|\, {}^t g J^+ g = J^+, \ \det g = 1\}.$$

If F is the standard Frobenius map Fr_q on G, $G^F = \mathrm{SO}^+(2n,q)$, with

$$|\mathrm{SO}^+(2n,q)| = q^{n^2-n}(q^n - 1)\prod_{j=1}^{n-1}(q^{2j} - 1).$$

$T = \{\,\text{diagonal matrices in } G\}$ is an F-stable maximal torus of G which is split over $\mathbb{F}_q$. The Weyl group $W = N(T)/T$ with respect to T is the subgroup of $W' = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ consisting of even signed permutations. A signed permutation $a\sigma \in W'$, with $a = (a_1,\ldots,a_n) \in (\mathbb{Z}/2\mathbb{Z})^n = \{\pm 1\}^n$, $\sigma \in S_n$, is called *even* (resp. *odd*) *signed* if $a_1\cdots a_n = 1$ (resp. $a_1\cdots a_n = -1$).

F acts trivially on W, and hence the F-conjugacy classes in W coincide with conjugacy classes in W. Since $[W':W] = 2$, the conjugacy class $\mathrm{conj}_{W'}(w)$ of each $w \in W$ in W' is either a single conjugacy class in W or the union of two conjugacy classes in W. More precisely, we have the following lemma.

Lemma 3.4. *Let $W \subset W'$ be as above. The conjugacy class $\mathrm{conj}_{W'}(w)$ of $w \in W$ in W' coincides with the conjugacy class $\mathrm{conj}_W(w)$ of w in W, except when the double partition of n corresponding to $\mathrm{conj}_{W'}(w)$ is of the form $(\mu,\mathbf{0}) \in \mathcal{P}_n \times \mathcal{P}_0$, with $\mu_1,\ldots,\mu_s$ all even. Here $\mu = (\mu_1,\ldots,\mu_s)$.*

Proof. Let $w = x\sigma \in W$ with $x \in (\mathbb{Z}/2\mathbb{Z})^n$, $\sigma \in S_n$. The conjugacy class of w in W' is a single W-conjugacy class if and only if $w = v^{-1}wv$ for some $v \in W' - W$. This is so if and only if there exist $y = (y_1, \ldots, y_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ and $\tau \in S_n$ such that $y_1 \cdots y_n = -1$, $\tau\sigma = \sigma\tau$, and $y^{-1} \cdot {}^\sigma y = x^{-1} \cdot {}^\tau x$, where ${}^\sigma y$ denotes the action of $\sigma \in S_n$ on $y \in (\mathbb{Z}/2\mathbb{Z})^n$. The lemma can be deduced from this observation. $\qquad\square$

The conjugacy classes $\mathrm{conj}_{W'}(w)$ of $w \in W$ in W' are in bijective correspondence with the subset $\mathcal{P}_{n,e}^{(2)}$ of $\mathcal{P}_n^{(2)}$ (cf. (20)) given by

$$(24) \qquad \mathcal{P}_{n,e}^{(2)} = \{(\mu, \nu) \in \mathcal{P}_n^{(2)} \,|\, l(\nu) \text{ is even}\}.$$

Let $\mathrm{conj}_{W'}(w)$ $(w \in W)$ correspond to $(\mu, \nu) = \big((\mu_1, \ldots, \mu_s), (\nu_1, \ldots, \nu_t)\big)$, with t even. Then, as in the case of symplectic group Sp_{2n},

$$T^{wF} \cong \mathbb{F}_{q^{\mu_1}}^\times \times \cdots \times \mathbb{F}_{q^{\mu_s}}^\times \times C_{q^{\nu_1}} \times \cdots \times C_{q^{\nu_t}},$$

and

$$\frac{1}{|T^{wF}|} \sum_{a \in T^{wF}} \lambda(\mathrm{tr}\, a) = \frac{K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{(q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}.$$

It is important to observe that the above sum depends only on the conjugacy classes of $w \in W$ in W'. In the RHS of the following expression, we group together w's in the same conjugacy classes in W'.

$$\sum_{g \in \mathrm{SO}^+(2n,q)} \lambda(\mathrm{tr}\, g) = \frac{|\mathrm{SO}^+(2n,q)|}{|W|} \sum_{w \in W} \frac{1}{|T^{wF}|} \sum_{a \in T^{wF}} \lambda(\mathrm{tr}\, a).$$

Then we obtain the next theorem from (21) and the above calculations. Here one must note that

$$|\mathrm{conj}_{W'}(w)| = \frac{|W'|}{2^{l(\mu)+l(\nu)} z_\mu z_\nu} = \frac{2|W|}{2^{l(\mu)+l(\nu)} z_\mu z_\nu}.$$

Theorem 3.5.

$$\sum_{g \in \mathrm{SO}^+(2n,q)} \lambda(\mathrm{tr}\, g)$$

$$= 2q^{n^2-n}(q^n - 1) \prod_{j=1}^{n-1} (q^{2j} - 1)$$

$$\times \sum_{(\mu,\nu) \in \mathcal{P}_{n,e}^{(2)}} \frac{K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{2^{l(\mu)+l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}.$$

3.4. $\mathrm{SO}^-(2n, q)$

Let J^- be the $2n \times 2n$ matrix with $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ along the first $n-1$ diagonal blocks, $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -\varepsilon \end{smallmatrix}\right]$ for the last diagonal block and zeros elsewhere, where ε is a fixed element in $\mathbb{F}_q^\times - \mathbb{F}_q^{\times^2}$. Let

$$G = \mathrm{SO}^-(2n, \overline{\mathbb{F}}_q) = \{ g \in \mathrm{GL}(2n, \overline{\mathbb{F}}_q) \,|\, {}^t g J^- g = J^-, \ \det g = 1 \}.$$

Then $G^{\mathrm{Fr}_q} = \mathrm{SO}^-(2n, q)$, where Fr_q is the standard Frobenius map on G.

G is a *form* of SO_{2n}^+ over $\mathbb{F}_q$. In other words, $G = \mathrm{SO}_{2n}^-$ is defined over $\mathbb{F}_q$, and is isomorphic to SO_{2n}^+ over $\overline{\mathbb{F}}_q$. Indeed, an isomorphism $\phi : \mathrm{SO}_{2n}^+ \to \mathrm{SO}_{2n}^-$ defined over $\mathbb{F}_{q^2}$ is given by $\phi(g) = AgA^{-1}$, where A is the block diagonal matrix with 1's along the diagonal and the last diagonal block is

$$2^{-\frac{1}{2}} \begin{bmatrix} 1 & 1 \\ \varepsilon^{-\frac{1}{2}} & -\varepsilon^{-\frac{1}{2}} \end{bmatrix}.$$

Hence $G = \mathrm{SO}_{2n}^-$ can be identified with SO_{2n}^+ equipped with the Frobenius map $F = \phi^{-1} \circ \mathrm{Fr}_q \circ \phi$. In particular,

$$(25) \qquad |\mathrm{SO}_{2n}^{+F}| = |\mathrm{SO}^-(2n, q)| = q^{n^2-n}(q^n+1) \prod_{j=1}^{n-1}(q^{2j}-1).$$

We note here that F is the twisted Frobenius map given by

$$F(g) = A^{-1}\mathrm{Fr}_q(AgA^{-1})A = B\,\mathrm{Fr}_q(g)B^{-1},$$

where $B = A^{-1}\mathrm{Fr}_q(A)$ is the block diagonal matrix with 1's along the diagonal except $\pm\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ for the last 2×2 diagonal block. Here the sign is plus or minus depending on whether 2 is a square in $\mathbb{F}_q^\times$ or not. So $F = \psi \circ \mathrm{Fr}_q$, where

$$(26) \qquad \psi : \mathrm{SO}_{2n}^+ \to \mathrm{SO}_{2n}^+ \text{ is given by } \psi(g) = BgB^{-1}.$$

The subgroup T of diagonal matrices in SO_{2n}^+ is again F-stable, but F acts nontrivially on the Weyl group $W = N(T)/T$.

Theorem 3.6.

$$\sum_{g \in \mathrm{SO}^-(2n,q)} \lambda(\mathrm{tr}\, g)$$

$$= -2q^{n^2-n}(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1)$$

$$\times \sum_{(\mu,\nu) \in \mathcal{P}_{n,o}^{(2)}} \frac{K(\lambda^{(\mu_1)})\cdots K(\lambda^{(\mu_s)})K(\lambda^{(\nu_1)})\cdots K(\lambda^{(\nu_t)})}{2^{l(\mu)+l(\nu)}z_\mu z_\nu(q^{\mu_1}-1)\cdots(q^{\mu_s}-1)(q^{\nu_1}+1)\cdots(q^{\nu_t}+1)},$$

where $\mathcal{P}_{n,o}^{(2)}$ denotes the subset of $\mathcal{P}_n^{(2)}$ given by

$$(27) \qquad \mathcal{P}_{n,o}^{(2)} = \{(\mu,\nu) \in \mathcal{P}_n^{(2)} \,|\, l(\nu) \text{ is odd}\}.$$

Proof. The action of ψ (cf. (26)) on T is given by $\psi|_T = w_0 = (1, \ldots, 1, -1) \in W' = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ if we view W' as a group of automorphisms of T. Let $w \in W$. Then $_wF = _{ww_0}\mathrm{Fr}_q$ on T, and hence $T^{wF} = T^{ww_0\,\mathrm{Fr}_q}$.

The Gauss sum for $\mathrm{SO}^-(2n, q)$ is

$$\sum_{g \in \mathrm{SO}^-(2n,q)} \lambda(\mathrm{tr}\,g) = \frac{|\mathrm{SO}^-(2n,q)|}{|W|} \sum_{w \in W} \frac{1}{|T^{ww_0\,\mathrm{Fr}_q}|} \sum_{a \in T^{ww_0\,\mathrm{Fr}_q}} \lambda(\mathrm{tr}\,a).$$

Observe now that Ww_0 is the subset of odd signed permutations in $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ and the conjugacy classes $\mathrm{conj}_{W'}(ww_0)$ of $ww_0 \in Ww_0$ in W' are in bijective correspondence with the subset $\mathcal{P}_{n,o}^{(2)}$ in (27) of $\mathcal{P}_n^{(2)}$.

As in the case of SO_{2n}^+, in the above summation, the sum corresponding to each $w \in W$ depends only on the conjugacy classes of $ww_0 \in Ww_0$ in W'. The result now follows from (22), (25). Here one must observe that (22) also holds for $ww_0 \in Ww_0$, if $\mathrm{conj}_{W'}(ww_0)$ corresponds to $(\mu, \nu) = \big((\mu_1, \ldots, \mu_s), (\nu_1, \ldots, \nu_t)\big) \in \mathcal{P}_{n,o}^{(2)}$, with t odd. Also, one must note that

$$|\mathrm{conj}_{W'}(ww_0)| = \frac{|W'|}{2^{l(\mu)+l(\nu)} z_\mu z_\nu} = \frac{2|W|}{2^{l(\mu)+l(\nu)} z_\mu z_\nu}. \qquad \square$$

3.5. $\mathrm{SO}(2n+1, q)$

Let

$$G = \mathrm{SO}(2n+1, \overline{\mathbb{F}}_q) = \{\, g \in \mathrm{GL}(2n+1, \overline{\mathbb{F}}_q) \,|\, {}^t g J g = J,\ \det g = 1 \,\}.$$

Here J denotes the $(2n+1) \times (2n+1)$ matrix

$$J = \begin{bmatrix} 0 & 1_n & 0 \\ 1_n & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

F is the standard Frobenius map Fr_q on G, and $G^F = \mathrm{SO}(2n+1, q)$, with

$$|\mathrm{SO}(2n+1, q)| = |\mathrm{Sp}(2n, q)| = q^{n^2} \prod_{j=1}^n (q^{2j} - 1).$$

$T = \{\text{diagonal matrices in } G\} = \{\mathrm{diag}(\alpha_1, \ldots, \alpha_n, \alpha_1^{-1}, \ldots, \alpha_n^{-1}, 1)\}$ is an F-stable maximal torus of G.

For the rest of this subsection, one is referred to the Section 3.1. As in the symplectic group case, $W = N(T)/T \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ is the Weyl group with respect to T, and F-conjugacy classes in W coincide with conjugacy classes in W.

If $\mathrm{conj}(w)$ ($w \in W$) corresponds to $(\mu, \nu) = \big((\mu_1, \ldots, \mu_s), (\nu_1, \ldots, \nu_t)\big) \in \mathcal{P}_n^{(2)}$, then

$$\begin{aligned} T^{wF} = \Big\{ &\mathrm{diag}\big(t_{\mu_1}(\alpha_1), \ldots, t_{\mu_s}(\alpha_s), t_{\nu_1}(\beta_1), \ldots, t_{\nu_t}(\beta_t), \\ &t_{\mu_1}(\alpha_1^{-1}), \ldots, t_{\mu_s}(\alpha_s^{-1}), t_{\nu_1}(\beta_1^{-1}), \ldots, t_{\nu_t}(\beta_t^{-1}), 1\big) \Big| \end{aligned}$$

$$\alpha_1 \in \mathbb{F}_{q^{\mu_1}}^\times, \ldots, \alpha_s \in \mathbb{F}_{q^{\mu_s}}^\times, \beta_1 \in C_{q^{\nu_1}}, \ldots, \beta_t \in C_{q^{\nu_t}}\Big\}.$$

So

$$\sum_{a \in T^{wF}} \lambda(\operatorname{tr} a) = \lambda(1)(-1)^t K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)}),$$

which is $\lambda(1)$ times the expression in (22).

Now, from (1), (21) and the above, the next theorem follows.

Theorem 3.7.

$$\sum_{g \in \mathrm{SO}(2n+1,q)} \lambda(\operatorname{tr} g) = \lambda(1) \sum_{g \in Sp(2n,q)} \lambda(\operatorname{tr} g)$$

$$= \lambda(1) q^{n^2} \prod_{j=1}^n (q^{2j} - 1)$$

$$\times \sum_{(\mu,\nu) \in \mathcal{P}_n^{(2)}} \frac{(-1)^{l(\nu)} K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{2^{l(\mu)+l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}.$$

Remark 3.8. The above relation was already observed in [11, (5.12)]. However, thanks to the main formula (1), the present argument gives more 'philosophical' reason that this should be so.

4. Identities

On the one hand, the expressions of Gauss sums in the last section were obtained by applying the main formula (1) to various finite classical groups. On the other hand, alternative expressions of those sums were gotten by using Bruhat decompositions with respect to certain maximal parabolic subgroups of the finite classical groups (cf. [7, 8, 9, 10, 11, 12, 15]). In this section, equating two different expressions of the Gauss sum for each finite classical group, we will be able to produce some new identities.

For $\gamma \in \mathbb{F}_q^\times$, we have defined the Kloosterman sum for $GL(t,q)$ in [10, (4.3)] by

$$(28) \qquad K_{\mathrm{GL}(t,q)}(\lambda; \gamma) = \sum_{g \in \mathrm{GL}(t,q)} \lambda(\operatorname{tr} g + \gamma \operatorname{tr} g^{-1}) \quad (t \geqslant 1),$$

$$K_{\mathrm{GL}(0,q)}(\lambda; \gamma) = 1,$$

$$K_{\mathrm{GL}(t,q)}(\lambda) = K_{\mathrm{GL}(t,q)}(\lambda; 1).$$

The expressions of the Gauss sum for $\mathrm{Sp}(2n,q)$ in Theorem 3.1 and the sentence following [10, (5.13)] yield the next theorem.

Theorem 4.1.

$$\sum_{(\mu,\nu) \in \mathcal{P}_n^{(2)}} \frac{(-1)^{l(\nu)} K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{2^{l(\mu)+l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}$$

$$= q^{-\binom{n}{2}} \prod_{j=1}^{n}(q^{2j}-1)^{-1} \times \sum_{\substack{0\leqslant r\leqslant n \\ r\,\text{even}}} q^{rn-r^2/4}\begin{bmatrix}n\\r\end{bmatrix}_q \prod_{j=1}^{r/2}(q^{2j-1}-1)K_{\mathrm{GL}(n-r,q)}(\lambda),$$

where $\mathcal{P}_n^{(2)}$, z_μ are respectively as in (20), (18).

Remark 4.2. In [10, Lemma 4.2], it is shown that the Kloosterman sums for general linear groups (28) satisfy the recursive relation

$$
\begin{aligned}
(29) \qquad K_{\mathrm{GL}(t,q)}(\lambda;\gamma) &= q^{t-1}K_{\mathrm{GL}(t-1,q)}(\lambda;\gamma)K(\lambda;\gamma) \\
&\quad + q^{2t-2}(q^{t-1}-1)K_{\mathrm{GL}(t-2,q)}(\lambda;\gamma) \;\; (t\geqslant 2).
\end{aligned}
$$

Here $K(\lambda;\gamma) = K_{\mathrm{GL}(1,q)}(\lambda;\gamma)$ is the usual Kloosterman sum as in (4) with $m=1$.

From this relation, for $t\geqslant 1$ one can show

$$
\begin{aligned}
(30) \qquad K_{\mathrm{GL}(t,q)}(\lambda;\gamma) &= q^{\frac{1}{2}(t-2)(t+1)} \\
&\quad \times \sum_{l=1}^{[(t+2)/2]} q^l K(\lambda;\gamma)^{t+2-2l} \sum_{j_1,\dots,j_{l-1}} \prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),
\end{aligned}
$$

where the second sum is over all integers $j_1,\dots,j_{l-1}$ satisfying

$$(31) \qquad 2l-1 \leqslant j_{l-1} \leqslant j_{l-2} \leqslant \cdots \leqslant j_1 \leqslant t+1,$$

and we agree that it is 1 for $l=1$.

A 'twisted average' version of Theorem 4.1 is obtained from the expressions of the Gauss sum for $\mathrm{GSp}(2n,q)$ in Theorem 3.2 and [10, (5.8)].

Theorem 4.3.

$$\sum_{(\mu,\nu)\in\mathcal{P}_n^{(2)}} \frac{(-1)^{l(\nu)}\sum_{\gamma\in\mathbb{F}_q^\times}\chi^n(\gamma)K(\lambda^{(\mu_1)};\gamma)\cdots K(\lambda^{(\mu_s)};\gamma)K(\lambda^{(\nu_1)};\gamma)\cdots K(\lambda^{(\nu_t)};\gamma)}{2^{l(\mu)+l(\nu)}z_\mu z_\nu(q^{\mu_1}-1)\cdots(q^{\mu_s}-1)(q^{\nu_1}+1)\cdots(q^{\nu_t}+1)}$$

$$= q^{-\binom{n}{2}} \prod_{j=1}^{n}(q^{2j}-1)^{-1}$$

$$\times \sum_{\substack{0\leqslant r\leqslant n \\ r\,\text{even}}} q^{rn-r^2/4}\begin{bmatrix}n\\r\end{bmatrix}_q \prod_{j=1}^{r/2}(q^{2j-1}-1) \sum_{\gamma\in\mathbb{F}_q^\times}\chi^n(\gamma)K_{\mathrm{GL}(t,q)}(\lambda;\gamma),$$

where $\mathcal{P}_n^{(2)}$, z_μ, $K_{GL(t,q)}(\lambda;\gamma)$ are respectively as in (20), (18), (28).

Remark 4.4. The twisted average Kloosterman sum $\sum_{\gamma\in\mathbb{F}_q^\times}\eta(\gamma)K_{\mathrm{GL}(t,q)}(\lambda;\gamma)$ for $\mathrm{GL}(t,q)$ satisfies similar relation as in (29) (cf. [10, (5.9)]). Here η is any multiplicative character of $\mathbb{F}_q$. From that relation, one can find an explicit expression for the sum (cf. (30)).

$$\sum_{\gamma \in \mathbb{F}_q^\times} \eta(\gamma) K_{\mathrm{GL}(t,q)}(\lambda; \gamma)$$

$$= q^{\frac{1}{2}(t-2)(t+1)} \sum_{l=1}^{[(t+2)/2]} q^l \sum_{\gamma} \eta(\gamma) K(\lambda; \gamma)^{t+2-2l} \sum_{j_1,\ldots,j_{l-1}} \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1),$$

where γ in the second sum is over $\mathbb{F}_q^\times$, the third sum runs over the same set of integers as in (31), and we understand that it is 1 for $l = 1$.

In the remark following [15, Theorem 4.3], it is shown that

$$(32) \qquad \sum_{g \in \mathrm{SO}^+(2n,q)} \lambda(\mathrm{tr}\, g) = q^{-n} \sum_{g \in \mathrm{Sp}(2n,q)} \lambda(\mathrm{tr}\, g)$$

$$(33) \qquad = q^{\binom{n}{2}} \sum_{\substack{0 \leqslant r \leqslant n \\ r \text{ even}}} q^{rn - r^2/4} \begin{bmatrix} n \\ r \end{bmatrix}_q \prod_{j=1}^{r/2} (q^{2j-1} - 1) K_{\mathrm{GL}(n-r,q)}(\lambda).$$

So we get the following identities, in view of (32), from Theorems 3.1, 3.5, and 4.1.

Theorem 4.5.

$$\sum_{(\mu,\nu) \in \mathcal{P}_{n,e}^{(2)}} \frac{K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{2^{l(\mu) + l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}$$

$$= \frac{1}{2}(q^n + 1) \sum_{(\mu,\nu) \in \mathcal{P}_n^{(2)}} \frac{(-1)^{l(\nu)} K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{2^{l(\mu) + l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}$$

$$= \frac{1}{2}(q^n + 1) q^{-\binom{n}{2}} \prod_{j=1}^n (q^{2j} - 1)^{-1} \sum_{\substack{0 \leqslant r \leqslant n \\ r \text{ even}}} q^{rn - r^2/4} \begin{bmatrix} n \\ r \end{bmatrix}_q \prod_{j=1}^{r/2} (q^{2j-1} - 1) K_{\mathrm{GL}(n-r,q)}(\lambda),$$

where $\mathcal{P}_n^{(2)}$, $\mathcal{P}_{n,e}^{(2)}$, z_μ, $K_{\mathrm{GL}(t,q)}(\lambda)$ are respectively as in (20), (24), (18), (28).

Put Σ, Σ_{even} and Σ_{odd} to be the summation of the following terms over (μ, ν) in $\mathcal{P}_n^{(2)}$, $\mathcal{P}_{n,e}^{(2)}$ and $\mathcal{P}_{n,o}^{(2)}$, respectively.

$$\frac{(-1)^{l(\nu)} K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{2^{l(\mu) + l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}$$

(cf. (27) for $\mathcal{P}_{n,o}^{(2)}$).

As $\mathcal{P}_n^{(2)} = \mathcal{P}_{n,e}^{(2)} \cup \mathcal{P}_{n,o}^{(2)}$ is a disjoint union, $\sum = \sum_{\text{even}} + \sum_{\text{odd}}$ by definition. As we saw in Theorem 4.5,

$$\sum_{\text{even}} = \frac{1}{2}(q^n + 1) \sum.$$

Hence, we obtain

$$\sum_{\text{odd}} = \frac{1}{2}(1 - q^n) \sum.$$

These results are summarized in the following corollary (cf. Theorem 4.1).

Corollary 4.6. (a)

$$\sum_{(\mu,\nu) \in \mathcal{P}_{n,o}^{(2)}} \frac{K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{2^{l(\mu)+l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}$$

$$= \frac{1}{2}(q^n - 1) \sum_{(\mu,\nu) \in \mathcal{P}_n^{(2)}} \frac{(-1)^{l(\nu)} K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{2^{l(\mu)+l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}$$

$$= \frac{1}{2}(q^n - 1) q^{-\binom{n}{2}} \prod_{j=1}^{n} (q^{2j} - 1)^{-1} \sum_{\substack{0 \leqslant r \leqslant n \\ r \text{ even}}} q^{rn - r^2/4} \begin{bmatrix} n \\ r \end{bmatrix}_q \prod_{j=1}^{r/2} (q^{2j-1} - 1) K_{\text{GL}(n-r,q)}(\lambda).$$

(b)

$$(q^n + 1)^{-1} \sum_{(\mu,\nu) \in \mathcal{P}_{n,e}^{(2)}} \frac{K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{2^{l(\mu)+l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}$$

$$= (q^n - 1)^{-1} \sum_{(\mu,\nu) \in \mathcal{P}_{n,o}^{(2)}} \frac{K(\lambda^{(\mu_1)}) \cdots K(\lambda^{(\mu_s)}) K(\lambda^{(\nu_1)}) \cdots K(\lambda^{(\nu_t)})}{2^{l(\mu)+l(\nu)} z_\mu z_\nu (q^{\mu_1} - 1) \cdots (q^{\mu_s} - 1)(q^{\nu_1} + 1) \cdots (q^{\nu_t} + 1)}.$$

Remark 4.7. It would be interesting if one could obtain the above identities purely in combinatorial manner.

From Theorems 3.5, 3.6 and the last corollary we get the following somewhat suprising identities:

$$(34) \qquad \sum_{g \in \text{SO}^-(2n,q)} \lambda(\text{tr}\, g) = 2q^{n^2-n}(q^n + 1) \prod_{j=1}^{n-1}(q^{2j} - 1) \sum_{\text{odd}}$$

$$= -2q^{n^2-n}(q^n - 1) \prod_{j=1}^{n-1}(q^{2j} - 1) \sum_{\text{even}}$$

$$= -\sum_{g \in \text{SO}^+(2n,q)} \lambda(\text{tr}\, g).$$

Now, we have enough information to have complete relations between Gauss sums for various special orthogonal groups and symplectic group. (cf. Theorem 3.7, (32), (34))

Theorem 4.8. (a) $\displaystyle \sum_{g \in \text{SO}(2n+1,q)} \lambda(\text{tr}\, g) = \lambda(1) \sum_{g \in \text{Sp}(2n,q)} \lambda(\text{tr}\, g),$

(b) $\displaystyle\sum_{g\in\mathrm{SO}^+(2n,q)}\lambda(\mathrm{tr}\,g)=q^{-n}\sum_{g\in\mathrm{Sp}(2n,q)}\lambda(\mathrm{tr}\,g),$

(c) $\displaystyle\sum_{g\in\mathrm{SO}^-(2n,q)}\lambda(\mathrm{tr}\,g)=-q^{-n}\sum_{g\in\mathrm{Sp}(2n,q)}\lambda(\mathrm{tr}\,g)=-\sum_{g\in\mathrm{SO}^+(2n,q)}\lambda(\mathrm{tr}\,g).$

On the one hand, $\sum_{g\in\mathrm{SO}^-(2n,q)}\lambda(\mathrm{tr}\,g)$ equals the negative of the expression in (33). On the other hand, it equals the expression in [8, (5.10)] (cf. [8, (4.13)], [13, (3.1)]). Thus the following identity holds, which also can be proved directly for example using induction on r.

Proposition 4.9.

$$K(\lambda)\sum_{\substack{0\leqslant r\leqslant n-1\\ r\,\mathrm{even}}}q^{nr-r^2/4}\begin{bmatrix}n-1\\r\end{bmatrix}_q\prod_{j=1}^{r/2}(q^{2j-1}-1)K_{\mathrm{GL}(n-1-r,q)}(\lambda)$$

$$+(q+1)\sum_{\substack{0\leqslant r\leqslant n-1\\ r\,\mathrm{odd}}}q^{nr-(r+1)^2/4}\begin{bmatrix}n-1\\r\end{bmatrix}_q\prod_{j=1}^{(r+1)/2}(q^{2j-1}-1)K_{\mathrm{GL}(n-1-r,q)}(\lambda)$$

$$=q^{1-n}\sum_{\substack{0\leqslant r\leqslant n\\ r\,\mathrm{even}}}\begin{bmatrix}n\\r\end{bmatrix}_q q^{nr-r^2/4}\prod_{j=1}^{r/2}(q^{2j-1}-1)K_{\mathrm{GL}(n-r,q)}(\lambda).$$

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