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INTEGRAL INEQUALITY REGARDING r-CONVEX AND r-CONCAVE FUNCTIONS

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Abstract. New integral inequalities concerning r-convex and r-concave functions are presented.

1. Introduction

The following open question was proposed in [5].

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\beta f^\alpha(x) \tag{1.1}$$

hold for α and β?

The authors in [4], [1], and [2] have been dealt gradually with this inequality assuming different conditions, but I think the idea of [1] is the best.

In [1], the authors gave an answer by establishing the following.

Theorem 1.1. *If the function f satisfies*

$$\int_0^1 f(t)dt \geq \frac{1-x^2}{2}, \quad \forall x \in [0,1], \tag{1.2}$$

then

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\beta f^\alpha(x)dx \tag{1.3}$$

for every real $\alpha \geq 1$ and $\beta > 0$.

Very recently, Liu, Cheng, and Li established a more general case by giving the following result.

Theorem 1.2. *Let $f(x) \geq 0$ be a continuous function on $[a,b]$ satisfying*

$$\int_x^b f^{\min(1,\beta)}(t)dt \geq \int_x^b (t-a)^{\min(1,\beta)}dt, \quad \forall x \in [a,b]. \tag{1.4}$$

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Then the inequality

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b (x-a)^\alpha f^\beta(x)dx, \tag{1.5}$$

holds for every positive real number $\alpha > 0$, $\beta > 0$.

The gamma function is denoted by $\Gamma(p)$ and is defined by

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx, \quad p > 0.$$

A function $f : [a, b] \to R$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad x, y \in [a, b], \quad t \in [0, 1]. \tag{1.6}$$

If the inequality is reversed, then f is said to be concave.

A positive function f is log-convex on a real interval $[a, b]$ if for all $x, y \in [a, b]$ and $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \leq f^t(x) + f^{(1-t)}(y). \tag{1.7}$$

A positive function f is r-*convex* on $[a, b]$ if for all $x, y \in [a, b]$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) = \begin{cases} \left(tf^r(x) + (1-t)f^r(y)\right)^{1/r}, & r \neq 0, \\ f^t(x)f^{1-t}(y), & r = 0. \end{cases} \tag{1.8}$$

If the above inequality reverses, then f is r-*concave*.

Clearly, the 0-convex functions are simply the log-convex functions and 1-convex functions are ordinary convex functions.

Hadamard inequality is as follows

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.9}$$

provided f is convex.

Very recently, concerning inequality (1.3), about more than 10 papers have been published. Since Hadamard inequality dealing with convex and concave functions is important in analysis, it is reasonable to incorporate inequalities similar to (1.3) with Hadamard inequality to find new inequalities that are probably important in analysis and applications.

2. A lemma and Hadamard inequality

Lemma 2.1. *Let $a, b > 0, t \in [0, 1]$. Then*

$$a^t b^{1-t} + a^{1-t}b^t \leq a + b. \tag{2.1}$$

Proof. Set

$$f(a) = a + b - a^t b^{1-t} - a^{1-t}b^t.$$

Then, we have

$$f'(a) = 1 - ta^{t-1}b^{1-t} - (1-t)a^{-t}b^t = 0, \text{ if } a = b,$$

and

$$[f''(a)]_{a=b} = [t(1-t)a^{t-2}b^{1-t} + t(1-t)a^{-t-1}b^t]_{a=b} = 2t(1-t)a^{-1} \geq 0.$$

This shows that f attains its minimum at $a = b$ which is zero. Therefore

$$f(a) \geq f(0) = 0. \qquad \square$$

Theorem 2.2. *Let $f : [a,b] \to R$ be a positive r-convex function $(r \neq 0)$. Then f^r satisfies the Hadamard inequality. That is,*

$$f^r\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f^r(x)dx \leq \frac{f^r(a)+f^r(b)}{2}. \tag{2.2}$$

Furthermore, if f is 0-convex, then (2.2) is satisfied, and also we have

$$\frac{1}{b-a}\int_a^b f^r(x)dx \leq \frac{f^r(a)-f^r(b)}{\ln f^r(a) - \ln f^r(b)}. \tag{2.3}$$

If f is r-concave or 0-concave, then (2.2) and (2.3) are reversed.

Proof. If f is r-convex, then f^r is convex, and therefore it satisfies the Hadamard inequality. If f is 0-convex, then

$$\begin{aligned}
f^r\left(\frac{a+b}{2}\right) &= \frac{1}{b-a}\int_a^b f^r\left(\frac{x+a+b-x}{2}\right) \\
&\leq \frac{1}{b-a}\int_a^b \left(f^{1/2}(x)f^{1/2}(a+b-x)\right)^r dx \\
&\leq \frac{1}{b-a}\left(\int_a^b f^r(x)dx\right)^{1/2}\left(\int_a^b f^r(a+b-x)dx\right)^{1/2} \\
&= \frac{1}{b-a}\int_a^b f^r(x)dx \\
&= \int_0^1 f^r(ta+(1-t)b)dt \\
&= \frac{1}{2}\left(\int_0^1 f^r(ta+(1-t)b)dt + \int_0^1 f^r((1-t)a+tb)dt\right) \\
&\leq \frac{1}{2}\left(\int_0^1 f^{tr}(a)f^{(1-t)r}(b)dt + \int_0^1 f^{(1-t)r}(a)f^{tr}(b)dt\right) \\
&= \frac{1}{2}\left(\int_0^1 \left(f^{tr}(a)f^{(1-t)r}(b) + f^{(1-t)r}(a)f^{tr}(b)\right)\right) dt \\
&\leq \frac{1}{2}\int_0^1 \left(f^r(a)+f^r(b)\right)dt \quad \text{(by Lemma 2.1)} \\
&= \frac{f^r(a)+f^r(b)}{2}.
\end{aligned}$$

Also, the 0-convexity of f implies

$$\begin{aligned}\frac{1}{b-a}\int_a^b f^r(x)dx &= \int_0^1 f^r\left(ta+(1-t)b\right)dt \\ &\le \int_0^1 f^{tr}(a)f^{(1-t)r}(b)dt \\ &= f^r(b)\int_0^1\left(\frac{f(a)}{f(b)}\right)^{rt}dt \\ &= \frac{f^r(a)-f^r(b)}{\ln f^r(a)-\ln f^r(b)}.\end{aligned}$$

□

3. Main results

We state and prove the following:

Theorem 3.1. *Let f, g be non-negative continuous functions defined on $[a,b]$, g is α-concave $(\alpha \ne 0)$ with $g'(x) \le 1, \forall x \in [a,b]$ and let $\alpha, \beta > 0$. If*

$$\int_x^b f^\alpha(t)dt \ge \int_x^b g^\alpha(t)dt, \quad \forall x \in [a,b], \tag{3.1}$$

then

$$\begin{aligned}&\int_a^b f^{\alpha+\beta}(x)dx - \int_a^b f^\alpha(x)\ln\left(\frac{b-a}{b-x}\right)dx \\ &\ge \frac{\beta}{\alpha}\left(\frac{1}{2}\left(\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}+(b-a)g^\alpha(b)\right)-\Gamma(2+\alpha/\beta)\right)\end{aligned} \tag{3.2}$$

If g is 0-concave with $g'(x) \ge 1$, $g(x)/g'(x)$ is non-increasing and (3.1) is satisfied, then

$$\begin{aligned}&\int_a^b f^{\alpha+\beta}(x)dx - \int_a^b (x-a)f^\alpha(x) \\ &\ge \frac{\beta}{\alpha}\left(\frac{g(b)}{\alpha g'(b)}\left((b-a)g^\alpha(b)-\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}\right)-\frac{\beta}{2\beta+\alpha}(b-a)^{2+\alpha/\beta}\right).\end{aligned} \tag{3.3}$$

Proof. In view of Theorem 2.2, we have

$$\begin{aligned}\int_a^b\int_x^b f^\alpha(t)\frac{1}{b-x}dtdx &= \int_a^b\left(\int_x^b f^\alpha(t)dt\right)\frac{1}{b-x}dx \\ &\ge \int_a^b\left(\int_x^b g^\alpha(t)dt\right)\frac{1}{b-x}dx\end{aligned}$$

$$\begin{aligned}
&\geq \int_a^b \frac{g^\alpha(x)+g^\alpha(b)}{2}dx \\
&\geq \frac{1}{2}\int_a^b \left(g^\alpha(x)g'(x)+g^\alpha(b)\right)dx \\
&= \frac{1}{2}\left(\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}+(b-a)g^\alpha(b)\right).
\end{aligned}$$

Also, via changing the order of integration, we have

$$\begin{aligned}
\int_a^b\int_x^b f^\alpha(t)\frac{1}{b-x}dtdx &= \int_a^b f^\alpha(t)dt\int_a^t\frac{1}{b-x}dx \\
&= \int_a^b \ln\left(\frac{b-a}{b-t}\right)f^\alpha(t)dt.
\end{aligned}$$

Collecting the above results, we obtain

$$\int_a^b \ln\left(\frac{b-a}{b-t}\right)f^\alpha(t)dt \geq \frac{1}{2}\left(\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}+(b-a)g^\alpha(b)\right).$$

Applying the AG inequality,

$$\frac{\alpha}{\alpha+\beta}f^{\alpha+\beta}(x)+\frac{\beta}{\alpha+\beta}h^{\alpha+\beta}(x)\geq f^\alpha(x)h^\beta(x), \quad h(x)>0.$$

On putting $h(x)=\ln^{1/\beta}\left(\frac{b-a}{b-x}\right)$, we obtain

$$\frac{\alpha}{\alpha+\beta}f^{\alpha+\beta}(x)+\frac{\beta}{\alpha+\beta}\ln^{1+\alpha/\beta}\left(\frac{b-a}{b-x}\right)\geq f^\alpha(x)\ln\left(\frac{b-a}{b-x}\right),$$

or

$$f^{\alpha+\beta}(x)-f^\alpha(x)\ln\left(\frac{b-a}{b-x}\right)\geq\frac{\beta}{\alpha}\left(f^\alpha(x)\ln\left(\frac{b-a}{b-x}\right)-\ln^{1+\alpha/\beta}\left(\frac{b-a}{b-x}\right)\right).$$

Integrating the above gives

$$\begin{aligned}
&\int_a^b f^{\alpha+\beta}(x)dx-\int_a^b f^\alpha(x)\ln\left(\frac{b-a}{b-x}\right)dx \\
&\geq\frac{\beta}{\alpha}\left(\int_a^b f^\alpha(x)\ln\left(\frac{b-a}{b-x}\right)dx-\int_a^b\ln^{1+\alpha/\beta}\left(\frac{b-a}{b-x}\right)dx\right).
\end{aligned} \tag{3.4}$$

Now, on putting $\ln\left(\frac{b-a}{b-x}\right)=u$, we have

$$\int_a^b\ln^{1+\alpha/\beta}\left(\frac{b-a}{b-x}\right)dx=\int_0^\infty u^{1+\alpha/\beta}e^{-u}du=\Gamma(2+\alpha/\beta),$$

and hence

$$\begin{aligned}&\int_a^b f^{\alpha+\beta}(x) - \int_a^b f^{\alpha}(x)\ln\left(\frac{b-a}{b-x}\right)dx \\ \geq\ & \frac{\beta}{\alpha}\left(\frac{1}{2}\left(\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}+(b-a)g^{\alpha}(b)\right)-\Gamma(2+\alpha/\beta)\right).\end{aligned}$$

If g is 0-concave and $g'(x)\geq 1$, then by Theorem 2.2,

$$\begin{aligned}&\int_a^b\int_x^b f^{\alpha}(t)\frac{\ln g^{\alpha}(b)-\ln g^{\alpha}(x)}{b-x}dtdx \\ &= \int_a^b\left(\int_x^b f^{\alpha}(t)dt\right)\frac{\ln g^{\alpha}(b)-\ln g^{\alpha}(x)}{b-x}dx \\ &\geq \int_a^b\left(\int_x^b g^{\alpha}(t)dt\right)\frac{\ln g^{\alpha}(b)-\ln g^{\alpha}(x)}{b-x}dx \\ &\geq \int_a^b\left(g^{\alpha}(b)-g^{\alpha}(x)\right)dx \\ &\geq \int_a^b\left(g^{\alpha}(b)-g^{\alpha}(x)g'(x)\right)dx \\ &= (b-a)g^{\alpha}(b)-\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}.\end{aligned}$$

By changing the order of integration, we have

$$\begin{aligned}&\int_a^b\int_x^b f^{\alpha}(t)\frac{\ln g^{\alpha}(b)-\ln g^{\alpha}(x)}{b-x}dtdx \\ &= \int_a^b f^{\alpha}(t)dt\int_a^t\frac{\ln g^{\alpha}(b)-\ln g^{\alpha}(x)}{b-x}dx \\ &= \alpha\frac{g'(c)}{g(c)}\int_a^b f^{\alpha}(t)dt\int_a^t dx \\ &\qquad \text{(for some } c,\ x<c<b, \text{ by the mean value theorem)} \\ &= \alpha\frac{g'(c)}{g(c)}\int_a^b (t-a)f^{\alpha}(t)dt.\end{aligned}$$

Collecting the above, we obtain

$$\begin{aligned}\int_a^b (t-a)f^{\alpha}(t)dt &\geq \frac{g(c)}{\alpha g'(c)}\left((b-a)g^{\alpha}(b)-\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}\right) \\ &\geq \frac{g(b)}{\alpha g'(b)}\left((b-a)g^{\alpha}(b)-\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}\right).\end{aligned}$$

Similarly, as we did in the previous result, we have

$$f^{\alpha+\beta}(x)-f^{\alpha}(x)(x-a)^{\beta}\geq\frac{\beta}{\alpha}\left(f^{\alpha}(x)(x-a)-(x-a)^{1+\alpha/\beta}\right).$$

Integrating the above to obtain

$$\begin{aligned}&\int_a^b f^{\alpha+\beta}(x) - \int_a^b (x-a)^\beta f^\alpha(x)\\ \geq\ &\frac{\beta}{\alpha}\left(\frac{g(b)}{\alpha g'(b)}\left((b-a)g^\alpha(b) - \frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}\right) - \frac{\beta}{2\beta+\alpha}(b-a)^{2+\alpha/\beta}\right).\end{aligned}$$

□

Theorem 3.2. *Let f, g be non-negative continuous functions defined on $[a,b]$, g is α-convex or 0-convex with $g'(x) \leq 1, \forall x \in [a,b]$, and let $\alpha, \beta > 0$. If (3.1) is satisfied, then*

(3.5)

$$\int_a^b f^{\alpha+\beta}(x)dx - \int_a^b f^\alpha(x)\ln\left(\frac{b-a}{b-x}\right)dx \geq \frac{\beta}{\alpha}\left(2\int_{\frac{a+b}{2}}^b f^\alpha(x)dx - \Gamma(2+\alpha/\beta)\right).$$

Proof. By virtue of Theorem 2.2, we have

$$\begin{aligned}\int_a^b\int_x^b f^\alpha(t)\frac{1}{b-x}dtdx &= \int_a^b\left(\int_x^b f^\alpha(t)dt\right)\frac{1}{b-x}dx\\ &\geq \int_a^b f^\alpha\left(\frac{x+b}{2}\right)dx.\end{aligned}$$

Also, by changing the order of integration, we have, as before

$$\int_a^b\int_x^b f^\alpha(t)\frac{1}{b-x}dtdx = \int_a^b \ln\left(\frac{b-a}{b-t}\right)f^\alpha(t)dt.$$

Therefore, we have

$$\int_a^b \ln\left(\frac{b-a}{b-t}\right)f^\alpha(t)dt \geq \int_a^b f^\alpha\left(\frac{t+b}{2}\right)dt. \tag{3.6}$$

If we proceeding exactly as we did in the proof of Theorem 3.1, using the above estimation, we have

$$\begin{aligned}&\int_a^b f^{\alpha+\beta}(x)dx - \int_a^b f^\alpha(x)\ln\left(\frac{b-a}{b-x}\right)dx\\ \geq\ &\frac{\beta}{\alpha}\left(\int_a^b f^\alpha\left(\frac{x+b}{2}\right)dx - \Gamma(2+\alpha/\beta)\right)\\ =\ &\frac{\beta}{\alpha}\left(2\int_{\frac{a+b}{2}}^b f^\alpha(x)dx - \Gamma(2+\alpha/\beta)\right).\end{aligned}$$

If g is 0-convex with $g'(x) \le 1, \forall x \in [a,b]$, then by making use of (3.4) and (3.6), we obtain

$$\int_a^b f^{\alpha+\beta}(x)dx - \int_a^b f^{\alpha}(x)\ln\left(\frac{b-a}{b-x}\right)dx$$
$$\ge \frac{\beta}{\alpha}\left(\int_a^b f^{\alpha}\left(\frac{x+b}{2}\right)dx - \Gamma(2+\alpha/\beta)\right)$$
$$= \frac{\beta}{\alpha}\left(2\int_{\frac{a+b}{2}}^b f^{\alpha}(x)dx - \Gamma(2+\alpha/\beta)\right).$$

□

Theorem 3.3. *Let f, g be non-negative continuous functions defined on $[a,b]$, g is α-convex $(\alpha \ne 0)$ with $g'(x) \ge 1$, and let $0 < \beta < \alpha < 2\beta$. If*

$$\int_a^x f^{\alpha}(t)dt \le \int_a^x g^{\alpha}(t)dt \quad \forall x \in [a,b], \tag{3.7}$$

then

$$\int_a^b f^{\alpha-\beta}(x)dx \le \frac{(1-\beta/\alpha)}{2}\left(\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}\right)+\frac{b-a}{2}g^{\alpha}(a)+\frac{\beta}{\alpha}\Gamma(2-\alpha/\beta). \tag{3.8}$$

Also, if g is 0-convex with $g'(x) \ge 1$ and $g(x)/g'(x)$ is non-increasing, we have

$$\begin{aligned}\int_a^b f^{\alpha-\beta}(x)dx \le \frac{(\alpha-\beta)g(a)}{\alpha^2 g'(a)}&\left(\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1} - (b-a)g^{\alpha}(a)\right)\\ &+ \frac{\beta}{2\alpha-\beta}(b-a)^{2-\alpha/\beta}.\end{aligned} \tag{3.9}$$

Proof. By Theorem 2.2,

$$\begin{aligned}\int_a^b\int_a^x f^{\alpha}(t)\frac{1}{x-a}dtdx &= \int_a^b\left(\int_a^x f^{\alpha}(t)dt\right)\frac{1}{x-a}dx\\ &\le \frac{1}{2}\int_a^b\left(g^{\alpha}(a)+g^{\alpha}(x)\right)dx\\ &\le \frac{1}{2}\int_a^b\left(g^{\alpha}(a)+g^{\alpha}(x)g'(x)\right)dx\\ &= \frac{1}{2}\left((b-a)g^{\alpha}(a)+\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}\right).\end{aligned}$$

Also, by changing the order of integration, we have

$$\begin{aligned}\int_a^b\int_a^x f^{\alpha}(t)\frac{1}{x-a}dtdx &= \int_a^b f^{\alpha}(t)dt\int_t^b\frac{1}{x-a}dx\\ &= \int_a^b f^{\alpha}(t)\ln\left(\frac{b-a}{t-a}\right)dt.\end{aligned}$$

The above result implies

$$\int_a^b f^\alpha(t)\ln\left(\frac{b-a}{t-a}\right)dt \le \frac{1}{2}\left(\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}+(b-a)g^\alpha(a)\right).$$

Making use of the AG inequality, we have

$$\frac{\alpha}{\alpha-\beta}f^{\alpha-\beta}(x)-\frac{\beta}{\alpha-\beta}k^{\alpha-\beta}(x)\le f^\alpha(x)k^{-\beta}(x), \quad k(x)>0.$$

On putting $k(x)=\ln^{-1/\beta}\left(\frac{b-a}{x-a}\right)$, we obtain

$$\frac{\alpha}{\alpha-\beta}f^{\alpha-\beta}(x)-\frac{\beta}{\alpha-\beta}\ln^{1-\alpha/\beta}\left(\frac{b-a}{x-a}\right)\le f^\alpha(x)\ln\left(\frac{b-a}{x-a}\right),$$

or

$$f^{\alpha-\beta}(x)\le(1-\beta/\alpha)f^\alpha(x)\ln\left(\frac{b-a}{x-a}\right)+\frac{\beta}{\alpha}\ln^{1-\alpha/\beta}\left(\frac{b-a}{x-a}\right).$$

Integrating the above inequality gives

$$\begin{aligned}&\int_a^b f^{\alpha-\beta}(x)dx\\ &\le (1-\beta/\alpha)\int_a^b f^\alpha(x)\ln\left(\frac{b-a}{x-a}\right)dx+\frac{\beta}{\alpha}\int_a^b \ln^{1-\alpha/\beta}\left(\frac{b-a}{x-a}\right)dx\\ &\le \frac{(1-\beta/\alpha)}{2}\left(\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}\right)+\frac{b-a}{2}g^\alpha(a)+\frac{\beta}{\alpha}\Gamma(2-\alpha/\beta).\end{aligned}$$

Concerning inequality (3.9), if we are assuming

$$I=\int_a^b\int_a^x f^\alpha(t)\frac{\ln g^\alpha(x)-\ln g^\alpha(a)}{x-a}dtdx,$$

then as we dealt before, it is not difficult to show that

$$I\le\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}-(b-a)g^\alpha(a),$$

and

$$I=\alpha\frac{g'(d)}{g(d)}\int_a^b(b-t)f^\alpha(t)dt,$$

(for some d, $a<d<x$, by the mean value theorem),

which together implies

$$\begin{aligned}\int_a^b(b-t)f^\alpha(t)dt &\le \frac{g(d)}{\alpha g'(d)}\left(\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}-(b-a)g^\alpha(a)\right)\\ &\le \frac{g(a)}{\alpha g'(a)}\left(\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}-(b-a)g^\alpha(a)\right).\end{aligned}$$

By using the AG inequality as before, we get

$$f^{\alpha-\beta}(x)\le(1-\beta/\alpha)f^\alpha(x)(b-x)+\frac{\beta}{\alpha}(b-x)^{1-\alpha/\beta}.$$

Integrating the above gives

$$\int_a^b f^{\alpha-\beta}(x)dx \le \frac{(\alpha-\beta)g(c)}{\alpha^2 g'(c)}\left(\frac{g^{\alpha+1}(b)-g^{\alpha+1}(a)}{\alpha+1}-(b-a)g^{\alpha}(a)\right)$$
$$+\frac{\beta}{2\alpha-\beta}(b-a)^{2-\alpha/\beta}.$$

□

Theorem 3.4. *Let f, g be non-negative continuous functions defined on $[a, b]$, g is α-concave with $g'(x) \ge 1$, and let $0 < \beta < \alpha < 2\beta$. If (3.1) is satisfied, then*

$$\int_a^b f^{\alpha-\beta}(x)dx \le (1-\beta/\alpha)\int_a^b f^{\alpha}\left(\frac{x+a}{2}\right)dx + \frac{\beta}{\alpha}\Gamma(2-\alpha/\beta). \tag{3.10}$$

Proof. By Theorem 2.2 we have

$$\int_a^b\int_a^x f^{\alpha}(t)\frac{1}{x-a}dtdx = \int_a^b\left(\int_a^x f^{\alpha}(t)dt\right)\frac{1}{x-a}dx$$
$$\le \int_a^b f^{\alpha}\left(\frac{a+x}{2}\right)dx.$$

As before

$$\int_a^b\int_a^x f^{\alpha}(t)\frac{1}{x-a}dtdx = \int_a^b f^{\alpha}(t)\ln\left(\frac{b-a}{t-a}\right)dt.$$

Therefore, we get

$$\int_a^b f^{\alpha}(t)\ln\left(\frac{b-a}{t-a}\right)dt \le \int_a^b f^{\alpha}\left(\frac{t+a}{2}\right)dt.$$

Making use of the AG inequality, we have as before

$$\int_a^b f^{\alpha-\beta}(x)dx$$
$$\le (1-\beta/\alpha)\int_a^b f^{\alpha}(x)\ln\left(\frac{b-a}{x-a}\right)dx + \frac{\beta}{\alpha}\int_a^b \ln^{1-\alpha/\beta}\left(\frac{b-a}{x-a}\right)dx$$
$$\le (1-\beta/\alpha)\int_a^b f^{\alpha}\left(\frac{x+a}{2}\right)dx + \frac{\beta}{\alpha}\Gamma(2-\alpha/\beta).$$

□

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