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THE GENERALIZED INVERSES $A^{(1,2)}_{T,S}$ OF THE ADJOINTABLE OPERATORS ON THE HILBERT C^*-MODULES

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Abstract. In this paper, we introduce and study the generalized inverse $A^{(1,2)}_{T,S}$ with the prescribed range T and null space S of an adjointable operator A from one Hilbert C^*-module to another, and get some analogous results known for finite matrices over the complex field or associated rings, and the Hilbert space operators.

1. Introduction

The generalized inverse $A^{(2)}_{T,S}$ with the prescribed range T and null space S for a matrix A over the complex field [6] or an associated ring [9], a bounded linear operator A from one Hilbert space to another [10], as well as an element A of a C^*-algebra [2] have been studied by many mathematicians. Since the matrix algebras, the Hilbert spaces and C^*-algebras can all be considered as certain Hilbert C^*-modules, it is meaningful to put forward the general theory of the $A^{(2)}_{T,S}$ inverse in the setting of the Hilbert C^*-module operators. As a Hilbert C^*-module takes its values in a C^*-algebra which may not be equal to the complex field, a bounded linear operator from one Hilbert C^*-module to another may fail to be adjointable, which leads to some new phenomena. For instance, in the Hilbert space case, condition (4) (resp. (12)) is enough to ensure the existence of $A^{(2)}_{T,S}$ (resp. $A^{(1,2)}_{T,S}$), whereas in the Hilbert C^*-module case, additional condition (5) (resp. (13)) should be added. Likewise, a representation for the $A^{(2)}_{T,S}$ inverse was given in [1, Theorem 4.2] for a bounded linear operator A from one Banach space to another, the analogous result only holds for the $A^{(1,2)}_{T,S}$ inverse of an adjointable Hilbert C^*-module operator A (see Theorem 2.6 below).

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The purpose of this paper is, in the general setting of the adjointable Hilbert C^*-module operators, to give a detailed description of the generalized inverse $A^{(1,2)}_{T,S}$. We use [5] and [4] for the general references of C^*-algebras and the Hilbert C^*-modules respectively, and [8] for the theory of the Moore-Penrose inverses of the adjointable operators between Hilbert C^*-modules. Throughout this paper, $\mathfrak{A}$ is a C^*-algebra, H and K are two Hilbert $\mathfrak{A}$-modules. Let $\mathcal{L}(H,K)$ be the set of the adjointable operators from H to K. For any $A\in\mathcal{L}(H,K)$, the range and the null space of A are denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$ respectively.

A closed submodule F of H is said to be *topologically complemented* if there exists a closed submodule G of H such that $H=F\dotplus G$ and $F\cap G=\{0\}$ (written briefly, $H=F\oplus G$). Furthermore, F is said to be *orthogonally complemented* if $H=F\oplus F^\perp$, where $F^\perp=\big\{x\in H\big|\big\langle x,y\big\rangle=0,\forall\, y\in F\big\}$.

By definition, if F is orthogonally complemented, then F is topologically complemented; but the reverse is not true (see [4] for a counterexample). However, an exception is that every null space of an element of $\mathcal{L}(H,K)$ with closed range is orthogonally complemented:

Lemma 1.1 (cf. [4, Theorem 3.2])**.** *Let $A\in\mathcal{L}(H,K)$ have closed range. Then A^* also has closed range, and the following orthogonal decompositions hold*:

$$H=\mathcal{N}(A)\oplus\mathcal{R}(A^*),\ K=\mathcal{R}(A)\oplus\mathcal{N}(A^*). \tag{1}$$

Remark 1.2. For any $A\in\mathcal{L}(H,K)$, by [8, Remark 1.1] we know that the closeness of any one of the following sets implies the closeness of the remaining three sets:

$$\mathcal{R}(A),\ \mathcal{R}(A^*),\ \mathcal{R}(AA^*),\ \mathcal{R}(A^*A).$$

In this case, $\mathcal{R}(A)=\mathcal{R}(AA^*)$ and $\mathcal{R}(A^*)=\mathcal{R}(A^*A)$.

Definition. Let F and G be two closed submodules of H such that $H=F\oplus G$. For any $x\in H$, x can be expressed uniquely as $x=x_1+x_2$ with $x_1\in F$ and $x_2\in G$. Let $P_{F,G}:H\to F$ be defined by

$$P_{F,G}(x)=x_1 \text{ for such } x. \tag{2}$$

Then the operator $P_{F,G}$ defined as above is idempotent, which is bounded by the closed graph theorem. However, the reader should be aware that $P_{F,G}$ may fail to be adjointable; in other words, $P_{F,G}$ may not be belong to $\mathcal{L}(H,H)$.

2. The generalized inverses $A^{(2)}_{T,S}$ and $A^{(1,2)}_{T,S}$

Throughout this section, T and S are submodules of H and K respectively, and A is an element of $\mathcal{L}(H,K)$. In this section, we will study the generalized inverses $A^{(2)}_{T,S}$ and $A^{(1,2)}_{T,S}$ respectively, and get some analogous results known for finite-matrices over the complex field [7] or associative rings [9], and the Hilbert space operators [10]. To simplify the notation, we use $\mathcal{L}(H)$ instead of $\mathcal{L}(H,H)$. The identity operator on H is denoted by I_H.

Theorem 2.1. *Let H and K be two Hilbert $\mathfrak{A}$-modules, and $A \in \mathcal{L}(H, K)$. Let T and S be submodules of H and K respectively. The following conditions are equivalent:*

(i) *There exists $X \in \mathcal{L}(K, H)$ such that*

$$XAX = X,\ \mathcal{R}(X) = T \text{ and } \mathcal{N}(X) = S. \tag{3}$$

(ii) *$S, T, AT, A^*S^\perp$ are all closed with*

$$K = AT \oplus S,\ \mathcal{N}(A) \cap T = \{0\}, \tag{4}$$

$$H = A^*S^\perp \oplus T^\perp,\ \mathcal{N}(A^*) \cap S^\perp = \{0\}. \tag{5}$$

In which case, $X \in \mathcal{L}(K, H)$ is unique which satisfies (3).

Proof. (i) $\Longrightarrow$ (ii): Suppose that there exists $X \in \mathcal{L}(K, H)$ such that (3) holds. Then XA and AX both are idempotents so that $T = \mathcal{R}(X) = \mathcal{R}(XA)$ and $AT = A\mathcal{R}(X) = \mathcal{R}(AX)$ are closed, and $S = \mathcal{N}(X)$ is also closed since X is bounded. As AX is idempotent, we have

$$K = \mathcal{R}(AX) \oplus \mathcal{N}(AX) = AT \oplus \mathcal{N}(X) = AT \oplus S.$$

Note that if $x = Xu \in T$ for some $u \in K$ such that $Ax = 0$, then $x = Xu = XAXu = XAx = 0$, so that $\mathcal{N}(A) \cap T = \{0\}$, and hence(4) holds. If we take $*$-operation on both sides of $XAX = X$ and apply Lemma 1.1, we get

$$X^*A^*X^* = X^*,\ \mathcal{R}(X^*) = \mathcal{N}(X)^\perp = S^\perp,\ \mathcal{N}(X^*) = \mathcal{R}(X)^\perp = T^\perp. \tag{6}$$

So if we replace A, X, T, S with $A^*, X^*, S^\perp$ and $T^\perp$ respectively, we conclude that $A^*S^\perp$ is closed and (5) also holds.

(ii) $\Longrightarrow$ (i): Suppose that $S, T, AT, A^*S^\perp$ are all closed such that (4) and (5) are satisfied. Since $T \cap \mathcal{N}(A) = \{0\}$, the restriction of A to T, $A|_T$ is a bijection from T onto AT. Let $X : K \to H$ be a linear operator which equals the inverse of $A|_T$ on AT, and is identically zero on S, so that

$$X(Au + v) = u \ \text{ for any } u \in T \text{ and } v \in S. \tag{7}$$

Similarly, there exists a linear operator $X^* : H \to K$ such that

$$X^*(A^*\xi + \eta) = \xi \ \text{ for any } \xi \in S^\perp \text{ and } \eta \in T^\perp. \tag{8}$$

For u, v in (7) and ξ, η in (8), we have

$$\begin{aligned} \big\langle X(Au + v), A^*\xi + \eta\big\rangle &= \big\langle u, A^*\xi + \eta\big\rangle = \big\langle u, A^*\xi\big\rangle = \big\langle Au, \xi\big\rangle \\ &= \big\langle Au + v, \xi\big\rangle = \big\langle Au + v, X^*(A^*\xi + \eta)\big\rangle, \end{aligned}$$

which means that X^* is the adjoint operator of X, so that $X \in \mathcal{L}(K, H)$. In view of (7) we have $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$. Furthermore,

$$XAX(Au + v) = XAu = X(Au + v)$$

so that $XAX = X$, and hence (3) holds.

The proof of the uniqueness of X is similar to that of [9, Theorem 1] and [10, Theorem 2.1]. $\square$

Definition. The element $X \in \mathcal{L}(K, H)$ which satisfies (3) is denoted by $A^{(2)}_{T,S}$. If in addition, $AXA = A$, then X is denoted by $A^{(1,2)}_{T,S}$.

Remark 2.2. By the proof of Theorem 2.1 (see (7) and (8)) we know that

$$A^{(2)}_{T,S} = (A|_T)^{-1} \circ P_{AT,S}, \tag{9}$$

$$\left(A^{(2)}_{T,S}\right)^* = (A^*|_{S^\perp})^{-1} \circ P_{A^*S^\perp,T^\perp} = (A^*)^{(2)}_{S^\perp,T^\perp}, \tag{10}$$

and $A^{(2)}_{T,S}$ is bounded as $(A|_T)^{-1}$ and $P_{AT,S}$ both are bounded. So if H and K are two Hilbert spaces, then conditions (4) and (5) can be reduced to (4); in other words, in the Hilbert space case, conditions (3), (4) and (5) in Theorem 2.1 are all equivalent [10, Theorem 2.1].

For the $A^{(1,2)}_{T,S}$ inverse, we have the following result:

Theorem 2.3. *Let H and K be two Hilbert $\mathfrak{A}$-modules, and $A \in \mathcal{L}(H, K)$. Let T and S be submodules of H and K respectively. The following conditions are equivalent*:

(i) *There exists $X \in \mathcal{L}(K, H)$ such that*

$$AXA = A, XAX = X, \mathcal{R}(X) = T \text{ and } \mathcal{N}(X) = S. \tag{11}$$

(ii) *$S, T, AT, A^*S^\perp$ are all closed with*

$$K = AT \oplus S \text{ and } H = \mathcal{N}(A) \oplus T, \tag{12}$$

$$H = A^*S^\perp \oplus T^\perp \text{ and } K = \mathcal{N}(A^*) \oplus S^\perp. \tag{13}$$

In which case, $X \in \mathcal{L}(K, H)$ is unique which satisfies (11).

Proof. (i) $\Longrightarrow$ (ii): Suppose that (11) holds. Then by Theorem 2.1 we get (4). Furthermore, since XA is idempotent, we have $H = \mathcal{R}(XA) \oplus \mathcal{N}(XA) = \mathcal{R}(X) \oplus \mathcal{N}(A) = T \oplus \mathcal{N}(A)$, so (12) holds. Replacing A and X with A^* and X^* respectively, we get (13).

(ii) $\Longrightarrow$ (i): Suppose that (12) and (13) hold. Then by Theorem 2.1 there exists $X \in \mathcal{L}(K, H)$ which satisfies (3). By assumption, $H = T \oplus \mathcal{N}(A)$, so for any $x \in H$, x can be expressed uniquely as $x = x_1 + x_2$ with $x_1 \in T$ and $x_2 \in \mathcal{N}(A)$, hence

$$AXAx = AXA(x_1 + x_2) = AXAx_1 = Ax_1 = A(x_1 + x_2) = Ax,$$

so $AXA = A$ and therefore (11) holds.

The uniqueness of X follows from Theorem 2.1. $\square$

Remark 2.4. In the Hilbert space case, conditions (12) and (13) can be reduced to (12).

We can clarify $A^{(1,2)}_{T,S}$ in terms of idempotents, and part of a technique result of [8, Theorem 2.1] can be restated as follows:

Theorem 2.5. *Let H and K be two Hilbert $\mathfrak{A}$-modules, T and S be closed submodules of H and K respectively, and $A \in \mathcal{L}(H,K)$. Then the following conditions are equivalent*:

(i) $A_{T,S}^{(1,2)}$ *exists*;

(ii) *There exist idempotents $P \in \mathcal{L}(K)$ and $Q \in \mathcal{L}(H)$ such that*

$$\mathcal{R}(P) = \mathcal{R}(A), \mathcal{N}(P) = S, \mathcal{N}(Q) = \mathcal{N}(A) \text{ and } \mathcal{R}(Q) = T. \tag{14}$$

Proof. (i) $\Longrightarrow$ (ii): Suppose that $A_{T,S}^{(1,2)}$ exists, so (12) and (13) hold. Let $P = P_{AT,S}$ and $Q = P_{T,\mathcal{N}(A)}$. Then clearly (14) holds, and $P^* = P_{S^\perp,\mathcal{N}(A^*)}$, $Q^* = P_{A^*S^\perp,T^\perp}$ so that $P \in \mathcal{L}(K)$ and $Q \in \mathcal{L}(H)$.

(ii) $\Longrightarrow$ (i): Suppose $P \in \mathcal{L}(K)$ and $Q \in \mathcal{L}(H)$ are idempotents such that (14) holds. Then

$$\begin{aligned}
H &= \mathcal{N}(Q) \oplus \mathcal{R}(Q) = \mathcal{N}(A) \oplus T, \text{ so } AT = \mathcal{R}(A) = \mathcal{R}(P) \text{ is closed,} \\
K &= \mathcal{R}(P) \oplus \mathcal{N}(P) = \mathcal{R}(A) \oplus S = AT \oplus S, \\
K &= \mathcal{N}(P^*) \oplus \mathcal{R}(P^*) = \mathcal{R}(P)^\perp \oplus \mathcal{N}(P)^\perp \\
&= \mathcal{R}(A)^\perp \oplus S^\perp = \mathcal{N}(A^*) \oplus S^\perp, \\
H &= \mathcal{R}(Q^*) \oplus \mathcal{N}(Q^*) = \mathcal{N}(Q)^\perp \oplus \mathcal{R}(Q)^\perp \\
&= \mathcal{N}(A)^\perp \oplus T^\perp = \mathcal{R}(A^*) \oplus T^\perp = A^*S^\perp \oplus T^\perp.
\end{aligned}$$

The existence of $A_{T,S}^{(1,2)}$ then follows from Theorem 2.3. □

Definition. An element X of $\mathcal{L}(K,H)$ is said to be a $\{1\}$-*inverse* of $A \in \mathcal{L}(H,K)$, written $X = A^{(1)}$, if $AXA = A$. If in addition $XAX = X$ and $AX = XA$, then X is called the *group inverse* of A. In this case, we write X as A_g. In view of [8, Theorem 2.2], A has a $\{1\}$-inverse if and only if $\mathcal{R}(A)$ is closed.

In the special case that $T = \mathcal{R}(G)$ and $S = \mathcal{N}(G)$ for some $G \in \mathcal{L}(K,H)$, we can clarify $A_{T,S}^{(1,2)}$ in terms of group inverses as follows:

Theorem 2.6. *Let H and K be two Hilbert $\mathfrak{A}$-modules, T and S be closed submodules of H and K respectively such that $T = \mathcal{R}(G)$ and $S = \mathcal{N}(G)$ for some $G \in \mathcal{L}(K,H)$. Let $A \in \mathcal{L}(H,K)$ and suppose that $A_{T,S}^{(1,2)}$ exists. Then*

$$G(AG)_g = A_{T,S}^{(1,2)} = (GA)_g G. \tag{15}$$

Proof. (1) By Theorem 2.3 we know that (12) and (13) hold. Since $K = S \oplus AT$, $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$, we know that the restriction of G to AT, $G\big|_{AT} : AT \to T$ is a bijection. Let

$$W_K = \left(G\big|_{AT}\right)^{-1} \circ (A|_T)^{-1} \circ P_{AT,S}. \tag{16}$$

Similarly, define

$$W_K^* = (A^*|_{S^\perp})^{-1} \circ \left(G^*\big|_{A^*S^\perp}\right)^{-1} \circ P_{S^\perp,\mathcal{N}(A^*)}. \tag{17}$$

Since $K = \mathcal{N}(G) \oplus AT$, we have $T = \mathcal{R}(G) = \mathcal{R}(GAT)$. Similarly, by $H = T^\perp \oplus A^*S^\perp = \mathcal{N}(G^*) \oplus A^*S^\perp$ we get $S^\perp = \mathcal{R}(G^*) = \mathcal{R}(G^*A^*S^\perp)$.

For any $x \in T, y \in S, u \in S^\perp$ and $v \in \mathcal{N}(A^*)$, let

$$x = GAl \text{ and } u = G^*A^*t \quad \text{for some } l \in T \text{ and } t \in S^\perp.$$

Then

$$\begin{aligned} \big\langle W_K(Ax+y), u+v\big\rangle &= \big\langle W_K AGAl, u+v\big\rangle = \big\langle Al, u+v\big\rangle = \big\langle Al, u\big\rangle \\ &= \big\langle Al, G^*A^*t\big\rangle = \big\langle GAl, A^*t\big\rangle = \big\langle x, A^*t\big\rangle = \big\langle Ax, t\big\rangle \\ &= \big\langle Ax, W_K^* G^*A^*t\big\rangle = \big\langle Ax, W_K^* u\big\rangle \\ &= \big\langle Ax+y, W_K^*(u+v)\big\rangle, \end{aligned}$$

which means that W_K^* is the adjoint operator of W_K so that $W_K \in \mathcal{L}(K)$.

Since $S = \mathcal{N}(G)$ and $K = S \oplus AT$, in view of (9) and (16) we have $W_k = (AG)_g$ and $GW_K = A_{T,S}^{(1,2)}$.

(2) Similarly, let

$$W_H = (A|_T)^{-1} \circ \big(G\big|_{AT}\big)^{-1} \circ P_{T,\mathcal{N}(A)}. \tag{18}$$

Then $W_H \in \mathcal{L}(H)$ with

$$W_H^* = \big(G^*\big|_{A^*S^\perp}\big)^{-1} \circ (A^*|_{S^\perp})^{-1} \circ P_{A^*S^\perp, T^\perp}. \tag{19}$$

Furthermore, we have $W_H = (GA)_g$ and $W_H G = A_{T,S}^{(1,2)}$. $\square$

Next we clarify $A_{T,S}^{(1,2)}$ inverse in terms of $\{1\}$-inverses. We need an auxiliary result as follows:

Lemma 2.7. *Let $\mathfrak{B}$ be a unital algebra and p be an idempotent of $\mathfrak{B}$. For any $x \in \mathfrak{B}$, the following two conditions are equivalent*:

(i) *There exists $y \in \mathfrak{B}$, such that $pxp \cdot pyp = p = pyp \cdot pxp$;*
(ii) *$pxp + (I_{\mathfrak{B}} - p)$ is invertible in $\mathfrak{B}$.*

Theorem 2.8 (cf. [9, Theorem 7])**.** *Let H and K be two Hilbert $\mathfrak{A}$-modules, $A \in \mathcal{L}(H,K)$ and $G \in \mathcal{L}(K,H)$ have closed ranges. Let $A^{(1)}$ be any $\{1\}$-inverse of A. Then the following conditions are equivalent*:

(i) *$U = AGAA^{(1)} + I_K - AA^{(1)}$ is invertible, and $\mathcal{N}(A) \cap \mathcal{R}(G) = \{0\}$;*
(ii) *$V = A^{(1)}AGA + I_H - A^{(1)}A$ is invertible, and $\mathcal{N}(A) \cap \mathcal{R}(G) = \{0\}$;*
(iii) *$A_{\mathcal{R}(G),\mathcal{N}(G)}^{(1,2)}$ exists.*

In which case, for any $\{1\}$-inverses $(GAG)^{(1)}$, $(GA)^{(1)}$ and $(AG)^{(1)}$ we have

$$\begin{aligned} A_{\mathcal{R}(G),\mathcal{N}(G)}^{(1,2)} &= GU^{-2}AG = GU^{-1}AV^{-1}G = GAV^{-2}G \\ &= G(GAG)^{(1)}G = G(AG)^{(1)}A(GA)^{(1)}G. \end{aligned} \tag{20}$$

Proof. (1) (i) $\Longleftrightarrow$ (ii): Suppose that U is invertible. Then there exists $X \in \mathcal{L}(K)$ such that

$$AGAA^{(1)} \cdot AA^{(1)}XAA^{(1)} = AA^{(1)},$$
$$AA^{(1)}XAA^{(1)} \cdot AGAA^{(1)} = AA^{(1)}.$$

Multiplying $A^{(1)}$ and A from the left and the right respectively, we get

$$A^{(1)}AGA \cdot A^{(1)}A\big(A^{(1)}XA\big)A^{(1)}A = A^{(1)}A,$$
$$A^{(1)}A\big(A^{(1)}XA\big)A^{(1)}A \cdot A^{(1)}AGA = A^{(1)}A,$$

which means that V is invertible by Lemma 2.7. The proof of (ii) $\Longrightarrow$ (i) is similar.

(iii) $\Longrightarrow$ (i): Suppose that $A^{(1,2)}_{\mathcal{R}(G),\mathcal{N}(G)}$ exists. Let $T = \mathcal{R}(G)$ and $S = \mathcal{N}(G)$. Then by (4) we have $\mathcal{N}(A) \cap T = \{0\}$. Let W_K be defined by (16) with $\mathcal{R}(W_K) = AT$. Since $H = T \oplus \mathcal{N}(A)$, we have $\mathcal{R}(AA^{(1)}) = \mathcal{R}(A) = AT$, so

$$AA^{(1)}W_K = W_K, AGW_KAA^{(1)} = AA^{(1)} = W_KAGAA^{(1)}.$$

It follows that

$$AGAA^{(1)} \cdot AA^{(1)}W_KAA^{(1)} = AGW_KAA^{(1)} = AA^{(1)},$$
$$AA^{(1)}W_KAA^{(1)} \cdot AGAA^{(1)} = W_KAGAA^{(1)} = AA^{(1)},$$

so U is invertible by Lemma 2.7.

(ii) $\Longrightarrow$ (iii): Suppose that V is invertible with

$$V^{-1} = A^{(1)}AXA^{(1)}A + I_H - A^{(1)}A \tag{21}$$

for some $X \in \mathcal{L}(H)$. Then

$$A^{(1)}AGA \cdot A^{(1)}AXA^{(1)}A = A^{(1)}A, \tag{22}$$
$$A^{(1)}AXA^{(1)}A \cdot A^{(1)}AGA = A^{(1)}A, \tag{23}$$
$$V^{-2} = A^{(1)}AXA^{(1)}AXA^{(1)}A + I_H - A^{(1)}A, \tag{24}$$
$$AV^{-2} = A \cdot A^{(1)}AXA^{(1)}AXA^{(1)}A. \tag{25}$$

Multiplying A from the left on both sides of (21)–(23), we get

$$AV^{-1} = AXA^{(1)}A, \tag{26}$$
$$A\big(GAXA^{(1)}\big)A = A = A\big(XA^{(1)}AG\big)A. \tag{27}$$

By the definition of V, we get

$$AV = AGA, \text{ so } AGAV^{-1} = A. \tag{28}$$

Let

$$Z = GAV^{-2}G \in \mathcal{L}(K,H). \tag{29}$$

Then by (27)–(29), we have

$$AZA = (AGA)V^{-2}GA = AV^{-1}GA = A(XA^{(1)}AG)A = A,$$

and

$$\begin{aligned} ZAZ &= GAV^{-2}G(AGA)V^{-2}G = GAV^{-2}GAV^{-1}G \\ &= GA\cdot A^{(1)}AXA^{(1)}AXA^{(1)}A\cdot GAV^{-1}G \\ &= GA\cdot A^{(1)}AXA^{(1)}AXA^{(1)}\cdot AG \\ &= G\big(A\cdot A^{(1)}AXA^{(1)}AXA^{(1)}A\big)G = G(AV^{-2})G = Z. \end{aligned}$$

For any $x\in K$, by (27) we have $\big(GAXA^{(1)}AG-G\big)(x)\in\mathcal{N}(A)\cap\mathcal{R}(G)=\{0\}$, which implies that

$$GAXA^{(1)}AG=G.$$

It follows from (28) and (26) that

$$GAZ=G(AGA)V^{-2}G=G(AV^{-1})G=GAXA^{(1)}AG=G, \tag{30}$$

and by (27) we have

$$\begin{aligned} ZAG &= GAV^{-2}GAG = GA\cdot A^{(1)}AXA^{(1)}A\big(XA^{(1)}AG\big)AG \\ &= GAXA^{(1)}AG = G. \end{aligned} \tag{31}$$

By (29), (30) and (31) we conclude that $\mathcal{R}(Z)=\mathcal{R}(G)$ and $\mathcal{N}(Z)=\mathcal{N}(G)$, so $Z=A^{(1,2)}_{\mathcal{R}(G),\mathcal{N}(G)}$. This completes the proof of the equivalence of conditions (i), (ii), and (iii).

(2) Suppose that $A^{(1,2)}_{\mathcal{R}(G),\mathcal{N}(G)}$ exists. Then by the proof of (ii) $\Longrightarrow$ (iii) we have $A^{(1,2)}_{\mathcal{R}(G),\mathcal{N}(G)}=GAV^{-2}G$. By the definitions of U and V, we have $AV=AGA=UA$, so $U^{-1}A=AV^{-1}$, and hence $U^{-2}A=U^{-1}AV^{-1}=AV^{-2}$, therefore (20) holds.

By Theorem 2.3 we have $H=\mathcal{N}(A)\oplus\mathcal{R}(G)$ and $K=\mathcal{N}(G)\oplus A\mathcal{R}(G)$, so

$$\begin{aligned} &\mathcal{R}(G)=GA\mathcal{R}(G)=\mathcal{R}(GAG), \mathcal{R}(AG)=A\mathcal{R}(G)=\mathcal{R}(A), \\ &\mathcal{R}(GA)=G\mathcal{R}(A)=GA\mathcal{R}(G)=\mathcal{R}(GAG)=\mathcal{R}(G), \end{aligned}$$

which means that $\mathcal{R}(AG),\mathcal{R}(GA)$ and $\mathcal{R}(GAG)$ are all closed. Let $(GAG)^{(1)}$, $(GA)^{(1)}$ and $(AG)^{(1)}$ be any $\{1\}$-inverses of GAG, GA and AG respectively. Then

$$\begin{aligned} AG &= U^{-1}(UA)G=(U^{-1}A)(GAG)=(U^{-1}A)(GAG)(GAG)^{(1)}(GAG) \\ &= (U^{-1}AGA)G(GAG)^{(1)}(GAG)=AG(GAG)^{(1)}(GAG). \end{aligned}$$

As $\mathcal{N}(A)\cap\mathcal{R}(G)=\{0\}$, we get

$$G=G(GAG)^{(1)}GAG. \tag{32}$$

Let W_K be defined by (16). Then it is easy to show that $GAGW_K=G$, so by (15) and (32) we have

$$A^{(1,2)}_{T,S}=GW_K=G(GAG)^{(1)}(GAGW_K)=G(GAG)^{(1)}G. \tag{33}$$

Finally, choose any $X \in \mathcal{L}(H)$ which satisfies (27). Then

$$A = A(XA^{(1)}A)GA = A(XA^{(1)}A)GA(GA)^{(1)}GA = A(GA)^{(1)}(GA),$$

therefore,

$$\begin{aligned} GAG \cdot (AG)^{(1)}A(GA)^{(1)} \cdot GAG &= GAG(AG)^{(1)}\Big(A(GA)^{(1)}GA\Big)G \\ &= GAG(AG)^{(1)}AG = GAG, \end{aligned}$$

which means that $(AG)^{(1)}A(GA)^{(1)}$ is a $\{1\}$-inverse of GAG. By (33) we conclude that $A^{(1,2)}_{T,S} = G(AG)^{(1)}A(GA)^{(1)}G$. $\square$

Remark 2.9. At this point, it is helpful to give an explanation of the preceding theorem. Let $H, K, A, G, A^{(1)}, U$ and V be as in Theorem 2.8 and suppose that $A^{(1,2)}_{\mathcal{R}(G),\mathcal{N}(G)}$ exists. Let W_K be defined by (16). Then by the proof of (iii) $\Longrightarrow$ (i) of Theorem 2.8 we know that

$$U^{-1} = AA^{(1)}W_KAA^{(1)} + I_K - AA^{(1)} = W_KAA^{(1)} + I_K - AA^{(1)},$$

so

$$\begin{aligned} GU^{-2}AG &= G\big(W_KAA^{(1)}W_KAA^{(1)}\big)AG = G(W_KW_KAG) \\ &= GW_K = A^{(1,2)}_{\mathcal{R}(G),\mathcal{N}(G)}. \end{aligned}$$

Corollary 2.10. *Let H and K be two Hilbert $\mathfrak{A}$-modules, $A \in \mathcal{L}(H,K)$ and $G \in \mathcal{L}(K,H)$ have closed ranges. Let $A^{(1)}$ be any $\{1\}$-inverse of A. Then the following conditions are equivalent*:

(i) $V = A^{(1)}AGA + I_H - A^{(1)}A$ *is invertible, and* $\mathcal{N}(A^*) \cap \mathcal{R}(G^*) = \{0\}$;
(ii) $U = AGAA^{(1)} + I_K - AA^{(1)}$ *is invertible, and* $\mathcal{N}(A^*) \cap \mathcal{R}(G^*) = \{0\}$;
(iii) $A^{(1,2)}_{\mathcal{R}(G),\mathcal{N}(G)}$ *exists.*

Proof. Note that if $(A^*)^{(1,2)}_{\mathcal{R}(G^*),\mathcal{N}(G^*)}$ exists, then its adjoint operator equals $A^{(1,2)}_{\mathcal{R}(G),\mathcal{N}(G)}$ (see (10)). Note also that the adjoint operator of $A^{(1)}$ is a $\{1\}$-inverse of A^*, so if we replace $H, K, A, G, \mathcal{R}(G)$, $\mathcal{N}(G)$ and $A^{(1)}$ with K, H, A^*, G^*, $\mathcal{R}(G^*)$, $\mathcal{N}(G^*)$ and $\big(A^{(1)}\big)^*$ respectively, then according to Theorem 2.8 we conclude that conditions (i) through (iii) are equivalent. $\square$

Remark 2.11. Let H, K be two Hilbert $\mathfrak{A}$-modules, and $A \in \mathcal{L}(H,K)$. As in the Hilbert space case, it can be proved that

(1) The Moore-Penrose inverse $A^\dagger$ exists if and only if $\mathcal{R}(A)$ is closed [8, Theorem 2.2]. In which case, $A^\dagger = A^{(1,2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}$.

(2) The (finite index) Drazin inverse A^D [3] exists if and only if $\mathrm{Ind}(A) = k < \infty$ and $\mathcal{R}(A^k)$ is closed. In which case, $A^D = A^{(1,2)}_{T,S}$ with $T = \mathcal{R}(A^k)$ and $S = \mathcal{N}(A^k)$.

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