

WEYL'S THEOREM AND TENSOR PRODUCT FOR OPERATORS SATISFYING $T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k$

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ABSTRACT. For a bounded linear operator T on a separable complex infinite dimensional Hilbert space \mathcal{H} , we say that T is a quasi-class (A, k) operator if $T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k$. In this paper we prove that if T is a quasi-class (A, k) operator and f is an analytic function on an open neighborhood of the spectrum of T , then $f(T)$ satisfies Weyl's theorem. Also, we consider the tensor product for quasi-class (A, k) operators.

1. Introduction

Weyl's theorem for an operator says that the complement in the spectrum of the Weyl spectrum coincides with the isolated points of the spectrum which are eigenvalues of finite multiplicity. H. Weyl [20] discovered that this property holds for Hermitian operators and it has been extended from Hermitian operators to hyponormal operators by L. A. Coburn [3], and to several classes of operators including seminormal operators by S. K. Berberian [2]. In [14], W. Y. Lee and S. H. Lee showed that if T is hyponormal operator, then Weyl's theorem holds for $f(T)$, where f is an analytic function on a neighborhood of spectrum of T . Recently, this result was extended to p -quasihyponormal operators, class A operators and quasi-class A operators in [19], [18] and [5], respectively.

In this paper we study Weyl theorem for $f(T)$ and tensor product for quasi-class A operators.

Throughout this paper, let \mathcal{H} be a separable complex infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on \mathcal{H} . Recall that $T \in \mathcal{L}(\mathcal{H})$ belongs to *class A* if $|T^2| \geq |T|^2$ ([6]), and recall that $T \in \mathcal{L}(\mathcal{H})$ is called a quasi-class A operator if $T^*|T^2|T \geq T^*|T|^2T$ ([10], [5]).

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In [17], K. Tanahashi, I. H. Jeon, I. H. Kim, and A. Uchiyama considered an extension of the notion of quasi-class A operators, similar in spirit to the extension of the notion of p -quasihyponormality to (p, k) -quasihyponormality.

Definition 1.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a quasi-class (A, k) operator if it satisfies the following operator inequality:

$$T^{*k} (|T^2| - |T|^2) T^k \geq 0,$$

where k is a positive integer.

The following example shows that there is a big gap between the set of quasi-class A operators and the set of quasi-class (A, k) operators.

Example 1.2. Consider the unilateral weighted shift operators as an infinite dimensional Hilbert space operator. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \dots$ (called weights), the unilateral weighted shift W_α associated with weight α is the operator on $\mathcal{H} = \ell^2$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2 . We easily see that W_α can be never normal, and so in general it is used to giving some easy examples of non-normal operators. It is well known that the followings are equivalent:

- (i) W_α is hyponormal.
- (ii) W_α is class A .
- (iii) α is monotonically increasing, i.e., $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \dots$.

Therefore it is meaningless to use this characterization for distinguishing some gaps between hyponormal operators and class A operators. However, for quasi-class (A, k) operators, W_α has a very useful characterization. Indeed, simple calculation shows that

$$W_\alpha = \begin{pmatrix} 0 & & & & & \\ \alpha_0 & 0 & & & & \\ & \alpha_1 & 0 & & & \\ & & \alpha_2 & 0 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \end{pmatrix}$$

is a quasi-class (A, k) operator if and only if $\alpha_0, \dots, \alpha_{k-1}$ are arbitrary and $\alpha_k \leq \alpha_{k+1} \leq \alpha_{k+2} \leq \dots$ for $k = 1, 2, \dots$

2. Weyl's theorem for $f(T)$

We shall denote the set of all complex numbers and the complex conjugate of a complex number λ by \mathbb{C} and $\bar{\lambda}$, respectively. The closure of a set \mathcal{M} is denoted by $\overline{\mathcal{M}}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. For $T \in \mathcal{B}(\mathcal{H})$, we write $\ker T$ and $\text{ran} T$ for the null space and the range of T , respectively. An operator $T \in \mathcal{B}(\mathcal{H})$ is called an upper semi-Fredholm if it has closed range and finite dimensional null space (i.e., $\alpha(T) := \dim \ker T < \infty$), and $T \in \mathcal{B}(\mathcal{H})$ is

called a lower semi-Fredholm if it has closed range and finite co-dimensional (i.e., $\beta(T) := \dim \ker T^* < \infty$). If $T \in \mathcal{B}(\mathcal{H})$ is both upper semi-Fredholm and lower semi-Fredholm, we call it *Fredholm*. If $T \in \mathcal{B}(\mathcal{H})$ is semi-Fredholm, then the *index* of T , denote $\text{ind}(T)$, is given by

$$\text{ind}(T) = \alpha(T) - \beta(T).$$

The index is an integer or $\{\pm\infty\}$. The ascent of $T \in \mathcal{B}(\mathcal{H})$, denote $\text{asc}(T)$, is the least non-negative integer n such that $\ker T^n = \ker T^{n+1}$ and the descent of T , denote $\text{dsc}(T)$, is the least non-negative integer n such that $\text{ran} T^n = \text{ran} T^{n+1}$. We say that $T \in \mathcal{B}(\mathcal{H})$ is of finite ascent (resp. finite descent) if $\text{asc}(T - \lambda) < \infty$ (resp. $\text{dsc}(T - \lambda) < \infty$) for all $\lambda \in \mathbb{C}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a *Weyl* if it is Fredholm of index zero. We denote the spectrum of $T \in \mathcal{B}(\mathcal{H})$ by $\sigma(T)$, and the sets of isolated points and accumulation points of $\sigma(T)$ are denoted by $\text{iso}\sigma(T)$ and $\text{acc}\sigma(T)$, respectively. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $w(T)$ are defined by

$$\begin{aligned}\sigma_e(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}, \\ w(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}.\end{aligned}$$

It is well known [7] that

$$\sigma_e(T) \subseteq w(T).$$

We let

$$\pi_{00}(T) := \{\lambda \in \mathbb{C} : \lambda \in \text{iso}\sigma(T) \text{ and } 0 < \alpha(T - \lambda) < \infty\}$$

denote the set of isolated eigenvalues of finite multiplicity.

Following [3], we say that $T \in \mathcal{B}(\mathcal{H})$ satisfies *Weyl's theorem* if there is equality

$$\sigma(T) \setminus w(T) = \pi_{00}(T).$$

Let $H(\sigma(T))$ be the set of all analytic functions on an open neighborhood of $\sigma(T)$. In [14], W. Y. Lee and S. H. Lee showed that if $T \in \mathcal{B}(\mathcal{H})$ is a hyponormal operator and $f \in H(\sigma(T))$, then Weyl's theorem holds for $f(T)$. Recently, this result was extended to p -quasihyponormal operators, class A operators and quasi-class A operators in [19], [18] and [5], respectively. In this section we show that if $T \in \mathcal{B}(\mathcal{H})$ is a quasi-class (A, k) operator and $f \in H(\sigma(T))$, then Weyl theorem holds for $f(T)$.

Lemma 2.1 (Hansen's inequality). *If $A, B \in \mathcal{B}(\mathcal{H})$ satisfy $A \geq 0$ and $\|B\| \leq 1$, then $(B^*AB)^\delta \geq B^*A^\delta B$ for all $0 < \delta \leq 1$.*

Lemma 2.2. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-class (A, k) operator and T does not have a dense range, then T has the following matrix representation:*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{\text{ran} T^k} \oplus \ker T^{*k},$$

where T_1 is a class A operator on $\overline{\text{ran} T^k}$ and T_3 is a nilpotent operator with nilpotency k . Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Consider the matrix representation of T with respect to the decomposition $\mathcal{H} = \overline{\text{ran}T^k} \oplus \ker T^{*k}$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

Let P be the orthogonal projection onto $\overline{\text{ran}T^k}$. Then

$$\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP.$$

Since T is a quasi-class (A, k) operator, we have

$$P(|T^2| - |T|^2)P \geq 0.$$

By Lemma 2.1 (Hansen's inequality), we obtain

$$P(|T^2|)P = P(T^{*2}T^2)^{\frac{1}{2}}P \leq (PT^{*2}T^2P)^{\frac{1}{2}} = \begin{pmatrix} |T_1^2| & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$P|T|^2P = PT^*TP = \begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = P|T|^2P \leq P|T^2|P \leq \begin{pmatrix} |T_1^2| & 0 \\ 0 & 0 \end{pmatrix},$$

and hence T_1 is a class A operator on $\overline{\text{ran}T^k}$. On the other hand, for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H}$, we have

$$\langle T_3^k x_2, x_2 \rangle = \langle T^k(I - P)x, (I - P)x \rangle = \langle (I - P)x, T^{*k}(I - P)x \rangle = 0,$$

which implies that $T_3^k = 0$. It is well known that $\sigma(T_1) \cup \sigma(T_3) = \sigma(T) \cup \mathfrak{G}$, where \mathfrak{G} is the union of certain of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1) \cap \sigma(T_3)$, and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points. Therefore we have

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\}. \quad \square$$

Corollary 2.3. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-class (A, k) operator and the restriction T_1 of T on $\overline{\text{ran}T^k}$ is invertible, then T is similar to a direct sum of a class A operator and a nilpotent operator.*

Proof. Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\text{ran}T^k} \oplus \ker T^{*k}$. By Lemma 2.2, T_1 is a class A operator and T_3 is a nilpotent operator with nilpotency k . Since $0 \notin \sigma(T)$ by assumption, we have $\sigma(T_1) \cap \sigma(T_3) = \emptyset$. Hence by Rosenblum's Corollary there exists an operator S for which $T_1S - ST_3 = T_2$. Therefore

$$\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix},$$

which completes the proof. \square

Corollary 2.4. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-class (A, k) operator, then T is an isoloid.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\text{ran}T^k} \oplus \ker T^{*k}$, and assume that $\lambda_0 \in \text{iso}\sigma(T)$. Then $\lambda_0 \in \text{iso}\sigma(T_1)$ or $\lambda_0 = 0$ by Lemma 2.2. If $\lambda_0 \in \text{iso}\sigma(T_1)$, then $\lambda_0 \in \sigma_p(T_1)$ because T_1 is a class A operator and a class A operator is an isoloid. Thus we may assume that $\lambda_0 = 0$ and $\lambda_0 \notin \sigma(T_1)$, so $\dim\ker(T_3) > 0$. Therefore if $x \in \ker(T_3)$, then $-T_1^{-1}T_2x \oplus x \in \ker(T)$. Hence λ_0 is an eigenvalue of T , which completes the proof. \square

Lemma 2.5. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-class (A, k) operator and $(T - \lambda)x = 0$ for $\lambda \neq 0$ and $x \in \mathcal{H}$, then $(T - \lambda)^*x = 0$.*

Lemma 2.6. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-class (A, k) operator, then T is of finite ascent.*

Proof. To prove this result we shall show that $\ker(T - \lambda)^{k+1} = \ker(T - \lambda)^{k+2}$. Since, by Lemma 2.5, $(T - \lambda)x = 0$ implies $(T - \lambda)^*x = 0$ for each non zero λ ,

$$\ker(T - \lambda) = \ker(T - \lambda)^2 \quad \text{for } \lambda \neq 0.$$

So it suffices to prove that $\ker(T)^{k+1} = \ker(T)^{k+2}$. Assume that $T^{k+2}x = 0$ but $T^{k+1}x \neq 0$ because if $T^{k+1}x = 0$, then we obviously get the conclusion. Using the Hölder-McCarthy inequality:

- (i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r \|x\|^{2(1-r)}$ for $r > 1$ and $x \in \mathcal{H}$,
- (ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r \|x\|^{2(1-r)}$ for $0 \leq r \leq 1$ and $x \in \mathcal{H}$.

We have

$$\begin{aligned} 0 &= \|T^{k+2}x\| \\ &= \langle T^{k+2}x, T^{k+2}x \rangle^{\frac{1}{2}} \\ &= \langle T^{*2}T^2T^kx, T^kx \rangle^{\frac{1}{2}} \\ &= \langle |T^2|^2T^kx, T^kx \rangle^{\frac{1}{2}} \\ &\geq \langle |T^2|T^kx, T^kx \rangle \|T^kx\|^{-1} \\ &\geq \langle |T|^2T^kx, T^kx \rangle \|T^kx\|^{-1} \\ &= \|T^{k+1}x\|^2 \|T^kx\|^{-1}, \end{aligned}$$

which implies $\ker(T^{k+2}) \subseteq \ker(T^{k+1})$. Consequently, $\ker(T)^{k+1} = \ker(T)^{k+2}$ which completes the proof. \square

The following lemma shows that the passage from $w(A) \cup w(B)$ to $w\left(\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\right)$.

Lemma 2.7 ([13, Theorem 6]). *For a given operator $A, B, C \in \mathcal{B}(\mathcal{H})$,*

$$w(A) \cup w(B) = w(M_C) \cup \mathfrak{G},$$

where $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and \mathfrak{G} is the union of certain of the holes in $w(M_C)$ which happen to be subset of $w(A) \cap w(B)$.

The following theorem shows that the spectral mapping theorem for Weyl spectrum holds for a quasi-class (A, k) operators.

Theorem 2.8. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-class (A, k) operator, then $w(f(T)) = f(w(T))$ for every analytic function f on a neighborhood of $\sigma(T)$.*

Proof. We need only to prove that $w(p(T)) = p(w(T))$ for any polynomial p . Since T has the matrix representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_1 is a class A operator and $T_3^k = 0$, and the spectral mapping theorem for Weyl spectrum holds for a class A operator, it follows that

$$\begin{aligned} w(p(T)) &= w(p(T_1)) \cup w(p(T_3)) \quad \text{by Lemma 2.7} \\ &= p(w(T_1)) \cup p(w(T_3)) \\ &= p(w(T_1) \cup w(T_3)) \\ &= p(w(T)). \end{aligned} \quad \square$$

It was known ([13], Lemma 10) that if A and B are isoloid and if Weyl’s theorem holds for A and B , then

$$\text{Weyl’s theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \iff w \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = w(A) \cup w(B).$$

The “spectral picture” of the operator $T \in \mathcal{B}(\mathcal{H})$, denoted by $SP(T)$, which consists of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes.

In general, Weyl’s theorem does not hold for an operator matrix $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ even though Weyl’s theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ (see [12]).

Lemma 2.9 ([12, Theorem 2.4]). *If either $SP(A)$ or $SP(B)$ has no pseudoholes and if A is an isoloid operator for which Weyl’s theorem holds, then for every $C \in \mathcal{B}(\mathcal{H})$,*

$$\text{Weyl’s theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \implies w \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = w(A) \cup w(B).$$

We have the following result from above lemma.

Corollary 2.10. *Weyl’s theorem holds for a quasi-class (A, k) operators.*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ is a quasi-class (A, k) operator. Then T has the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{\text{ran}T^k} \oplus \ker T^{*k},$$

where T_1 is a class A operator on $\overline{\text{ran}T^k}$ and T_3 is a nilpotent operator with nilpotency k . Therefore Weyl’s theorem holds for $\begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}$ because Weyl’s theorem holds for class A operators and nilpotent operators, and both class A operators and nilpotent operators are isoloid. Hence by Lemma 2.9, Weyl’s theorem holds for $\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ because $SP(T_3)$ has no pseudoholes. \square

Theorem 2.11. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-class (A, k) , then $f(T)$ satisfies Weyl's theorem for every analytic function f on a neighborhood of $\sigma(T)$.*

Proof. Recall ([14], Theorem 2) that if $A \in \mathcal{B}(\mathcal{H})$ is isoloid, then

$$f(\sigma(A) \setminus \pi_{00}(A)) = \sigma(f(A)) \setminus \pi_{00}(f(A)) \quad \text{for every } f \in H(\sigma(A)).$$

Thus it follows from Corollary 2.4, Theorem 2.8 and Corollary 2.10 that

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(w(T)) = w(f(T)),$$

which implies that $f(T)$ satisfies Weyl's theorem. \square

3. Tensor product for quasi-class (A, k) operators

Assume that $\mathcal{J} \subseteq \mathbb{R}$ is an interval and that $f : \mathcal{J} \rightarrow \mathbb{R}$ is an operator monotone function. Consider an operator $T \in \mathcal{B}(\mathcal{H})$ for which the spectrum of $|T|$ is contained in \mathcal{J} , where $|T|$ denotes $(T^*T)^{\frac{1}{2}}$. The operator $T \in \mathcal{B}(\mathcal{H})$ is said to be f -hyponormal if

$$f(T^*T) \geq f(TT^*).$$

Especially, $T \in \mathcal{B}(\mathcal{H})$ is called a hyponormal if f is identity function, and is called a p -hyponormal if $f(x) = x^p$ for $0 < p \leq 1$. For given non-zero $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$, let $T \otimes S$ denote the tensor product on the product space $\mathcal{H} \otimes \mathcal{K}$.

The following open question is very interesting.

Question. Which intervals $\mathcal{J} \subseteq \mathbb{R}_0^+$ and operator monotone functions f have the property that $T \otimes S$ is f -hyponormal if and only if T and S are f -hyponormal?

The operation of taking tensor products $T \otimes S$ preserves many properties of $T, S \in \mathcal{B}(\mathcal{H})$, but by no means all of them. Thus, whereas the normaloid property is invariant under tensor products (see [15], p. 623); again, whereas $T \otimes S$ is normal if and only if so are T and S [8], [16], there exist paranormal operators T and S such that $T \otimes S$ is not paranormal [1]. In [4], Duggal showed that for non-zero $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$, $T \otimes S$ is p -hyponormal if and only if $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ are p -hyponormal. This result was extended to p -quasihyponormal operators, class A operators and quasi-class A operators in [11], [9] and [10], respectively.

In this section we prove analogous results for a quasi-class (A, k) operators.

The following key lemma is due to J. Stochel [16].

Lemma 3.1 ([16, Proposition 2.2]). *Let $A_1, A_2 \in \mathcal{B}(\mathcal{H}), B_1, B_2 \in \mathcal{B}(\mathcal{K})$ be non-negative operators. If A_1 and B_1 are non-zero, then the following assertions are equivalent:*

- (1) $A_1 \otimes B_1 \leq A_2 \otimes B_2$.
- (2) There exists $c > 0$ such that $A_1 \leq cA_2$ and $B_1 \leq c^{-1}B_2$.

Theorem 3.2. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ are non-zero operators. Then $T \otimes S$ is a class A operator if and only if T and S are class A operators.*

Proof. We begin with two observations. The first is that $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$ and so, by the uniqueness of positive square roots, $|T \otimes S|^r = |T|^r \otimes |S|^r$ for any positive rational number r . From the density of the rationales in the real, we obtain $|T \otimes S|^p = |T|^p \otimes |S|^p$ for any positive real number p . The second observation is that if $T_1 \geq T_2$ and $S_1 \geq S_2$, then $T_1 \otimes S_1 \geq T_2 \otimes S_2$ (see, [16]).

Assume that T and S are class A operators. Then

$$|(T \otimes S)^2| = |T^2 \otimes S^2| = |T^2| \otimes |S^2| \geq |T|^2 \otimes |S|^2 = |T \otimes S|^2$$

which implies $T \otimes S$ is a class A operator.

Conversely, assume that $T \otimes S$ is a class A operator. We aim to show that T and S are class A operators. Without loss of generality, it is enough to show that T is a class A operator. Since $T \otimes S$ is a class A operator, we obtain

$$|T|^2 \otimes |S|^2 \leq |(T \otimes S)^2|.$$

Therefore, by Lemma 3.1, there exists a positive real number c for which

$$|T|^2 \leq c|T^2| \quad \text{and} \quad |S|^2 \leq c^{-1}|S^2|.$$

Consequently, for arbitrary $x \in \mathcal{H}$ and $y \in \mathcal{K}$

$$\begin{aligned} \|T\|^2 &= \sup_{\|x\|=1} \langle |T|^2 x, x \rangle \\ &\leq \sup_{\|x\|=1} \langle c|T^2| x, x \rangle \\ &= c \| |T^2| \| = c \|T^2\| \leq c \|T\|^2 \end{aligned}$$

and

$$\begin{aligned} \|S\|^2 &= \sup_{\|y\|=1} \langle |S|^2 y, y \rangle \\ &\leq \sup_{\|y\|=1} \langle c^{-1}|S^2| y, y \rangle \\ &= c^{-1} \| |S^2| \| = c^{-1} \|S^2\| \leq c^{-1} \|S\|^2. \end{aligned}$$

Thus, $c = 1$, and hence T is a class A operator. \square

Theorem 3.3. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ are non-zero operators. Then $T \otimes S$ is a quasi-class (A, k) operator if and only if one of the following holds:*

- (i) T and S are quasi-class (A, k) operators.
- (ii) $T^{k+1} = 0$ or $S^{k+1} = 0$.

Proof. By simple calculation we have $T \otimes S$ is a quasi-class (A, k) operator if and only if

$$(T \otimes S)^{*k} (|(T \otimes S)^2| - |T \otimes S|^2) (T \otimes S)^k \geq 0$$

or, equivalently, if and only if

$$(T \otimes S)^{*k} \left((|T^2| - |T|^2) \otimes |S^2| + |T|^2 \otimes (|S^2| - |S|^2) \right) (T \otimes S)^k \geq 0$$

or, equivalently, if and only if

$$T^{*k} (|T^2| - |T|^2) T^k \otimes S^{*k} |S^2| S^k + T^{*k} |T|^2 T^k \otimes S^{*k} (|S^2| - |S|^2) S^k \geq 0.$$

Thus the sufficiency is easily proved. Conversely, suppose that $T \otimes S$ is a quasi-class (A, k) operator. Then for every $x \in \mathcal{H}$ and $y \in \mathcal{K}$ we have

$$(1) \quad \begin{aligned} & \left\langle T^{*k} (|T^2| - |T|^2) T^k x, x \right\rangle \left\langle S^{*k} |S^2| S^k y, y \right\rangle \\ & + \left\langle T^{*k} |T|^2 T^k x, x \right\rangle \left\langle S^{*k} (|S^2| - |S|^2) S^k y, y \right\rangle \geq 0. \end{aligned}$$

It suffices to show that if the statement (i) does not hold, then the statement (ii) holds. Thus, assume to the contrary that neither of S^k and T^k is the zero operator, and T is not quasi-class (A, k) operator. Then there exists $x_k \in \mathcal{H}$ such that

$$\left\langle T^{*k} (|T^2| - |T|^2) T^k x_k, x_k \right\rangle := \alpha < 0$$

and

$$\left\langle T^{*k} |T|^2 T^k x_k, x_k \right\rangle := \beta > 0.$$

From (1) we have

$$(2) \quad (\alpha + \beta) \left\langle S^{*k} |S^2| S^k y, y \right\rangle \geq \beta \left\langle S^{*k} |S|^2 S^k y, y \right\rangle.$$

Thus S is a quasi-class (A, k) operator because $\alpha + \beta < \beta$. Using the Hölder-McCarthy inequality we have

$$\begin{aligned} \left\langle S^{*k} |S^2| S^k y, y \right\rangle &= \left\langle (S^{*2} S^2)^{\frac{1}{2}} S^k y, S^k y \right\rangle \\ &\leq \left\langle S^{*2} S^2 S^k y, S^k y \right\rangle^{\frac{1}{2}} \|S^k y\|^{2(1-\frac{1}{2})} \\ &= \|S^k y\| \|S^{k+2} y\| \end{aligned}$$

and

$$\left\langle S^{*k} |S|^2 S^k y, y \right\rangle = \left\langle S^{k+1} y, S^{k+1} y \right\rangle = \|S^{k+1} y\|^2.$$

Therefore, we have

$$(3) \quad (\alpha + \beta) \|S^k y\| \|S^{k+2} y\| \geq \beta \|S^{k+1} y\|^2.$$

On the other hand, since S is a quasi-class (A, k) operator, from Lemma 2.2 we have a decomposition of S as the following:

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\text{ran} S^k} \oplus \ker S^{*k},$$

where S_1 is a class A operator on $\overline{\text{ran} S^k}$ and S_3 is a nilpotent operator with nilpotency k .

Hence by (3) we obtain

$$(4) \quad (\alpha + \beta) \|S_1^k \eta\| \|S_1^{k+2} \eta\| \geq \beta \|S_1^{k+1} \eta\|^2 \text{ for all } \eta \in \overline{\text{ran}(S^k)}.$$

Since S_1 is class A operator (hence it is normaloid), and thus taking supremum on both sides of the above inequality, we have

$$(\alpha + \beta) \|S_1\|^{2(k+1)} \geq \beta \|S_1\|^{2(k+1)}.$$

This inequality forces that $S_1 = 0$. Therefore $S^{k+1} = 0$ because $S^{k+1}y = S_1 S^k y = 0$ for all $y \in \mathcal{K}$. This contradicts the assumption $S^{k+1} \neq 0$. Hence T must be a quasi-class (A, k) operator. A similar argument shows that S is also a quasi-class (A, k) operator, which completes the proof. \square

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