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CHARACTERIZATION OF THE GROUPS $D_{p+1}(2)$ AND $D_{p+1}(3)$ USING ORDER COMPONENTS

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Abstract. In this paper we will prove that the groups $D_{p+1}(2)$ and $D_{p+1}(3)$, where p is an odd prime number, are uniquely determined by their sets of order components. A main consequence of our result is the validity of Thompson's conjecture for the groups $D_{p+1}(2)$ and $D_{p+1}(3)$.

1. Introduction

For a positive integer n, let $\pi(n)$ denote the set of all prime divisors of n. If G is a finite group, we set $\pi(G) = \pi(|G|)$. The Gruenberg-Kegel graph of G, or the prime graph of G, is denoted by $GK(G)$ and is defined as follows. The vertex set of $GK(G)$ is the set $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G contains an element of order pq. We denote the connected components of $GK(G)$ by $\pi_1, \pi_2, \ldots, \pi_{s(G)}$, where $s(G)$ denotes the number of connected components of $GK(G)$. If the order of G is even, the notation is chosen so that $2 \in \pi_1$. It is clear that the order of G can be expressed as the product of the numbers $m_1, m_2, \ldots, m_{s(G)}$, where $\pi(m_i) = \pi_i$, $1 \leq i \leq s(G)$. If the order of G is even and $s(G) \geq 2$, according to our notation $m_2, \ldots, m_{s(G)}$ are odd numbers. The positive integers $m_1, m_2, \ldots, m_{s(G)}$ are called the order components of G and $OC(G) = \{m_1, m_2, \ldots, m_{s(G)}\}$ is called the set of order components of G. If the finite groups G and H have the same order components we are interested to know if G is isomorphic to H. For many simple groups H with $s(H) \geq 2$, the answer to the above question is affirmative. However if $s(H) = 1$ the answer is negative. The simple groups $B_n(q)$ and $C_n(q)$, where $n = 2^m \geq 4$ and q is odd, have the same order components but they are not isomorphic. Hence it is natural to adopt the following definition.

Definition 1. Let G be a finite group. The number of non-isomorphic finite groups with the same order components as G is denoted by $h(G)$ and is called the h-function of G. For any natural number k we say the finite group G is k-recognizable by its set of order components if $h(G) = k$. If $h(G) = 1$ we

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say that G is characterizable by its set of order components or briefly G is a characterizable group. In this case G is uniquely determined by the set of its order components.

Obviously for any finite groups G we have $h(G) \geq 1$. The components of the Gruenberg-Kegel graph $GK(P)$ of any non-abelian finite simple group P with $GK(P)$ disconnected are found in [21] and [24] from which we can deduce the component orders of P. These information which will be used in proving our main result are listed in Tables 1, 2, and 3.

Table 1. The order components of finite simple groups P with $s(P) = 2$ (p an odd prime)

P	Restrictions on P	m_1	m_2
$\mathbb{A}_n$	$6 < n = p, p+1, p+2$; one of $n, n-2$ is not a prime	$\frac{n!}{2p}$	p
$A_{p-1}(q)$	$(p,q) \neq (3,2), (3,4)$	$q^{\binom{p}{2}} \prod_{i=1}^{p-1}(q^i - 1)$	$\frac{(q^p-1)}{(q-1)(p,q-1)}$
$A_p(q)$	$(q-1) \mid (p+1)$	$q^{\binom{p+1}{2}}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-1)$	$\frac{(q^p-1)}{(q-1)}$
${}^2A_{p-1}(q)$		$q^{\binom{p}{2}} \prod_{i=1}^{p-1}(q^i - (-1)^i)$	$\frac{(q^p+1)}{(q+1)(p,q+1)}$
${}^2A_p(q)$	$(q+1) \mid (p+1)$, $(p,q) \neq (3,3), (5,2)$	$q^{\binom{p+1}{2}}(q^{p+1}-1)$ $\prod_{i=1}^{p-1}(q^i - (-1)^i)$	$\frac{(q^p+1)}{(q+1)}$
${}^2A_3(2)$		$2^6 \cdot 3^4$	5
$B_n(q)$	$n = 2^m \geq 4,\ q$ odd	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{(q^n+1)}{2}$
$B_p(3)$		$3^{p^2}(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{(3^p-1)}{2}$
$C_n(q)$	$n = 2^m \geq 2$	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{(q^n+1)}{(2,q-1)}$
$C_p(q)$	$q = 2, 3$	$q^{p^2}(q^p+1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{(q^p-1)}{(2,q-1)}$
$D_p(q)$	$p \geq 5,\ q = 2, 3, 5$	$q^{p(p-1)}\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{(q^p-1)}{(q-1)}$
$D_{p+1}(q)$	$q = 2, 3$	$q^{p(p+1)}(q^p+1)$ $(q^{p+1}-1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{(q^p-1)}{(2,q-1)}$
${}^2D_n(q)$	$n = 2^m \geq 4$	$q^{n(n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{(q^n+1)}{(2,q+1)}$
${}^2D_n(2)$	$n = 2^m + 1 \geq 5$	$2^{n(n-1)}(2^n+1)(2^{n-1}-1)$ $\prod_{i=1}^{n-2}(2^{2i}-1)$	$2^{n-1}+1$
${}^2D_{p+1}(2)$	$5 \leq p \neq 2^m - 1$	$2^{p(p+1)}(2^p+1)(2^{p+1}+1)$ $\prod_{i=1}^{p-1}(2^{2i}-1)$	$2^p - 1$
${}^2D_p(3)$	$5 \leq p \neq 2^m + 1$	$3^{p(p-1)}\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{(3^p+1)}{4}$
${}^2D_n(3)$	$9 \leq n = 2^m + 1 \neq p$	$\frac{1}{2}3^{n(n-1)}(3^n+1)(3^{n-1}-1)$ $\prod_{i=1}^{n-2}(3^{2i}-1)$	$\frac{(3^{n-1}+1)}{2}$
$G_2(q)$	$2 < q \equiv \epsilon \pmod 3,\ \epsilon = \pm 1$	$q^6(q^3-\epsilon)(q^2-1)(q+\epsilon)$	$q^2 - \epsilon q + 1$
${}^3D_4(q)$		$q^{12}(q^6-1)(q^2-1)$ (q^4+q^2+1)	$q^4 - q^2 + 1$
$F_4(q)$	q odd	$q^{24}(q^8-1)(q^6-1)^2(q^4-1)$	$q^4 - q^2 + 1$
${}^2F_4(2)'$		$2^{11} \cdot 3^3 \cdot 5^2$	13

Table 1. (continued)

$E_6(q)$		$q^{36}(q^{12}-1)(q^8-1)(q^6-1)(q^5-1)(q^3-1)(q^2-1)$	$\frac{(q^6+q^3+1)}{(3,q-1)}$
${}^2E_6(q)$	$q>2$	$q^{36}(q^{12}-1)(q^8-1)(q^6-1)(q^5+1)(q^3+1)(q^2-1)$	$\frac{(q^6-q^3+1)}{(3,q+1)}$
M_{12}		$2^6\cdot 3^3\cdot 5$	11
J_2		$2^7\cdot 3^3\cdot 5^2$	7
Ru		$2^{14}\cdot 3^3\cdot 5^3\cdot 7\cdot 13$	29
He		$2^{10}\cdot 3^3\cdot 5^2\cdot 7^3$	17
McL		$2^7\cdot 3^6\cdot 5^3\cdot 7$	11
Co_1		$2^{21}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13$	23
Co_3		$2^{10}\cdot 3^7\cdot 5^3\cdot 7\cdot 11$	23
Fi_{22}		$2^{17}\cdot 3^9\cdot 5^2\cdot 7\cdot 11$	13
HN		$2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 11$	19

Table 2. The order components of finite simple groups P with $s(P)=3$ (p an odd prime)

P	Restrictions on P	m_1	m_2	m_3
$\mathbb{A}_n$	$n>6,\ n=p,\ p-2$ are primes	$\frac{n!}{2n(n-2)}$	p	$p-2$
$A_1(q)$	$3<q\equiv\epsilon \pmod 4$, $\epsilon=\pm1$	$q-\epsilon$	q	$\frac{(q+\epsilon)}{2}$
$A_1(q)$	$q>2$, q even	q	$q-1$	$q+1$
${}^2A_5(2)$		$2^{15}\cdot 3^6\cdot 5$	7	11
${}^2D_p(3)$	$p=2^m+1\geq 5$	$2\cdot 3^{p(p-1)}(3^{p-1}-1)\prod_{i=1}^{p-2}(3^{2i}-1)$	$\frac{(3^{p-1}+1)}{2}$	$\frac{(3^p+1)}{4}$
${}^2D_{p+1}(2)$	$p=2^n-1,\ n\geq 2$	$2^{p(p+1)}(2^p-1)\prod_{i=1}^{p-1}(2^{2i}-1)$	2^p+1	$2^{p+1}+1$
$G_2(q)$	$q\equiv 0 \pmod 3$	$q^6(q^2-1)^3$	q^2-q+1	q^2+q+1
${}^2G_2(q)$	$q=3^{2m+1}>3$	$q^3(q^2-1)$	$q-\sqrt{3}q+1$	$q+\sqrt{3}q+1$
$F_4(q)$	q even	$q^{24}(q^6-1)^2(q^4-1)^2$	q^4+1	q^4-q^2+1
${}^2F_4(q)$	$q=2^{2m+1}>2$	$q^{12}(q^4-1)(q^3+1)$	$q^2-\sqrt{2q^3}+q-\sqrt{2q}+1$	$q^2+\sqrt{2q^3}+q+\sqrt{2q}+1$
$E_7(2)$		$2^{63}\cdot 3^{11}\cdot 5^2\cdot 7^3\cdot 11\cdot 13\cdot 17\cdot 19\cdot 31\cdot 43$	73	127
$E_7(3)$		$2^{23}\cdot 3^{63}\cdot 5^2\cdot 7^3\cdot 11^2\cdot 13^2\cdot 19\cdot 37\cdot 41\cdot 61\cdot 73\cdot 547$	757	1093
M_{11}		$2^4\cdot 3^2$	5	11
M_{23}		$2^7\cdot 3^2\cdot 5\cdot 7$	11	23
M_{24}		$2^{10}\cdot 3^3\cdot 5\cdot 7$	11	23
J_3		$2^7\cdot 3^5\cdot 5$	17	19
HiS		$2^9\cdot 3^2\cdot 5^3$	7	11
Suz		$2^{13}\cdot 3^7\cdot 5^2\cdot 7$	11	13
Co_2		$2^{18}\cdot 3^6\cdot 5^3\cdot 7$	11	23
Fi_{23}		$2^{18}\cdot 3^{13}\cdot 5^2\cdot 7\cdot 11\cdot 13$	17	23
F_3		$2^{15}\cdot 3^{10}\cdot 5^3\cdot 7^2\cdot 13$	19	31
F_2		$2^{41}\cdot 3^{13}\cdot 5^6\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23$	31	47

Table 3. The order components of finite simple groups P with $s(P) > 3$

P	Restrictions on P	m_1	m_2	m_3	m_4	m_5	m_6
$A_2(4)$		2^6	3	5	7		
${}^2B_2(q)$	$q = 2^{2m+1} > 2$	q^2	$q-1$	$q-\sqrt{2q}$ $+1$	$q+\sqrt{2q}$ $+1$		
${}^2E_6(2)$		$2^{36}\cdot 3^9\cdot 5^2\cdot 7^2\cdot 11$	13	17	19		
$E_8(q)$	$q \equiv 2,3$ (mod 5)	$q^{120}(q^{20}-1)(q^{18}-1)$ $(q^{14}-1)(q^{12}-1)$ $(q^{10}-1)(q^8-1)$ $(q^4+1)(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4 $+1$		
M_{22}		$2^7\cdot 3^2$	5	7	11		
J_1		$2^3\cdot 3\cdot 5$	7	11	19		
$O'N$		$2^9\cdot 3^4\cdot 5\cdot 7^3$	11	19	31		
LyS		$2^8\cdot 3^7\cdot 5^6\cdot 7\cdot 11$	31	37	67		
Fi'_{24}		$2^{21}\cdot 3^{16}\cdot 5^2\cdot 7^3\cdot 11\cdot 13$	17	23	29		
F_1		$2^{46}\cdot 3^{20}\cdot 5^9\cdot 7^6\cdot 11^2\cdot 13^3\cdot$ $17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 47$	41	59	71		
$E_8(q)$	$q \equiv 0,1,4$ (mod 5)	$q^{120}(q^{18}-1)(q^{14}-1)$ $(q^{12}-1)^2(q^{10}-1)^2$ $(q^8-1)^2(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4 $+1$	$\frac{q^{10}+1}{q^2+1}$	
J_4		$2^{21}\cdot 3^3\cdot 5\cdot 7\cdot 11^3$	23	29	31	37	43

In [19] and [20] it is proved that if $n = 2^m \geq 4$, then $h(B_n(q)) = h(C_n(q)) = 2$ for q odd and $h(B_n(q)) = h(C_n(q)) = 1$ for q even. In [9] it is proved that $h(B_p(3)) = h(C_p(3)) = 2$, where p is an odd prime number. The following groups have been proved to be characterizable by their order components by various authors: All the sporadic simple groups [2], $PSL_2(q)$, ${}^2D_n(3)$, where $9 \leq n = 2^m + 1$ is not a prime, ${}^2D_{p+1}(2)$, where p is a prime number and $5 < p \neq 2^m - 1$, in [3], [6] and [23], respectively. Some projective special linear (unitary) groups have been characterized in [11], [13], [14], [15] and [17]. A few of the alternating or symmetric groups are proved to be characterizable by their order components in [1] and [18]. The groups ${}^2D_p(3)$, where $p \geq 5$ is a prime number not of the form $2^m + 1$, and ${}^2D_n(2)$, where $n = 2m + 1 \geq 5$ are characterized by the set of order components in [10] and [12], respectively. Based on these results we put forward the following conjecture.

Conjecture 1. *Let P be a non-abelian finite simple group with $s(P) \geq 2$. If G is a finite group and $OC(G) = OC(P)$, then either $G \cong P$ or $G \cong B_n(q)$ or $C_n(q)$, where $n = 2^m \geq 4$ and q is an odd prime power, or $G \cong B_p(3)$, $C_p(3)$, where p is an odd prime number.*

A motivation for characterizing finite groups by the set of their order components is the following conjecture due to J. G. Thompson.

Conjecture 2 (Thompson)**.** *For a finite group G let $N(G) = \{n \in \mathbb{N} \mid G$ has a conjugacy class of size $n\}$. Let $Z(G) = 1$ and M be a non-abelian finite simple group satisfying $N(G) = N(M)$. Is it true that $G \cong M$?*

In [4] it is proved that if $s(M) \geq 3$, then the above conjecture holds. Also in [4] it is proved that if G and M are finite groups with $s(M) \geq 2$, $Z(G) = 1$, $N(G) = N(M)$, then $|G| = |M|$, in particular $s(M) = s(G)$ and $OC(G) = OC(M)$. Therefore if the simple group M is characterizable by the set of its order components, then Thompson's Conjecture holds for M.

There is another conjecture due to W. Shi and J. Bi which states:

Conjecture 3. *Let G be a group and M a finite simple group. Then $G \cong M$ if and only if*

(a) $|G| = |M|$ *and*

(b) $\pi_e(G) = \pi_e(M)$, *where $\pi_e(G)$ denotes the set of order elements of G.*

Clearly conditions (a) and (b) above imply $OC(G) = OC(M)$. Therefore if the group G is characterizable by its order components, then we will deduce $G \cong M$ and Conjecture 3 is true for M. According to the main theorem of this paper which is stated below, Conjectures 2 and 3 are true for the simple groups $D_{p+1}(2)$ and $D_{p+1}(3)$, where p is an odd number.

In this paper we consider the simple groups $D_{p+1}(2)$ and $D_{p+1}(3)$, and prove that these groups are characterizable by their order components. Another name for these group is $O^{+}_{2(p+1)}(q)$ or $P\Omega^{+}_{2(p+1)}(q)$, where $q = 2, 3$. More precisely we will prove:

Main Theorem. *If a finite group G has the same set of order components as $D_{p+1}(2)$ or $D_{p+1}(3)$, then $G \cong D_{p+1}(2)$ or $D_{p+1}(3)$.*

2. Preliminary results

The structure of finite groups with disconnected Gruenberg-Kegel graph follows from Theorem A of [24] which will be stated below:

Lemma 1. *Let G be a finite group with $s(G) \geq 2$. Then one of the following holds*:

(1) *G is either a Frobenius or 2-Frobenius group.*

(2) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H is a nilpotent π_1-group, K/H is a non-abelian simple group, G/K is a π_1-group, $|G/K|$ divides $|\mathrm{Out}(K/H)|$ and any odd order component of G is equal to one of the odd order components of K/H.*

To deal with the first case in the above lemma we need the following results which are taken from [5] and [2], respectively.

Lemma 2. (a) *Let G be a Frobenius group of even order with kernel and complements K and H, respectively. Then $s(G) = 2$ and the prime graph components of G are $\pi(H)$ and $\pi(K)$.*

(b) *Let G be a 2-Frobenius group of even order. Then $s(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $|K/H| = m_2$, $|H|\,|G/K| = m_1$ and $|G/K|$ divides $|K/H| - 1$ and H is a nilpotent π_1-group.*

Lemma 3. *Let G be a finite group with $s(G) \geq 2$. If $H \trianglelefteq G$ is a π_i-group, then* $(\prod_{j=1,\ j\neq i}^{s(G)} m_j) \mid (|H|-1)$.

By the above lemma if H is a π_1-subgroup of G, $H \trianglelefteq G$, and $s(G)=2$, then $m_2 \mid |H|-1$ or $|H| \equiv 1 \pmod{m_2}$. The following result of Zsigmondy [25] is important in some number theoretical considerations.

Lemma 4. *Let n and a be integers greater than* 1. *Then there exists a prime divisor p of a^n-1 such that p does not divide a^i-1 for all $1 \leq i < n$, except in the following cases*:

(1) $n=2$, $a=2^k-1$, *where* $k \geq 2$,

(2) $n=6$, $a=2$.

The prime p in Lemma 4 is called a Zsigmondy prime for a^n-1.

Next we consider the simple groups $D_n(q)$. Using [7] we have $|D_n(q)| = \frac{1}{(4,q^n-1)}q^{n(n-1)}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$ and for $n>3$ all these groups are simple. The outer automorphism group of $D_n(q)$ has order $(2,q-1)^2 \cdot f \cdot \mathbb{S}_3$ for $n=4$, $(2,q-1)^2 \cdot f \cdot 2$ for $n>4$ even and $(4,q^n-1)^2 \cdot f \cdot 2$, n odd, where $q=r^f$, r is a prime number. By [21] if p is an odd prime number, then $s(D_{p+1}(2)) = s(D_{p+1}(3)) = 2$. Therefore in these cases the prime graphs of $D_{p+1}(q)$, $q=2,3$, have two components. The two order components by Table 1 are: $m_1 = q^{p(p+1)}(q^p+1)(q^{p+1}-1)\prod_{i=1}^{p-1}(q^{2i}-1)$ and $m_2 = \frac{q^p-1}{(4,q^p-1)} = \frac{q^p-1}{(2,q-1)}$, where $q=2$ or 3. The prime components of the graph $GK(D_{p+1}(q))$ are $\pi_1 = \pi(q(q^p+1)\prod_{i=1}^{p-1}(q^{2i}-1))$ and $\pi_2 = \pi(\frac{q^p-1}{(2,q-1)})$, where $q=2,3$.

3. Proof of Main Theorem

We assume G is a finite group with $OC(G)=\{m_1,m_2\}$, where m_1 and m_2 are the order components of the group $D_{p+1}(q)$, where p is an odd prime number and $q=2$ or 3. Therefore $m_2 = 2^p-1$ or $\frac{3^p-1}{2}$ in the respective cases of $q=2$ or 3. We will use the Lemma 1. Therefore we start with the following lemma.

Lemma 5. *If G is a finite group with $OC(G)=\{m_1,m_2\}$, then G is neither a Frobenius nor a 2-Frobenius group.*

Proof. First we assume G is a Frobenius group with complement H and kernel K and derive a contradiction. By Lemma 2 we have $OC(G)=\{|H|,|K|\}$. Since $|H| \mid |K|-1$ we must have $|H|<|K|$, hence $|K|=m_1=q^{p(p+1)}(q^p+1)(q^{p+1}-1)\prod_{i=1}^{p-1}(q^{2i}-1)$ and $|H|=m_2=\frac{q^p-1}{(2,q-1)}$, where $q=2$ or 3. Let r be a Zsigmondy prime for $q^{2(p-1)}-1$, which exists by Lemma 4 because $p \geq 3$. Then $r \mid q^{p-1}+1$ and from the order of $D_{p+1}(q)$, $q=2,3$, we observe that the order of a Sylow r-subgroup S of G, and hence K, is a divisor of $q^{p-1}+1$. Since K is a nilpotent normal subgroup of G we deduce $S \trianglelefteq G$ and using Lemma 3 the divisibility relation $m_2 \mid |S|-1$ must hold. Considering $m_2 = \frac{3^p-1}{2}$ or 2^p-1 and $|S| \mid q^{p-1}+1$ in respective cases of $q=3$ or 2, a contradiction is obtained.

Next assume that G is a 2-Frobenius group. By Lemma 2(b) there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ for G such that H is a nilpotent π_1-group, $|K/H| = m_2$ and $|G/K| \mid (|K/H| - 1) = 2^p - 2$ or $\frac{3^p-3}{2}$. Therefore $|G/K| \mid 2^{p-1} - 1$ or $3^{p-1} - 1$. Again if r is a Zsigmondy prime for $q^{2(p-1)} - 1$, then $r \nmid q^{p-1} - 1$, where $q = 2$ or 3, hence $r \nmid |G/K|$ and from the Lemma 2 we deduce $r \mid |H|$. Since a Sylow r-subgroup of H has order r^k and H is a nilpotent normal subgroup of G, using the Lemma 3 we deduce $m_2 \mid r^k - 1$. But $r^k \mid q^{p-1}+1$ and considering the cases $q = 2$ or 3 and $m_2 = 2^p - 1$ or $\frac{3^p-1}{2}$, respectively, a contradiction is obtained. Therefore G is not a 2-Frobenius group and the lemma is proved. □

The following lemma is useful in our further investigations. We remind that for a prime number r and a positive integer n, n_r denotes the n-part of n, i.e., $n = mn_r$, where $(m, r) = 1$.

Lemma 6. *Let r be a prime divisor of $a = \prod_{i=1}^{n}(3^{2i} - 1)$. Then $a_r < 2^{\frac{5}{2}n}$ if r is odd and $a_r < 2^{4n}$ if $r = 2$. If $r \geq 7$, then $a_r < 2^{\frac{7}{3}n}$.*

Proof. First we assume $r = 2$. It is easy to verify that $(3^{2i}-1)_2 = 8(i)_2$. Hence when i varies in the interval $1 \leq i \leq n$, we obtain $2^k \mid 8^n \cdot 2^{[\frac{n}{2}]+[\frac{n}{4}]+\cdots}$, where $a_2 = 2^k$, therefore $a_2 < 2^{3n+\frac{n}{2}+\frac{n}{4}+\cdots} < 2^{4n}$.

Next we will assume that r is odd. Let e be the least positive integer for which $r \mid 3^{2e} - 1$, and set $3^{2e} = 1 + kr^l$, $r \nmid k$. It is clear that if $r \mid 3^{2i} - 1$, $1 \leq i \leq n$, then $e \mid i$. Therefore if we set $s = [\frac{n}{e}]$, then for the r-part of a we have:

$$a_r = r^{ls+[\frac{s}{r}]+[\frac{s}{r^2}]+\cdots} < r^{ls+\frac{s}{r}+\frac{s}{r^2}+\cdots} < r^{ls+\frac{s}{r-1}}.$$

The last inequality can be written as:

$$(*) \qquad a_r < r^{ls} r^{\frac{s}{r-1}}.$$

Since $r^l \mid 3^{2e} - 1 = (3^e - 1)(3^e + 1)$ and r is odd we deduce that $r^l \mid 3^e - 1$ or $r^l \mid 3^e + 1$ implying that in any case $r^l \leq 3^e + 1$. Now using $(*)$ we can write:

$$a_r < r^{ls} r^{\frac{s}{r-1}} < (3^e+1)^s(3^e+1)^{\frac{s}{r-1}} = (3^e+1)^{\frac{rs}{r-1}} < (3^e+1)^{\frac{5}{4}s} < (4^e)^{\frac{5}{4}s} \leq 2^{\frac{5}{2}n},$$

because $r \neq 3$. This proves $a_r < 2^{\frac{5}{2}n}$. If $r \geq 7$, then $(3^e+1)^{\frac{rs}{r-1}} \leq (3^e+1)^{\frac{7}{6}s} < 4^{\frac{7}{6}se} \leq 2^{\frac{7}{3}n}$ proving $a_r < 2^{\frac{7}{3}n}$. □

Now we continue the proof of the main theorem. By Lemmas 1 and 5, if G is a finite group with $OC(G) = OC(D_{p+1}(q))$, $q = 2, 3$, then there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ for G such that K/H is a non-abelian simple group, H and G/K are π_1-groups and H is nilpotent. Moreover $|G/K|$ divides $|\mathrm{Out}(K/H)|$ and the odd order component of G which was denoted by m_2 is equal to one of the odd order components of K/H and $s(K/H) \geq 2$.

Since $P = K/H$ is a non-abelian simple group with disconnected Gruenberg-Kegel graph, by the classification of finite simple groups we have one of the possibilities in Tables 1, 2 or 3 for P. In the following we deal with these groups.

Note that we are characterizing two groups $D_{p+1}(2)$ and $D_{p+1}(3)$, where p is an odd prime number, and the odd order components of these groups are 2^p-1 and $\frac{3^p-1}{2}$, respectively. Therefore in each of the following case by case investigation we have subcases dealing with two different groups.

Case 1. $P \cong A_2(4)$, ${}^2A_3(2)$, ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, ${}^2E_6(2)$, ${}^2F_4^{'}(2)$ or one of the 26 sporadic simple groups listed in Tables 1, 2 or 3.

The odd order component of $D_{p+1}(2)$ or $D_{p+1}(3)$ are $m_2 = 2^p - 1$ or $\frac{3^p-1}{2}$, respectively. Using Tables 1-3, we obtain the following possibilities.

(a) $OC(G) = OC(D_{p+1}(2))$.

$(J_2,\ p = 3)$, $(E_7(2),\ p = 7)$, $(HiS,\ p = 3)$, $(F_2,\ p = 5)$, $(A_2(4), p = 3)$, $(M_{22}, p = 3)$, $(J_1, p = 3)$, $(O'N, p = 5)$, $(Ly, p = 5)$, $(J_4, p = 5)$.

If $p = 3$, then $OC(G) = OC(D_4(2))$, from which we obtain $|G| = 2^{12}{\cdot}3^5{\cdot}5^2{\cdot}7$. Using the condition $|P| \mid |G|$, only J_2 and $A_2(4)$ need further investigation.

By [7] we have $|\mathrm{Out}(A_2(4))| = 12$, $|\mathrm{Out}(J_2)| = 2$, hence from $|G/H| \mid |\mathrm{Aut}(P)|$ we deduce $5^2 \mid |H|$ in the case of $K/H \cong A_2(4)$. Since H is nilpotent it is the direst product of its Sylow subgroups. Hence if Q is a Sylow 5-subgroup of H, then Q is characteristic in K, hence $Q \trianglelefteq G$. Now an element of order 7 in G acts on Q by conjugation, where $|Q| = 5^2$, and we must obtain a fixed element which results an element of order 35 in G. But then the prime graph of G would be connected, a contradiction.

Next we assume $K/H \cong J_2$. In this case similar consideration as above shows that $3^2 \mid |H|$ and a Sylow 3-subgroup R of H has order 3^k, where $2 \leq k \leq 5$. But now an element of order 7 of G acting by conjugation on R must fix a non-trivial element of order 3 resulting an element of order 21 which makes the prime graph of G connected. This final contradiction rules out the case $p = 3$.

In the case $(E_7(2),\ p = 7)$ it is easy to see that $|E_7(2)| \nmid |D_8(2)|$. For $p = 5$, it can be verified the orders of the groups F_2, $O'N$, Ly and J_4 don't divide the order of $D_6(2)$. This final contradiction rules out case (a).

(b) $OC(G) = OC(D_{p+1}(3))$.

$({}^2F_4^{'}(2),\ p = 3)$, $(Fi_{22}(2),\ p = 3)$, $(Suz,\ p = 3)$, $({}^2E_6(2),\ p = 3)$.

We have $|G| = |D_4(3)| = 2^{13} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$, and only the order of ${}^2F_4^{'}(2)$ divides the order of $D_4(3)$. Now from $|G/H| \mid |\mathrm{Aut}(P)|$ we obtain $7 \mid |H|$, hence a Sylow 7-subgroup of H has order 7. Therefore an element of order 13 must commute with an element of order 7 contradicting disconnectedness of the prime graph of G. Therefore Case 1 can not happen.

Case 2. $P \cong \mathbb{A}_n$ and either $n = p',\ p' + 1,\ p' + 2$, one of n or $n - 2$ is not prime; or $n = p',\ p' - 2$ are both prime, where $p' > 6$ is a prime number.

By Tables 1 and 2, the odd order components of $\mathbb{A}_n$ are p' and(or) $p' - 2$.

(a) $OC(G) = OC(D_{p+1}(2))$.

If $m_2 = 2^p - 1 = p' - 2$, then $p' = 2^p + 1$ is a multiple of 3 and $p' > 3$, contradicting primality of p'. Hence we assume $m_2 = 2^p - 1 = p'$. If $p = 3$, then $p' = 7$ and $|G| = 2^{12}{\cdot}3^5{\cdot}5^2{\cdot}7$. Since $|\mathrm{Out}(\mathbb{A}_7)| = 2$, from $|G/H| \mid |\mathrm{Aut}(\mathbb{A}_7)|$

we deduce that $5 \mid |H|$, hence a Sylow 5-subgroup of H has order 5 or 5^2. But then an element of order 7 in G must fix an element of order 5 because a Sylow 5-subgroup of H is normal in G, resulting the connectivity of the prime graph of $D_4(2)$, a contradiction.

Therefore we assume $p \geq 5$, hence $p' \geq 31$. By Lemma 1 in [22] there are at least three distinct primes p_i such that $2^{p-1} < p_i < 2^p - 1$. Of course we have $p_i \mid |\mathbb{A}_{p'}|$. But considering the order of the group $|D_{p+1}(2)|$ we observe that there are at most two prime divisors of $|D_{p+1}(2)|$ between 2^{p-1} and $2^p - 1$, a contradiction.

(b) $OC(G) = OC(D_{p+1}(3))$.

If $m_2 = \frac{3^p-1}{2} = p' - 2$, then $p' = \frac{3^p+3}{2}$ which is not a prime number. Hence we assume $m_2 = \frac{3^p-1}{2} = p'$. If $p = 3$, then $p' = 13$. We have $|G| = |D_4(3)| = 2^{13}.3^{12}.5^2.7.13$ and obviously $|\mathbb{A}_{13}|$ does not divide $|D_4(3)|$. Hence we may assume $p \geq 5$. The 3-part of $|G|$ is $3^{p(p+1)}$ and we must have $|\mathbb{A}_{p'}|_3 \mid 3^{p(p+1)}$. But the largest power of 3 dividing $p'!$ is $[\frac{p'}{3}] + [\frac{p'}{9}] + \cdots = \frac{3^p-2p-1}{4}$. It can be verified that if $p \geq 5$, then $\frac{3^p-2p-1}{4} > p^2 + p$ and hence $|\mathbb{A}_{p'}|_3$ does not divide $|G|_3$, a contradiction.

Case 3. $P \cong A_{p'-1}(q)$, $(p', q) \neq (3, 2), (3, 4)$.

From now on we use Lemma 9 in [23]. According to this lemma if r is an odd prime divisor of $a = \prod_{i=1}^{n}(2^{2i} - 1)$, then the r-part of a, i.e., a_r, satisfies $a_r < 2^{3n}$ and moreover if $r \geq 5$, then $a_r < 2^{2n}$.

(a) $OC(G) = OC(D_{p+1}(2))$.

In this case we have:

$$(*) \qquad \frac{q^{p'} - 1}{(q-1)(p', q-1)} = 2^p - 1.$$

From which we deduce $q^{p'} \geq 2^p$. If $p = 3$, then it is easy to verify that the only solution of $\frac{q^{p'}-1}{(q-1)(p',q-1)} = 7$ are $(p', q) = (3, 2)$ and $(3, 4)$ which are not the case by assumption. Therefore we assume $p \geq 5$. Suppose $p' \geq 7$ and let $q = r^f$, r prime. From $|P| \mid |G| = |D_{p+1}(2)|$ we deduce that $r \mid |D_{p+1}(2)|$. The r-part of $|P|$ is $r^{fp'(p'-1)/2}$ and we have $q^{p'(p'-1)/2} > q^{2(p'+1)} > 2^{2(p+1)}$, since $p \geq 5$ by Lemma 9 in [23] r must be even, i.e., $r = 2$.

Now since $(p', q-1) \leq q - 1 < q$, from $(*)$ we can deduce:

$$2^p > \frac{q^{p'} - 1}{(q-1)q} = (q^{p'-1} + \cdots + q + 1)/q = q^{p'-2} + \cdots + 1 + q^{-1} > q^{p'-2}$$

which implies $2^p > q^5$, because $p' \geq 7$. Since q is a power of 2, the inequality $2^p > q^5$ implies $q^5 \mid 2^p$.

Now again we rearrange $(*)$ as $q^{p'} = 2^p(q-1)(p', q-1) - q(p', q-1) + (p', q-1) + 1$ from which it follows that $q^5 \mid -q(p', q-1) + (p', q-1) + 1$ which in turn implies that $q = 2$, and then $(*)$ implies $p' = p$. Therefore $|G| = |D_{p+1}(2)|$ and

$|P| = |A_{p-1}(2)|$, hence from $|G/H| \mid |\mathrm{Aut}(P)| = 2\,|A_{p-1}(2)|$ we obtain that a Zsigmondy prime for $2^{p+1}-1$ must divide $|H|$. Since H is a nilpotent normal subgroup of G its Sylow subgroups are normal in G. If $p = 5$, then $7 \mid |H|$ and a Sylow 7-subgroup of H has order at most 7^2. Now an element of order $2^5-1 = 31$ must fix an element of order 7, contradicting disconnectedness of the prime graph of $D_6(2)$. Hence we will assume $p > 5$. Let s be a Zsigmondy prime for $2^{p+1}-1$ which exists because $p \neq 5$. Therefore for a Sylow s-subgroup S of G, and hence of H, by Lemma 3, we must have $|S| \equiv 1 \pmod{m_2}$, implying $|S| = 1 + k(2^p - 1)$ for some k. Since $|S| \leq 2^{p+1} - 1$ we deduce $k = 1$ or 2. If $k = 1$, then $s = 2^p$ is not a prime, and if $k = 2$, then $s = 2^{p+1} - 1$, again not a prime number. This final contradiction rules out the possibility of $P \cong A_{p'}(q)$ for $p' \geq 7$. The special cases $p' = 2, 3, 5$ may be treated similarly which end to contradiction as well.

(b) $OC(G) = OC(D_{p+1}(3))$.

In this case we have

$$(**) \qquad \frac{q^{p'}-1}{(q-1)(p',q-1)} = \frac{3^p-1}{2}.$$

If $q = 2$, then from $(**)$ we obtain $2^{p'+1} - 3^p = 1$ which is impossible because $3^p + 1$ is not a multiple of 8. Hence we assume $q \geq 3$. Now $(**)$ can be written as $q^{p'} = 1 + \frac{3^p-1}{2}(q-1)(p',q-1)$ from which we obtain $q^{p'} \geq 3^p$. Now if $p' \geq 11$, then using the above inequality we can write:

$$q^{p'(p'-1)/2} > q^{4(p'+1)} \geq 3^{4(p+1)}.$$

Therefore the r-part of $|P|$ is more than $3^{4(p+1)}$, where $q = r^f$. By Lemma 6 we obtain $r = 3$.

Now using the same method as used in (a) we can show that 3^p is divisible by q^9. If we rewrite $(**)$ as $2q^{p'} = 3^p(q-1)(p',q-1) - q(p',q-1) + (p',q-1) + 2$, then $q^9 \mid -q(p',q-1) + (p',q-1) + 2$, from which we obtain $q = 3$ and the by $(**)$ we get $p' = p$. We have $|G/H| \mid |\mathrm{Aut}(A_{p-1}(3))|$ from which we obtain that a Zsigmondy prime s for $3^{p+1}-1$ must divide $|H|$. Since H is a nilpotent normal subgroup of G the Sylow s-subgroup S of H is normal in G, and by Lemma 3 we obtain $|S| \equiv 1 \pmod{m_2}$, implying $|S| = 1 + k(\frac{3^p-1}{2})$. Now from $s \mid 3^{p+1} - 1$ we deduce that $s \mid 3^{\frac{p+1}{2}} + 1$, but $1 + k(\frac{3^p-1}{2}) > 1 + 3^{\frac{p+1}{2}}$. This contradiction proves the impossibility of the case when $p' \geq 11$.

If $p' = 7$, then from $q^7 \geq 3^p$ we obtain $q^{21} \geq 3^{3p} > 2^{4(p+1)}$, hence by Lemma 6, q must be a power of 2. Now the same method as above can be applied to obtain a contradiction.

Assume $p' = 5$. If $(5, q-1) = 5$, then $q^5 = 1 + 5(q-1)(\frac{3^p-1}{2}) > 1 + 5(3^p - 1) = 5 \cdot 3^p - 4 > 2^{\frac{5(p+1)}{4}}$ hence $q^{10} > 2^{\frac{5}{2}(p+1)}$ and by Lemma 6 either $r = 2$ or $r = 3$. Hence the previous method can be applied. If $(5, q-1) = 1$, then $q^5 = 1 + (q-1)\frac{3^p-1}{2}$, and if $q \neq 3$ we will obtain $q^5 > \frac{3}{2}(3^p - 1) + 1 > 2^{\frac{5}{4}(p+1)}$, hence $q^{10} > 2^{\frac{5}{2}(p+1)}$. Therefore by Lemma 6, $r = 2$ or 3 where $q = r^f$. Hence the previous method can be applied.

Finally assume $p' = 3$. In this case the equation $\frac{q^3-1}{(q-1)(p',q-1)} = \frac{3^p-1}{2}$ has no solution for the prime power q.

Case 4. $P \cong A_{p'}(q),\ q-1 \mid p'+1$.

In this case we have $m_2 = \frac{q^{p'}-1}{q-1}$ and either $\frac{q^{p'}-1}{q-1} = 2^p - 1$ or $\frac{q^{p'}-1}{q-1} = \frac{3^p-1}{2}$. The equations in this case are the same as in the Case 3 and it is ruled out similarly.

Case 5. $P \cong {}^2A_{p'-1}(q)$ or ${}^2A_{p'}(q)$, $q+1 \mid p'+1$, $(p', q) \neq (3,3), (5,2)$.

We have $m_2 = \frac{q^{p'}+1}{(q+1)(p',q+1)}$ or $\frac{q^{p'}+1}{q+1}$, respectively. Because of similarity we give details related to $P \cong {}^2A_{p'}(q)$, where $q+1 \mid p'+1$, $(p', q) \neq (3,3), (5,2)$.

(a) $OC(G) = OC(D_{p+1}(2))$.

In this case we have $\frac{q^{p'}+1}{q+1} = 2^p - 1$. If $p = 3$, then from this equation we deduce $p' = 3$ and $q = 3$ which is not the case. Therefore we assume $p \geq 5$. Since $q \geq 2$, from the last equality we obtain $q^{p'} > 2^{p+1}$. Therefore from $p' \geq 3$ we obtain the following inequality:

$$q^{p'(p'+1)/2} \geq q^{2p'} > 2^{2(p+1)}.$$

Since $p \geq 5$, by Lemma 9 in [23] r must be even, i.e., $r = 2$, where $q = r^f$ is a power of the prime number r. From the equation $\frac{q^{p'}+1}{q+1} = 2^p - 1$ we deduce $2^p = \frac{q^{p'}+q+2}{q+1} \geq q$, and since q is a power of 2 we obtain $q \mid 2^p$. Now we can rewrite $\frac{q^{p'}+1}{q+1} = 2^p - 1$ as $q^{p'} = (q+1)2^p - q - 2$, from which we deduce $q \mid q+2$, implying $q = 2$. Therefore we obtain the equation $2^{p'} = 3 \cdot 2^p - 4$ which is a contradiction because p and p' are at lest 3. Hence this case is ruled out.

(b) $OC(G) = OC(D_{p+1}(3))$.

In this case we have

$$(*) \qquad \frac{q^{p'}+1}{q+1} = \frac{3^p-1}{2}.$$

If $q = 2$, then $(*)$ can be written as $3^{p+1} - 2^{p'+1} = 5$, and because $p+1$ and $p'+1$ are even we can write $3^{2(\frac{p+1}{2})} - 2^{2(\frac{p'+1}{2})} = 5$. Reducing both sides modulo 5 we obtain $(-1)^{\frac{p+1}{2}} - (-1)^{\frac{p'+1}{2}} = 0$. Therefore $p+1 = 4k$ and $p'+1 = 4k'$ and the original equation becomes $3^{4k} - 2^{4k'} = 5$. Now reducing both sides modulo 16 we obtain $1 = 5$, a contradiction. The same arguments prove the equality $(*)$ is impossible for $q = 3$, 4 and 5. Therefore we will assume $q \geq 7$. Now $(*)$ can be rewritten as: $q^{p'} = -1 + (q+1)(\frac{3^p-1}{2}) \geq -1 + 4(3^p - 1) = 4 \cdot 3^p - 5 > 3^{p+1}$ hence $q^{p'} > 3^{p+1}$. If $p' \geq 7$ we have the following inequality: $q^{p'(p'+1)/2} \geq q^{4p'} > 3^{4(p+1)}$. Therefore the r-part of $|P|$ is more than $3^{4(p+1)}$, where $q = r^f$. By Lemma 6 we obtain $r = 3$.

Now from $(*)$ we obtain $3^p = \frac{2q^{p'}+q+3}{q+1} > q$ and since q is a power of 3 we obtain $q \mid 3^p$. Next we rewrite $(*)$ as $2q^{p'} = (q+1)3^p - q - 3$, from which we obtain $q \mid q+3$, hence $q = 3$. Therefore we end with the equation $2 \cdot 3^{p'} = 4 \cdot 3^p - 6$ which is impossible because p and p' are both at least 3. This final contradiction rules out the possibility $P = {}^2A_{p'}(q)$ for $p' \geq 7$.

If $p' = 5$ or 3, then from $q+1 \mid p'+3$ we obtain $q = 2, 5$ for $p' = 5$ and $q = 3$ for $p' = 3$, which are not the case because we are assuming $q \geq 7$. Therefore the Case 5 is completely ruled out.

Case 6. $P \cong B_n(q),\ n = 2^m \geq 4$, q odd; $C_n(q),\ n = 2^m \geq 2$; ${}^2D_n(q),\ n = 2^m \geq 4$.
In this case because of the similarity of arguments we give the details of the case $P \cong B_n(q)$. In this case we have $\frac{q^n+1}{2} = 2^p - 1$ or $\frac{3^p-1}{2}$ and we deal with them separately. As usual we set $q = r^f$, where r is a prime number.

(a) $OC(G) = OC(D_{p+1}(2))$.
We have $\frac{q^n+1}{2} = 2^p - 1$ which implies $q^n = 2^{p+1} - 3 > 2^p$. If $p = 3$, then $q = 13$ and $n = 1$ which is not the case. Hence we assume $p \geq 5$. A Sylow r-subgroup of $B_n(q)$ has order q^{n^2} and we can write:

$$q^{n^2} > q^{3n} > 2^{3p} \geq 2^{2(p+1)}$$

which by Lemma 9 in [23] we deduce $r = 2$. Now it is easy to see that for $q = 2^f$ the equation $q^n = 2^{p+1} - 3$ is impossible.

(b) $OC(G) = OC(D_{p+1}(3))$.
In this case we have $\frac{q^n+1}{2} = \frac{3^p-1}{2}$ which implies $q^n = 3^p - 2 > 3^{p-1}$. It is clear that $q^n = 3^p - 2$ does not hold for $p = 3$. Hence we assume $p \geq 5$. Similar to case (a) we can write $q^{n^2} \geq q^{4n} > 3^{4(p-1)} > 3^{\frac{5}{2}(p+1)}$, which by Lemma 6 implies $r = 3$ or 2. Now it is easy to see that the equation $q^n = 3^p - 2$ does not hold for $q = 2^f$ and $q = 3^f$.

Case 7. $P \cong B_{p'}(3)$.
In this case we have either $\frac{3^{p'}-1}{2} = 2^p - 1$ or $\frac{3^{p'}-1}{2} = \frac{3^p-1}{2}$. In the first case we obtain $3^{p'} - 2^{p+1} = -1$ which is impossible [8]. In the second case we obtain $p' = p$. The outer automorphism group of P has order 2 and from $|G/K| \mid |\mathrm{Out}(P)| = 2$ we obtain $G = K$ or $|G/K| = 2$. If $G = K$, then $H = K$ which is a contradiction. If $|G/K| = 2$, then $|G/K| \cdot |H| = \frac{|G|}{|P|}$ implying $2\,|H| = 3^p(3^{p+1} - 1)$, hence $|H| = \frac{3^p(3^{p+1}-1)}{2}$. Let s be a Zsigmondy prime for $3^{p+1} - 1$ (the case $p = 5$ is ruled out similarly), then a Sylow s-subgroup S of H is normal in G and by Lemma 3 we have $\frac{3^p-1}{2} \mid |S| - 1$. Therefore we may write $|S| - 1 = k(\frac{3^p-1}{2})$ for some $k \in \mathbb{N}$. Now from $|S| \leq 3^{p+1} - 1$ we obtain $k \leq 6$ and examination of each k leads to a contradiction. Therefore this case is ruled out.

Case 8. $P \cong C_{p'}(q)$, $q = 2, 3$; $P \cong D_{p'}(q)$, $p' \geq 5$, $q = 2, 3, 5$.

In this case because of the similarity in arguments we give the details in the case $P \cong D_{p'}(5)$, $p' \geq 5$. In this case we have $\frac{5^{p'}-1}{4} = 2^p - 1$ or $\frac{3^p-1}{2}$. If $\frac{5^{p'}-1}{4} = 2^p - 1$, then $5^{p'} = 2^{p+2} - 3 > 2^{p+1}$ and since a Sylow 5-subgroup of P has order $5^{p'(p'-1)}$, $p' \geq 5$, we can write $5^{p'(p'-1)} \geq 5^{3p'} > 2^{3(p+1)}$ which by Lemma 9 in [23] is impossible.

Therefore we assume $\frac{5^{p'}-1}{4} = \frac{3^p-1}{2}$ which implies $5^{p'}+1 = 2\cdot 3^p$. The prime p' has one of the following forms $p' = 6k+1$ or $p' = 6k+5$. Since $5^6 \equiv 1 \pmod 9$ we obtain $(5^{6k+1}+1) \equiv 6 \pmod 9$ and $(5^{6k+5}+1) \equiv 3 \pmod 9$, hence $5^{p'}+1$ is not divisible by 9 in both cases, a contradiction.

Case 9. $P \cong D_{p'+1}(q),\ q = 2, 3$.

In this case the odd order component of $D_{p'+1}(2)$ is $2^{p'} - 1$ and $\frac{3^{p'}-1}{2}$ in the respective cases $q = 2$ and $q = 3$. Therefore $2^{p'} - 1 = 2^p - 1$ or $\frac{3^p-1}{2}$, and in the second case using [8] we get a contradiction, but in the first case we obtain $p = p'$. But then $P \cong D_{p+1}(2)$ and $|P| = |H/K| = |G|$ implies $G \cong P \cong D_{p+1}(q)$ which is our desire case.

If $\frac{3^{p'}-1}{2} = 2^p - 1$ or $\frac{3^p-1}{2}$, then only the first case is possible from which we obtain $p = p'$ and similar to above $G \cong P \cong D_{p+1}(3)$ which is desired.

Case 10. $P \cong {}^2D_n(2),\ n = 2^m+1 \geq 5$; $P \cong {}^2D_{p'}(3),\ 5 \leq p' \neq 2^m+1$ or $P \cong {}^2D_n(3),\ 9 \leq n = 2^m+1 \neq p'$.

If m_2 is the odd order component of any of the above groups, then considering the equations $m_2 = 2^p - 1$ or $m_2 = \frac{3^p-1}{2}$ we obtain a contradiction by number theoretic methods.

Case 11. $P \cong G_2(q),\ 2 < q \equiv \epsilon \pmod 3$, $\epsilon = \pm 1$.

(a) $OC(G) = OC(D_{p+1}(2))$.

In this case we have $q^2 - \epsilon q + 1 = 2^p - 1$ which implies $q^2 - \epsilon q = 2^p - 2$. Obviously q can not be any power of 2 and $p = 3$ is impossible. Therefore $p \geq 5$. If $\epsilon = 1$ then $q^2 = 2^p + q - 2 > 2^{p-1}$ which implies $q^6 > 2^{3(p-1)} \geq 2^{2(p+1)}$, hence by Lemma 9 in [23] a contradiction is obtained.

If $\epsilon = -1$, then $q^2 + q = 2^p - 2$, and can be verified that for $p = 3$ and 5 we obtain contradictions. Therefore we assume $p \geq 7$. We can write $q^2 + q = 2^p - 2$ as $(2q+1)^2 = 2^{p+2} - 9 > 2^{p+1}$ which implies $2q+1 > 2^{\frac{p+1}{2}}$, hence $2q > 2^{\frac{p+1}{2}} - 1$ or $q > 2^{\frac{p-1}{2}} - \frac{1}{2} > 2^{\frac{p-2}{2}}$. Therefore $q^6 > 2^{3(p-2)} \geq 2^{2(p+1)}$ and again by Lemma 9 in [23] we obtain a contradiction.

(b) $OC(G) = OC(D_{p+1}(3))$.

In this case we have $q^2 - \epsilon q + 1 = \frac{3^p-1}{2}$. We will give the details only when $\epsilon = 1$. We have $q^2 - q = \frac{3^p-3}{2}$. If $p = 3$, then $q = 4$. Therefore $|G/K|\cdot|H| = \frac{|G|}{|P|} = \frac{|D_4(3)|}{|G_2(4)|} = 2\cdot 3^9$. But $|G/K| \mid |\mathrm{Out}(G_2(4))| = 2$, hence if $|G/K| = 1$ we get $G = K$ which implies $|H| = 2\cdot 3^9$. Hence by Lemma 3, $m_2 = 13 \mid 2\cdot 3^9 - 1$, a contradiction. Therefore $|G/K| = 2$ implying $|H| = 3^9$. Now by the Brauer character table of $G_2(4)$ in [16] an element of order 13 in $G_2(4)$ fixes a non-trivial element in every $G_2(4)$-module over $GF(3)$, hence G has an element of order 13×3 which violates the disconnectedness of the prime graph of G.

Hence we may assume $p \geq 5$. Now from $q^2 = q + \frac{3^p-3}{2} > 3^{p-1}$ we deduce $q^6 > 3^{3(p-1)} > 2^{\frac{5}{2}(p+1)}$ which holds because of $p \geq 5$. Now by Lemma 6 we obtain $q = 2^f$. Because of $q \equiv 1 \pmod 3$ and the fact that $9 \nmid q^2 - q$ we obtain $q \equiv 4$ or $7 \pmod 9$, hence $|\mathrm{Out}(G_2(q))| = f$ is prime to 3. Let $t = |G/K|$. Then $t\,|H| = \frac{|D_{p+1}(3)|}{|G_2(q)|}$ whose 3-part is 3^{p^2+p-3}. Since $(t, 3) = 1$, a Sylow 3-subgroup of H must have order 3^{p^2+p-3}, and by Lemma 3, $m_2 \mid 3^{p^2+p-3} - 1$ from which we can deduce $p = 3$, and this was already dismissed.

Case 12. $P \cong {}^3D_4(q)$ or $P \cong F_4(q)$, q odd.
In both cases we have $m_2 = q^4 - q^2 + 1$. Using similar methods as in Case 11 it is possible to rule out the above possibilities.

Case 13. $P \cong E_6(q)$ or $P \cong {}^2E_6(q)$, $q > 2$.
The odd order component is either $\frac{q^6+q^3+1}{(3,q-1)}$ or $\frac{q^6-q^3+1}{(3,q+1)}$ for the above respective cases. Using the following inequalities together with Lemma 9 in [23] and Lemma 6 the second case is ruled out.

$$q^6 = q^3 - 1 + (3, q+1)(2^p - 1) > 2^p,$$
$$q^6 = q^3 - 1 + (3, q+1)\Big(\frac{3^p - 1}{2}\Big) > 3^{p-1}.$$

For the first case we use the following inequalities:

$$q^9 + 1 = (q^3 + 1)(q^6 - q^3 + 1) > \frac{q^6 - q^3 + 1}{(3, q+1)} + 2 > 2^p + 1,$$
$$q^9 + 1 = (q^3 + 1)(q^6 - q^3 + 1) > \frac{2(q^6 - q^3 + 1)}{(3, q+1)} + 2 > 3^p + 1$$

to obtain $q^9 > 2^p$ and $q^9 > 3^p$ and then $q^{36} > 2^{4p} > 2^{3(p+1)}$ or $q^{36} > 3^{4p} > 2^{4(p+1)}$ which enables us to use Lemma 6 of [23] and Lemma 3 to deduce $r = 2$ or 3 in the respective cases. Then using the method of Case 11 one is able to rule out this case.

Case 14. $P \cong {}^2D_{p'+1}(2)$, $5 \leq p' \neq 2^m - 1$.
In this case $m_2 = 2^{p'} - 1 = 2^p - 1$ or $\frac{3^p-1}{2}$. In the first case $p' = p$, and then $2^{p+1} + 1 \mid \left|{}^2D_{p+1}(2)\right|$ and a Zsigmondy argument rules out this possibility. If $2^{p'+1} = \frac{3^p-1}{2}$, then $2^{p'+1} - 3^p = 1$ which is impossible.

Next we consider simple groups with three prime graph components listed in Table 2. We remark that the case of the alternating group is already dealt with.

Case 15. $P \cong A_1(q)$, $3 < q \equiv \epsilon \pmod 4$, $\epsilon = \pm 1$.
The odd order components are q and $\frac{q+\epsilon}{2}$.

(a) $OC(G) = OC(D_{p+1}(2))$.

Let $q = r^f$, where r is a prime number. $2^p - 1 = q = r^f$ is impossible by [8]. If $2^p - 1 = \frac{q-1}{2}$, then $q = 2^{p+1} - 1$ and cannot be a prime number. Hence we assume $2^p - 1 = \frac{q+1}{2}$ which implies $q = 2^{p+1} - 3$, hence $r^f = 2^{p+1} - 3$ from which it is easy to see that r must be odd. But $q + 1 = 2^{p+1} - 2$ and $q - 1 = 2^{p+1} - 4$, therefore $|A_1(q)|_2 = 4$ and from $|G/K| \cdot |H| = \frac{|G|}{|P|}$ we deduce that the 2-part of $\frac{|G|}{|P|}$ is 2^{p^2+p-2}. Since $\frac{|G|}{|K|} \mid 2f$ and f is odd, the 2-part of $|G/K|$ is at most 2. Therefore a Sylow 2-subgroup of H has order either 2^{p^2+p-2} or 2^{p^2+p-3}. Now from Lemma 3 we must have $2^p - 1 \mid 2^{p^2+p-2} - 1$ or $2^p - 1 \mid 2^{p^2+p-3} - 1$, from which only the second divisibility with $p = 3$ is possible. But if $p = 3$, then $q = 13$ and this implies that $13 \nmid |D_4(2)|$, which is a contradiction.

(b) $OC(G) = OC(D_{p+1}(3))$.

In this case we have to deal with each of the following possibilities: $q = \frac{3^p-1}{2}$, $\frac{q+\epsilon}{2} = \frac{3^p-1}{2}$, where $\epsilon = \pm 1$. If $t = |G/K|$, then we know $t\,|H| = \frac{|G|}{|P|}$ and $t \mid |\mathrm{Out}(P)|$. In the case of $A_1(q)$, q odd, we have $|\mathrm{Out}(P)| = 2f$, where $q = r^f$ is a power of the prime r. Using the above information we will deal with appropriate cases as follows.

(i) $q = \frac{3^p-1}{2}$.
In this case it is easy to see that $q \equiv 4 \pmod 9$ from which it follows that $3 \nmid f$. The 3-part of $\frac{|G|}{|P|}$ is 3^{p^2+p-1} and since $(f, 3) = 1$ we deduce that a Sylow 3-subgroup of H has order 3^{p^2+p-1}. By Lemma 3 we must have $\frac{3^p-1}{2} \mid 3^{p^2+p-1} - 1$ from which a contradiction can be derived.

(ii) $\frac{q-1}{2} = \frac{3^p-1}{2}$.
In this case we have $q = 3^p$. If $p = 3$, then $q = 27$ and from $t\,|H| = \frac{|D_4(3)|}{|A_1(27)|}$ we find that $5^2 \mid |H|$. But then an element of order 13 has to fix an element of order 5 upon its action by conjugation on a Sylow 5-subgroup of H. This will violate the disconnectedness of the prime graph of $D_4(3)$. Therefore we assume $p > 3$, hence $|\mathrm{Out}(P)| = 2p$ is prime to 3. Now from $t\,|H| = \frac{|G|}{|P|}$ we deduce that $3^{2(p-1)} - 1 \mid |H|$. Now using a Zsigmondy prime argument for $3^{2(p-1)} - 1$ and the fact that $p > 3$, one can obtain a contradiction.

(iii) $\frac{q+1}{2} = \frac{3^p-1}{2}$.
In this case we have $q = 3^p - 2$. It is easy to see that $q \equiv 7 \pmod 9$ and in $q = r^f$, f is not a multiple of 3. The 3-part of $\frac{|G|}{|P|}$ is 3^{p^2+p-1} and from $t\,|H| = \frac{|G|}{|P|}$ and the fact that $(3, f) = 1$, we deduce that a Sylow 3-subgroup of H has order 3^{p^2+p-1} from which a contradiction can be derived.

Case 16. $P \cong A_1(q)$, $q > 2$ even.
In this case the odd order components of $A_1(q)$ are $q - 1$ and $q + 1$ and all the possibilities are ruled out easily and only $q - 1 = 2^p - 1$ needs investigation. Therefore $q = 2^p$ and if $p = 3$, then it is easy to see that $5^2 \mid |H|$ and this possibility is ruled out. Hence we assume $p > 3$. Now from $t\,|H| = \frac{|G|}{|P|}$ and the

fact that $t \mid 2p$ we find $2^{2(p-1)} - 1 \mid |H|$ and a Zsigmondy prime argument for $2^{2(p-1)} - 1$ rules out this case.

Case 17. $P \cong {}^2D_{p'}(3)$, $p' = 2^m + 1 \geq 5$.
The odd order components of ${}^2D_{p'}(3)$ are $\frac{3^{p'-1}+1}{2}$ and $\frac{3^{p'}+1}{4}$ and all the possibilities are ruled out easily and only $\frac{3^{p'}+1}{4} = 2^p - 1$ needs further investigation. We have $3^{p'} = 2^{p+2} - 5$. If $p' = 3$, then $p = 3$ and it is easy to verify that $|P| \nmid |G|$. Hence we assume $p' > 3$. From $3^{p'} = 2^{p+2} - 5$ we obtain $3^{p'} > 2^{p+1}$, hence $3^{p'(p'-1)} > 3^{3p'} > 2^{3(p+1)}$ which is a contradiction by Lemma 9 in [23].

Case 18. $P \cong {}^2D_{p'+1}(2)$, $p' = 2^n - 1$, $n \geq 2$.
The odd order components of ${}^2D_{p'+1}(2)$ are $2^{p'} + 1$ and $2^{p'+1} + 1$ and using number theoretic methods the possibilities are ruled out.

Case 19. $P \cong G_2(q)$, $q \equiv 0 \pmod 3$.
The odd order components of $G_2(q)$ are $q^2 - q + 1$ and $q^2 + q + 1$. If $q^2 \pm q + 1 = \frac{3^p+1}{2}$, then $q^2 \pm q = \frac{3^p-3}{2}$ and from the fact that q is a power of 3 we obtain $q = 3$. Hence in the case of $q^2 + q + 1 = \frac{3^p-1}{2}$ we obtain $p = 3$. Now $|G_2(3)| = 2^9 \cdot 3^6 \cdot 7 \cdot 13$ and $|D_4(3)| = 2^{13} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$ and similar reasoning as before shows $5^2 \mid |H|$ which results a contradiction. Therefore we should consider the case $q^2 \pm q + 1 = 2^p - 1$.

If $q^2 \pm q + 1 = 2^p - 1$, then $q^2 \pm q = 2^p - 2$. Let $q = 3^f$. It can be checked that $f = 1$ leads to a contradiction, hence $f \geq 2$ and $9 \mid q^2 \pm q$. But $9 \mid 2^p - 2$ if and only if p is of the form $p = 6k + 1$, where $k \in \mathbb{N}$. First we consider $q^2 + q + 1 = 2^p - 1$ which implies $q^2 + q = 2^p - 2$. Taking $q^2 + q$ modulo 4 we obtain $q(q+1) \equiv (-1)^f((-1)^f + 1)$ which must be equal to $2 \pmod 4$. Hence f must be even. Let $f = 2h$ and find $q(q+1)$ modulo 8. We have $q(q+1) = 3^{2h}(3^{2h} + 1) \equiv 2 \pmod 8$ which contradicts $2^p - 2 \equiv 6 \pmod 8$. Therefore the equality $q^2 + q + 1 = 2^p - 1$ is impossible.

If $q^2 - q + 1 = 2^p - 1$, then $q^2 - q = 2^p - 2$ and considering $q(q-1) \equiv (-1)^f((-1)^f - 1) \pmod 4 \equiv 2 \pmod 4$, we deduce that f must be odd, i.e., $f = 2h+1$. Since $2^p - 1 = 2^{6k+1} - 1 \equiv 1 \pmod 5$, and $q(q-1) = 3^{2h+1}(3^{2h+1} - 1) \equiv 3(-1)^h(3(-1)^h - 1) \pmod 5$, we must have h even. Hence $f = 4l + 1$ for some $l \in \mathbb{N}$. Now $q(q-1) = 3^{4l+1}(3^{4l+1} - 1) \equiv 6 \pmod{16}$ which violates $2^p - 2 = 2^{6k+1} - 2 \equiv 14 \pmod{16}$. In this way the possibility of $P \cong G_2(q)$, $q \equiv 0 \pmod 3$ is ruled out.

Case 20. $P \cong {}^2G_2(q)$, $q = 3^{2m+1} > 3$.
The odd order components are $q - \sqrt{3q} + 1$ and $q + \sqrt{3q} + 1$. Clearly $q \pm \sqrt{3q} + 1 = \frac{3^p-1}{2}$ leads to a contradiction because q is a power of 3. Therefore we consider $q \pm \sqrt{3q} + 1 = 2^p - 1$ which implies $3^{m+1}(3^3 \pm 1) = 2^p - 2$. Now similar methods as in Case 19 rule out this possibility.

Case 21. $P \cong F_4(q)$, q even; $P \cong {}^2F_4(q)$, $q = 2^{2m+1} > 2$.
In both cases if we equate the odd order components with $2^p - 1$ a contradiction is obtained because q is a power of 2. In the case of $P \cong F_4(q)$ the odd order

components are q^4+1 and q^4-q^2+1. If $q^4+1=\frac{3^p-1}{2}$, then $q^4=\frac{3^p-3}{2}>3^{p-1}$. Since $p>3$ we have $q^6>3^{6(p-1)}>2^{4(p+1)}$ and by Lemma 6, q must be a power of 3, a contradiction. Other cases are dealt with similarly.

Next our final cases are finite simple groups P with $s(P)>3$ which are listed in Table 3.

Case 22. $P\cong {}^2B_2(q)$, $q=2^{2m+1}>2$.

(a) $OC(G)=OC(D_{p+1}(2))$.

In this case 2^p-1 can be equal to either of $q-1$, $q-\sqrt{2q}+1$ or $q+\sqrt{2q}+1$. If $q\pm\sqrt{2q}+1=2^p-1$, then $q\pm\sqrt{2q}=2^p-2$ and we get a contradiction since q is a power of 2. If $2^p-1=q-1$, then $q=2^p$, hence $2^p-1\mid \left|{}^2B_2(q)\right|$. Now if s is a Zsigmondy prime for $2^{4p}-1$, then $s\mid 2^{2p}+1$ and hence s is a prime divisor of ${}^2B_2(q)$ that obviously does not divide $|G|$. This rules out the possibility of $P\cong {}^2B_2(q)$ in the case of $D_{p+1}(2)$.

(b) $OC(G)=OC(D_{p+1}(3))$.

The case $q-1=\frac{3^p-1}{2}$ is impossible, because it implies $q=\frac{3^p+1}{2}$ which is not a prime number. Therefore $q\pm\sqrt{2q}+1=\frac{3^p-1}{2}$. After substituting $q=2^{2m+1}$ in the last equality we obtain $2^{m+2}(2^m\pm1)=3(3^{p-1}-1)$.

First we deal with $2^{m+2}(2^m+1)=3(3^{p-1}-1)$. If $p=3$, then $m=2$ and $P\cong {}^2B_2(8)$. Now from $|G/K|\,|H|=\frac{|G|}{|P|}$ we deduce that a Sylow 5-subgroup of H has order 5, hence by Lemma 3 we must have $m_2=13\mid 5-1=4$, a contradiction. Therefore we assume $p>3$. Then p is in the form of $p=6l+1$ or $6l+5$ and $3(3^{p-1}-1)\equiv 0$ or $2 \pmod 7$, respectively. On the other hand, since 2^m+1 is divisible by 3 to the first power m must be of the form $m=6k+1$ or $6k+5$. But then $2^{m+2}(2^m+1)\equiv 3 \pmod 7$ in both cases contradicting $3(3^{p-1}-1)\equiv 0$ or $2 \pmod 7$.

Next we consider $2^{m+2}(2^m-1)=3(3^{p-1}-1)$. In this case p cannot be equal to 3, hence $p>3$. On the other hand since 2^m-1 is divisible by 3 to the first power we obtain $m=6k+2$ or $6k+4$. Then $2^{m+2}(2^m+1)\equiv 6$ or $1 \pmod 7$, respectively, contradicting $3(3^{p-1}-1)\equiv 0$ or $2 \pmod 7$. This final contradiction rules out the possibility $P\cong {}^2B_2(q)$.

Case 23. $P\cong E_8(q)$.

In this case if $q\equiv 2,3 \pmod 5$, then P has 3 odd order components, otherwise it has 4 odd order components. Since the method works for all of the odd order components we will consider only one of them, i.e., $m_2=\frac{q^{10}-q^5+1}{q^2-q+1}$.

If $m_2=2^p-1$, then $q^{10}-q^5+1=(q^2-q+1)(2^p-1)>2^p-1$ implying $q^{10}>2^p$. Hence $q^{120}>2^{12p}>2^{3(p+1)}$ and by Lemma 9 in [23] we deduce that q must be a power of 2. But $\frac{q^{10}-q^5+1}{q^2-q+1}=q^8+q^7-q^5-q^4-q^3+q+1=2^p-1$ from which we deduce $q=2$ and in this case there is no prime p satisfying the above equality.

If $m_2=\frac{3^p-1}{2}$, then $q^{10}-q^5+1=(q^2-q+1)(\frac{3^p-1}{2})>3^p-1$ implying $q^{10}>3^p$. Hence $q^{120}>3^{12p}>2^{4(p+1)}$, and by Lemma 6 q must be a power of 3. But then from $\frac{q^{10}-q^5+1}{q^2-q+1} = \frac{3^p-1}{2}$ we obtain $q = 3$ and there is no prime p satisfying the above equation with $q = 3$. This final contradiction rules out the possibility of $P \cong E_8(q)$.

Finally since we have considered all the simple groups listed in Tables 1, 2, and 3, the Main Theorem is proved now.

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