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INVARIANT RINGS AND DUAL REPRESENTATIONS OF DIHEDRAL GROUPS

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Abstract. The Weyl group of a compact connected Lie group is a reflection group. If such Lie groups are locally isomorphic, the representations of the Weyl groups are rationally equivalent. They need not however be equivalent as integral representations. Turning to the invariant theory, the rational cohomology of a classifying space is a ring of invariants, which is a polynomial ring. In the modular case, we will ask if rings of invariants are polynomial algebras, and if each of them can be realized as the mod p cohomology of a space, particularly for dihedral groups.

Suppose G is a compact connected Lie group. The Weyl group $W(G)$ acts on a maximal torus T^n, and the integral representation $W(G)\longrightarrow GL(n,\mathbb{Z})$ obtained makes $W(G)$ a reflection group. Recall that if such Lie groups are locally isomorphic, the representations of the Weyl groups are equivalent over $\mathbb{Q}$. They need not however be equivalent as $\mathbb{Z}$-representations. For instance, the integral representation of $W(PU(n))$ is not equivalent to that of $W(SU(n))$. Let $W(G)^*$ denote the dual representation of $W(G)$. Then we see $W(PU(n)) = W(SU(n))^*$ [11].

Turning to the rings of invariants, for the cohomology of classifying spaces BG, it is well-known that $H^*(BG;\mathbb{Q}) = H^*(BT^n;\mathbb{Q})^{W(G)}$, which is a polynomial ring. We recall that $\mathbb{Q}$ can be replaced by a finite field $\mathbb{F}_p$ when the prime p is large. Here W is a pseudoreflection subgroup of $GL(n,\mathbb{F}_p)$, [21, Ch 7] and [13, Part VI]. If the order of W is prime to p, according to [7, Theorem 1.5 and Lemma 5.2], we see $H^*(BT^n;\mathbb{F}_p)^W \cong H^*(BT^n;\mathbb{F}_p)^{W^*}$. So we will consider mainly the case that $|W| \equiv 0 \bmod p$. For dihedral groups and symmetric groups, we will consider dual representations, invariant rings and the realizability in the modular case.

We start with an example $H^*(BT^n;\mathbb{F}_p)^{W(G)}$ that is not isomorphic to the ring of invariants by the dual representation $H^*(BT^n;\mathbb{F}_p)^{W(G)^*}$. Recall that,

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if we write $H^*(BT^2;\mathbb{F}_3)=\mathbb{F}_3[t_1,t_2]$ with $\deg(t_i)=2$, then

$$H^*(BT^2;\mathbb{F}_3)^{W(SU(3))}=\mathbb{F}_3[y_4,y_6],$$

where $y_4=(t_1-t_2)^2$ and $y_6=t_1t_2(t_1+t_2)$, and for the dual representation we have

$$H^*(BT^2;\mathbb{F}_3)^{W(SU(3))^*}=\mathbb{F}_3[z_2,z_{12}],$$

where $z_2=t_1+t_2$ and $z_{12}=t_1^2t_2^2(t_1-t_2)^2$. On the other hand, there is an example of $H^*(BT^n;\mathbb{F}_p)^{W(G)}\cong H^*(BT^n;\mathbb{F}_p)^{W(G)^*}$. Such an example is given by $W(G_2)$ and $W(G_2)^*$. Since the exceptional Lie group G_2 is 3-torsion free, the mod 3 cohomology $H^*(BG_2,\mathbb{F}_3)$ is isomorphic to the ring of invariants $H^*(BT^2;\mathbb{F}_3)^{W(G_2)}$. The Weyl group $W(G_2)$ is the dihedral group of order 12 presented as $D_{12}=\langle r,s\mid r^6=s^2=1, srs=r^5\rangle$. The matrix (integral) representation can be taken as follows:

$$r=\begin{pmatrix}1&-1\\1&0\end{pmatrix}\quad\text{and}\quad s=\begin{pmatrix}1&-1\\0&-1\end{pmatrix}.$$

The ring of invariants is the following polynomial ring:

$$H^*(BT^2;\mathbb{F}_3)^{W(G_2)}=\mathbb{F}_3[x_4,x_{12}],$$

where $x_4=(t_1-t_2)^2$ and $x_{12}=t_1^2t_2^2(t_1+t_2)^2$. According to a result of [12], we see that $H^*(BT^2;\mathbb{F}_3)^{W(G_2)}\cong H^*(BT^2;\mathbb{F}_3)^{W(G_2)^*}$ as unstable algebras. Here we note that an isomorphism of cohomology rings need not imply the equivalence of the two representations. There is $\phi\in GL(2,\mathbb{Z})$ such that $\phi^{-1}W(G_2)\phi=W(G_2)^*$.

We will generalize this result. Let p be an odd prime. Consider a modular representation of the dihedral group $D_{4p}=\langle r,s\mid r^{2p}=s^2=1, srs=r^{-1}\rangle$:

$$\rho: D_{4p}\longrightarrow GL(2,\mathbb{F}_p)$$

defined by $\rho(r)=\left(\begin{smallmatrix}-1&b\\0&-1\end{smallmatrix}\right)$ and $\rho(s)=\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$, where $2b+1=0$. The action of D_{4p} on $H^*(BT^2;\mathbb{F}_p)=\mathbb{F}_p[t_1,t_2]$ is given by matrix multiplication: $\rho(r)(t_1)=-t_1$ and $\rho(r)(t_2)=bt_1-t_2$, and so on. When $p=3$, this representation is equivalent to $W(G_2)$ via conjugation with $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$. The representations of $\rho(D_{4p})$ and its dual $\rho(D_{4p})^*$ are not isomorphic.

Theorem 1. *Let p be an odd prime. Suppose $\rho(D_{4p})^*$ denotes the dual representation of $\rho(D_{4p})$ as above. The unstable algebra $H^*(BT^2;\mathbb{F}_p)^{\rho(D_{4p})}$ is isomorphic to $H^*(BT^2;\mathbb{F}_p)^{\rho(D_{4p})^*}$.*

Related results will be obtained as follows. We will show that

$$H^*(BT^2;\mathbb{F}_p)^{\rho(D_{4p})}$$

is a polynomial algebra in Theorem 2, and its nonrealizability for $p\geq 5$ in Theorem 3. The case $p=2$ is treated separately, and Theorem 4 shows analogous results.

The dihedral group $W(SU(3))$ of order 6 may also be regarded as the symmetric group Σ_3. For $\Sigma_n=W(SU(n))$, we recall that, except for $n=2$ and

$p = 2$, $H^*(BT^{n-1};\mathbb{F}_p)^{\Sigma_n} = H^*(BSU(n);\mathbb{F}_p)$. We will see in §5 that if p does not divide n, then $H^*(BT^{n-1};\mathbb{F}_p)^{\Sigma_n} \cong H^*(BT^{n-1};\mathbb{F}_p)^{\Sigma_n^*}$, which is a polynomial algebra. When $n = p$, according to [9] or [15] we see that $H^*(BT^{p-1};\mathbb{F}_p)^{\Sigma_p^*}$ is not a polynomial algebra.

Here is a table summarizing our results:

W	$W \sim W^*$	$S(V)^W \cong S(V)^{W^*}$	polynomial	realizable
D_6	0	0	1	1
D_6^*	0	0	1	0
D_{2p} $(p \geq 5)$	0	0	1	0
D_{2p}^* $(p \geq 5)$	0	0	1	0
D_{12}	0	1	1	1
D_{4p} $(p \geq 5)$	0	1	1	0
D_8	0	1	1	0
Σ_p $(p \geq 5)$	0	0	1	1
Σ_p^* $(p \geq 5)$	0	0	0	0
Σ_n $(p \nmid n)$	1	1	1	1

In this table, we use the symbol 1 to denote that the indicated property is true and 0 to indicate it is false. Note that the group W actually means its representation discussed in this paper. The symbol $W \sim W^*$ indicates the equivalence of the two representations and $S(V) = H^*(BT^m;\mathbb{F}_p)$ for a suitable number m and a prime p. We note that $S(V)^W \cong S(V)^{W^*}$ over the Steenrod algebra if and only if W is conjugate to W^* in $GL(V)$ [18].

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1. Modular representations of dihedral groups

Let p be an odd prime. For a non-zero element $b \in \mathbb{F}_p$, we consider a faithful representation of the dihedral group $D_{4p} = \langle r, s \mid r^{2p} = s^2 = 1, srs = r^{-1}\rangle$:

$$\rho_b : D_{4p} \longrightarrow GL(2,\mathbb{F}_p)$$

defined by $\rho_b(r) = \left(\begin{smallmatrix} -1 & b \\ 0 & -1 \end{smallmatrix}\right)$ and $\rho_b(s) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Using $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right)^k = \left(\begin{smallmatrix} \alpha^k & k\alpha^{k-1}\beta \\ 0 & \alpha^k \end{smallmatrix}\right)$, it is easy to show that the dihedral relations hold. Any two of the representations are equivalent. If $2b + 1 = 0 \bmod p$, then $\rho_b = \rho$, which is the case discussed in our introduction. Let $R = \rho(r)$ and $S = \rho(s)$. When $p = 3$, we have $R = \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For $\phi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, we see that $\phi^{-1}R\phi = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\phi^{-1}S\phi = \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$. Thus, as mentioned before, the representation ρ is a generalization of that of $W(G_2)$. We notice here that the representations ρ and its dual ρ^* are not equivalent. In fact, a composition series for the $\rho(D_{4p})$-module V is

$$0 \longrightarrow \mathbb{F}_p\langle t_1\rangle \longrightarrow V \longrightarrow \mathbb{F}_p\langle [t_2]\rangle \longrightarrow 0,$$

while a composition series for the $\rho(D_{4p})^*$-module U is

$$0 \longrightarrow \mathbb{F}_p\langle t_2\rangle \longrightarrow U \longrightarrow \mathbb{F}_p\langle [t_1]\rangle \longrightarrow 0.$$

The action of S on $\mathbb{F}_p\langle t_2\rangle$ is trivial, and that on $\mathbb{F}_p\langle t_1\rangle$ is non-trivial. So U and V have different fixed point sets. We note that the $\mathbb{F}_p$-representation of $W(G_2)$ and its dual are equivalent if and only if $p \neq 3$.

The group D_{4p} generated by r and s includes D_{2p} as a subgroup, which is generated by r^2 and s. Considering the opposite direction, we ask if the representation ρ of D_{4p} can be a restriction of a representation of D_{8p} in $GL(2, \mathbb{F}_p)$, which is generated by Q and S with $Q^2 = R$ and $SQS = Q^{-1}$. The following shows that the answer is no.

Proposition 1.1. *Let p be an odd prime. There is no Q in $GL(2, \mathbb{F}_p)$ satisfying the two conditions $Q^2 = R$ and $SQS = Q^{-1}$.*

Proof. For $Q = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, suppose $Q^2 = R$ so that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = R.$$

Since the $(1,2)$-entry of Q^2 is non-zero, we see $a + d \neq 0$. Consequently, comparing the $(2,1)$-entries, we see $c = 0$. And therefore $a^2 = -1$ and $d^2 = -1$. Since $SQS = Q^{-1}$, it follows that $(SQ)^2$ must be the identity matrix. We have, however, the following:

$$(SQ)^2 = \begin{pmatrix} a^2 & ab-bd \\ 0 & d^2 \end{pmatrix} = \begin{pmatrix} -1 & ab-bd \\ 0 & -1 \end{pmatrix}$$

Thus $(SQ)^2$ is not the identity matrix. This contradiction completes the proof. $\square$

If $GL(2, \mathbb{F}_p)$ contains D_{8p} as a subgroup, it must have an element of order $4p$. The following shows $GL(2, \mathbb{F}_p)$ has an element of order $4p$ if and only if 4 divides $p - 1$.

Proposition 1.2. *For an odd prime p, the following hold*:

(1) *If m divides $p-1$, there is an element of order mp in $GL(2, \mathbb{F}_p)$.*

(2) *If $p-1$ is not divisible by 4, there is no element of order $4p$ in $GL(2, \mathbb{F}_p)$.*

Proof. (1) Since $m \mid p-1$, there is an element $\alpha \in \mathbb{F}_p$ of order m. Suppose $\beta \in \mathbb{F}_p$ is nonzero. Consider the 2×2 matrix $A = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right)$. Since $A^k = \left(\begin{smallmatrix} \alpha^k & k\alpha^{k-1}\beta \\ 0 & \alpha^k \end{smallmatrix}\right)$, the matrix A is of order mp.

(2) Next, assume $4 \nmid p-1$, and A is an element of order $4p$ in $GL(2, \mathbb{F}_p)$. Since $\det A^{p-1} = (\det A)^{p-1} = 1$, we see $A^{p-1} \in SL(2, \mathbb{F}_p)$. Recall that p is odd. Since $(A^{p-1})^{2p} = (A^{4p})^{\frac{p-1}{2}}$, the order of A^{p-1} must be $2p$. The conjugacy classes in the unimodular group $SL(2, \mathbb{F}_p)$ are known, [22, Ch 9]. The order of each element is less than or equal to $p + 1$. Thus no element in $SL(2, \mathbb{F}_p)$ has order $2p$. This contradiction completes the proof. $\square$

Proof of Theorem 1. Recall that $H^*(BT^n;\mathbb{F}_p)$ is a polynomial ring generated by n elements of degree 2. A map of unstable algebras $\phi : H^*(BT^n;\mathbb{F}_p)\longrightarrow H^*(BT^m;\mathbb{F}_p)$ is determined by the images of the n generators. So ϕ is often expressed by a matrix. Now consider unstable algebras $H^*(BT^n;\mathbb{F}_p)^W$ and $H^*(BT^n;\mathbb{F}_p)^{W'}$, and a map θ between them. According to [2, Proposition 1.10] for $\theta : H^*(BT^n;\mathbb{F}_p)^W\longrightarrow H^*(BT^n;\mathbb{F}_p)^{W'}$ we can find a homomorphism ϕ which makes the following diagram commutative:

$$\begin{array}{ccc} H^*(BT^n;\mathbb{F}_p)^W & \xrightarrow{\theta} & H^*(BT^n;\mathbb{F}_p)^{W'} \\ \downarrow & & \downarrow \\ H^*(BT^n;\mathbb{F}_p) & \xrightarrow{\phi} & H^*(BT^n;\mathbb{F}_p) \end{array}$$

In this diagram the vertical maps are inclusions, and ϕ is an admissible map [1]. If $\phi W\phi^{-1} = W'$, then for any $w' \in W'$ there is $w \in W$ such that $w'\phi = \phi w$. Thus for $x \in H^*(BT^n;\mathbb{F}_p)^W$, we see that $w'\phi(x) = \phi w(x) = \phi(x)$. Consequently, $\phi(H^*(BT^n;\mathbb{F}_p)^W) \subset H^*(BT^n;\mathbb{F}_p)^{W'}$. Since ϕ is invertible, we also see that $\phi^{-1}(H^*(BT^n;\mathbb{F}_p)^{W'}) \subset H^*(BT^n;\mathbb{F}_p)^W$, and hence $H^*(BT^n;\mathbb{F}_p)^W \cong H^*(BT^n;\mathbb{F}_p)^{W'}$. Therefore, it remains to find ϕ such that $\phi\rho(D_{4p})\phi^{-1} = \rho(D_{4p})^*$.

The representation ρ of D_{4p} is generated by $\rho(r) = \left(\begin{smallmatrix} -1 & b \\ 0 & -1 \end{smallmatrix}\right)$ and $\rho(s) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where $2b + 1 = 0 \bmod p$. Since p is odd, there is $d \in \mathbb{F}_p$ such that $b + 2d = 0 \bmod p$. If $\phi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & d \end{smallmatrix}\right)$, then a calculation shows

$$\phi\rho(r)\phi^{-1} = {}^t\rho(r) \quad \text{and} \quad \phi\rho(s)\phi^{-1} = {}^t(\rho(s)\rho(r)),$$

where ${}^t(\rho(r)) = \left(\begin{smallmatrix} -1 & 0 \\ b & -1 \end{smallmatrix}\right)$ and ${}^t(\rho(s)\rho(r)) = \left(\begin{smallmatrix} 1 & 0 \\ -b & -1 \end{smallmatrix}\right)$. This completes the proof. $\square$

2. Invariant rings and dihedral groups

We recall how to see if a ring of invariants $H^*(BT^n;\mathbb{F}_p)^W$ is polynomial, [21], [14] and [10]. A set of n elements $x_1, x_2, \ldots, x_n \in H^*(BT^n;\mathbb{F}_p)^W$ is said to be a system of parameters if the solution of the following system of equations

$$\begin{cases} x_1(t_1, t_2, \ldots, t_n) = 0 \\ x_2(t_1, t_2, \ldots, t_n) = 0 \\ \qquad\vdots \\ x_n(t_1, t_2, \ldots, t_n) = 0 \end{cases}$$

is trivial. Namely $t_1 = t_2 = \cdots = t_n = 0$. As usual, we write $H^*(BT^n;\mathbb{F}_p) = \mathbb{F}_p[t_1, t_2, \ldots, t_n]$. Let $d(x)$ denote $\frac{1}{2}\deg(x)$ so that $d(t_i) = 1$ for $1 \le i \le n$. According to [21, Proposition 5.5.5], for a finite group W, if we can find a system of parameters $\{x_1, x_2, \ldots, x_n\}$ with $\prod_{i=1}^n d(x_i) = |W|$, then $H^*(BT^n;\mathbb{F}_p)^W = \mathbb{F}_p[x_1, x_2, \ldots, x_n]$.

The exceptional Lie group G_2 contains $SU(3)$, and $W(SU(3))$ can be a subgroup of $W(G_2)$ such that $W(SU(3)) = \{1, r^2, r^4, s, sr^2, sr^4\}$. In this way, we consider the subgroup D_{2p}, generated by r^2 and s, of D_{4p}. For the representation of D_{2p} and its dual, the invariant rings are discussed in [21, §5.6]. In particular, $H^*(BT^2;\mathbb{F}_p)^{\rho(D_{2p})} \ncong H^*(BT^2;\mathbb{F}_p)^{\rho(D_{2p})^*}$.

Theorem 2. *The following hold*:

(1) $H^*(BT^2;\mathbb{F}_p)^{\rho(D_{4p})} = \mathbb{F}_p[x_4, x_{4p}]$, *where* $x_4 = t_1^2$ *and* $x_{4p} = \prod_{a=0}^{p-1}(at_1 + t_2)^2$.

(2) $H^*(BT^2;\mathbb{F}_p)^{\rho(D_{4p})^*} = \mathbb{F}_p[x_4^*, x_{4p}^*]$, *where* $x_4^* = t_2^2$ *and* $x_{4p}^* = \prod_{b=0}^{p-1}(t_1 + bt_2)^2$.

(3) $H^*(BT^2;\mathbb{F}_p)^{\rho(D_{2p})} = \mathbb{F}_p[y_4, y_{2p}]$, *where* $y_4 = t_1^2$ *and* $y_{2p} = \prod_{a=0}^{p-1}(at_1 + t_2)$.

(4) $H^*(BT^2;\mathbb{F}_p)^{\rho(D_{2p})^*} = \mathbb{F}_p[z_2, z_{4p}]$, *where* $z_2 = t_2$ *and* $z_{4p} = \prod_{b=0}^{p-1}(t_1 + bt_2)^2$.

Proof. We show only the case of $\rho(D_{4p})$, since the other cases are similarly proved. Let $x_1 = t_1^2$ and $x_2 = \prod_{a=0}^{p-1}(at_1 + t_2)^2$, the top orbit Chern classes. It follows that $x_1, x_2 \in H^*(BT^2;\mathbb{F}_p)^{\rho(D_{4p})}$. Since $\{x_1, x_2\}$ is a system of parameters with $d(x_1) \cdot d(x_2) = 2 \cdot 2p = |D_{4p}|$, we obtain the desired result. □

3. Nonrealizability of polynomial rings

For a subgroup W of $GL(n, \mathbb{F}_p)$, the unstable algebra $H^*(BT^n;\mathbb{F}_p)^W$ is realizable if $H^*(BT^n;\mathbb{F}_p)^W \cong H^*(X;\mathbb{F}_p)$ for a space X. When $W = W(G)$, it is well-known that $H^*(BT^n;\mathbb{F}_p)^{W(G)} \cong H^*(BG;\mathbb{F}_p)$ if $p \nmid |W(G)|$. The unstable algebra $H^*(BT^2;\mathbb{F}_3)^{W(SU(3))}$ is realizable, since it is isomorphic to $H^*(BSU(3), \mathbb{F}_3)$. However for the dual representation, $H^*(BT^2;\mathbb{F}_3)^{W(SU(3))^*}$ is not realizable, [11]. Other cases are found in [5], and nonrealizability is discussed in [20] and [7]. See [17, §3], [21, Ch 10], and [4] for a detail of this realization problem.

Theorem 3. *Let W be a subgroup of $GL(n, \mathbb{F}_p)$ such that $|W| \equiv 0$ mod p. If a polynomial ring $H^*(BT^n;\mathbb{F}_p)^W$ is realizable, then $n \geq p - 1$. In particular, for $p \geq 5$, none of $H^*(BT^2;\mathbb{F}_p)^{\rho(D_{4p})}$, $H^*(BT^2;\mathbb{F}_p)^{\rho(D_{2p})}$, and $H^*(BT^2;\mathbb{F}_p)^{\rho(D_{2p})^*}$ are realizable.*

We will need the following result to prove this theorem.

Proposition 3.1. *There is a matrix $A \in GL(n, \mathbb{Z}_p^\wedge)$ of order p if and only if $n \geq p - 1$.*

Proof. First, assume that $A^p = I$ and $A \neq I$, where I is the identity matrix. Let $f(x)$ be the characteristic polynomial of A. Then $f(x) \in \mathbb{Z}_p^\wedge[x]$, and $\deg(f) = n$.

By our assumption, one of the eigenvalues, say ζ, must be a primitive p-th root of unity. Note [19, §3.2] that $(\mathbb{Q}_p^\wedge)^* \cong \mathbb{Z} \times \mathbb{Z}_p^\wedge \times \mathbb{Z}/(p-1)$. So the minimal polynomial of ζ is $p(x) = x^{p-1} + x^{p-2} + \cdots + 1$. Since $p(x)$ divides $f(x)$, we see $n \geq p-1$.

Conversely, assume $n \geq p-1$. We consider the integral representation $W(SU(p)) \hookrightarrow GL(p-1, \mathbb{Z})$. Since

$$\mathbb{Z}/p \subset \Sigma_p = W(SU(p)) \hookrightarrow GL(p-1, \mathbb{Z}) \hookrightarrow GL(n, \mathbb{Z}_p^\wedge),$$

we can find $A \in GL(n, \mathbb{Z}_p^\wedge)$ of order p. □

Proof of Theorem 3. If a polynomial ring $H^*(BT^n; \mathbb{F}_p)^W$ is realizable for an odd prime p, a result of [7] shows that the modular representation $W \longrightarrow GL(n, \mathbb{F}_p)$ lifts to a p-adic representation. Since $|W| \equiv 0 \bmod p$, Proposition 3.1 implies $n \geq p-1$. □

4. Results for $p = 2$

For $p = 2$, the group D_{4p} does not have a faithful representation in $GL(2, \mathbb{F}_2)$. However there is a faithful 3-dimensional representation:

$$\rho : D_8 \longrightarrow GL(3, \mathbb{F}_2)$$

defined by $\rho(r) = \begin{pmatrix} 1&1&1\\0&1&1\\0&0&1 \end{pmatrix}$ and $\rho(s) = \begin{pmatrix} 1&0&0\\0&1&1\\0&0&1 \end{pmatrix}$, where $D_8 = \langle r, s \mid r^4 = s^2 = 1, srs = r^{-1} \rangle$. It turns out that $\rho(D_8)$ is the unipotent subgroup of $GL(3, \mathbb{F}_2)$. The representations of $\rho(D_8)$ and its dual $\rho(D_8)^*$ are not isomorphic.

Theorem 4. *Let $H^*(BT^3; \mathbb{F}_2) = \mathbb{F}_2[t_1, t_2, t_3]$ with $\deg(\mathrm{t_i}) = 2$. The following hold*:

(1) *$H^*(BT^3; \mathbb{F}_2)^{\rho(D_8)} = \mathbb{F}_2[x_2, x_4, x_8]$ where $x_2 = t_1$, $x_4 = t_2(t_1 + t_2)$ and $x_8 = t_3(t_1 + t_3)(t_2 + t_3)(t_1 + t_2 + t_3)$.*

(2) *$H^*(BT^3; \mathbb{F}_2)^{\rho(D_8)} \cong H^*(BT^3; \mathbb{F}_2)^{\rho(D_8)^*}$.*

(3) *The unstable algebra $H^*(BT^3; \mathbb{F}_2)^{\rho(D_8)}$ is not realizable.*

Proof. (1) Recall that $\rho(D_8)$ is the unipotent subgroup of $GL(3, \mathbb{F}_2)$. From [21, Theorem 8.3.5] as well as [16, §4.5, Example 2], we see that our invariant ring $H^*(BT^3; \mathbb{F}_2)^{\rho(D_8)}$ is a polynomial algebra generated by $x_2 = t_1$, $x_4 = t_2(t_1+t_2)$ and $x_8 = t_3(t_1 + t_3)(t_2 + t_3)(t_1 + t_2 + t_3)$.

(2) Let $\phi = \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}$. Then a calculation shows

$$\phi\rho(r)\phi^{-1} = {}^t\rho(r) \text{ and } \phi\rho(s)\phi^{-1} = {}^t(\rho(rs)).$$

Consequently, $H^*(BT^3; \mathbb{F}_2)^{\rho(D_8)} \cong H^*(BT^3; \mathbb{F}_2)^{\rho(D_8)^*}$.

(3) If the unstable algebra $H^*(BT^3; \mathbb{F}_2)^{\rho(D_8)}$ is realizable, there is a 2-compact group X such that

$$H^*(BT^3; \mathbb{F}_2)^{\rho(D_8)} \cong H^*(BX; \mathbb{F}_p).$$

Since the polynomial algebra is generated by even-degree elements, the classifying space BX is 2-torsion free. So the 2-adic cohomology is also a polynomial algebra generated by elements of the same degree. We can find, [3], a compact connected Lie group G such that $H^*(BX;\mathbb{Z}_2^\wedge) \cong H^*(BG;\mathbb{Z}_2^\wedge)$. For degree reason, we see $G = S^1 \times SP(2)/C$ where C is a finite 2-subgroup of the center of $S^1 \times SP(2)$. Passing to the cohomology with the coefficients in the field of 2-adic numbers $\mathbb{Q}_2^\wedge$, we have the following commutative diagram:

$$\begin{array}{ccc} H^*(BX;\mathbb{Q}_2^\wedge) & \xrightarrow{\cong} & H^*(B(S^1 \times SP(2));\mathbb{Q}_2^\wedge) \\ \downarrow & & \downarrow \\ H^*(BT^3;\mathbb{Q}_2^\wedge) & \xrightarrow{\phi_0} & H^*(BT^3;\mathbb{Q}_2^\wedge) \end{array}$$

Recall that $H^*(B(S^1 \times SP(2));R) = R[t_1, t_2^2 + t_3^2, t_2^2t_3^2]$ for $R = \mathbb{Z}_p^\wedge, \mathbb{Q}_p^\wedge$, where $H^*(BT^3;R) = R[t_1,t_2,t_3]$. We note that $H^*(BX;\mathbb{Z}_2^\wedge) \cong \mathbb{Z}_2^\wedge[\widetilde{x}_2, \widetilde{x}_4, \widetilde{x}_8]$ with $\widetilde{x}_2 = t_1$, and $\phi_0(t_1) = kt_1$ for some $k \in \mathbb{Z}_2^\wedge$ for degree reason.

Let $J_{\widetilde{G}}$ denote the Jacobian of the generators of the polynomial algebra $H^*(B\widetilde{G};\mathbb{Z}_2^\wedge)$ for $\widetilde{G} = S^1 \times SP(2)$. Then

$$J_{\widetilde{G}} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2t_2 & 2t_3 \\ 0 & 2t_2t_3^2 & 2t_2^2t_3 \end{vmatrix} = 4t_2t_3(t_2+t_3)(t_2-t_3).$$

On the other hand, let J_X denote the Jacobian for $H^*(BX;\mathbb{Z}_2^\wedge)$, and we consider it modulo 2. Then

$$J_X = \begin{vmatrix} 1 & 0 & 0 \\ * & t_1 & 0 \\ * & * & t_1t_2(t_1+t_2) \end{vmatrix} = t_1^2t_2(t_1+t_2) \bmod 2.$$

The Jacobian is well-defined modulo scalar multiples. Notice that J_X has a t_1-term, and $J_{\widetilde{G}}$ doesn't. This means that the admissible map ϕ_0 can not induce an isomorphism between $H^*(BX;\mathbb{Q}_2^\wedge)$ and $H^*(B(S^1 \times SP(2));\mathbb{Q}_2^\wedge)$. This contradiction completes the proof. □

5. Related results for symmetric groups

The representation of $\Sigma_n = W(SU(n))$ is generated by the permutation matrices together with the following $(n-1) \times (n-1)$ matrix:

$$\begin{pmatrix} 1 & & & -1 \\ & \ddots & & \vdots \\ & & 1 & \vdots \\ & & & -1 \end{pmatrix}$$

For example, the symmetric group Σ_4 is generated by the three reflections:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

For the following $(n-1)\times(n-1)$ matrix,

$$\phi = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{pmatrix}$$

we see [11] that $\phi^{-1}\sigma\phi = {}^t\sigma$ for each of the generators σ of the reflection group Σ_n. We note that det $\phi = n$. Consequently, if p does not divide n, then the modular representation of Σ_n is equivalent to its dual representation. It follows that $H^*(BT^{n-1};\mathbb{F}_p)^{\Sigma_n} \cong H^*(BT^{n-1};\mathbb{F}_p)^{\Sigma_n^*}$, which is a polynomial algebra.

Now we explain the truth table in our introduction. For the dihedral groups, the table is merely the summary of the results that have been discussed. So we consider Σ_p and its dual Σ_p^*. Note that $H^*(BT^{p-1};\mathbb{F}_p)^{\Sigma_p} \cong H^*(BSU(p);\mathbb{F}_p) \cong \mathbb{F}_p[\widetilde{c_2},\ldots,\widetilde{c_p}]$, and that $H^*(BT^{p-1};\mathbb{F}_p)^{\Sigma_p^*}$ has an invariant vector $c_1 = t_1+\cdots+t_{p-1}$. Consequently, the two unstable algebras are not isomorphic, and hence the representations are inequivalent.

Moreover, we will see that $H^*(BT^{p-1};\mathbb{F}_p)^{\Sigma_p^*}$ is not a polynomial algebra. We use a result of Dwyer-Wilkerson [9, Theorem 1.4]. Let $V = \oplus^{p-1}\mathbb{F}_p$ and $W = \Sigma_p^*$. For a subset U of V, let W_U denote the subgroup of W consisting of elements which fix U pointwise. To prove the desired result, we need to find a subset U such that W_U is not generated by pseudoreflections. Take U to be the 1-dimensional subspace spanned by the vector $\boldsymbol{x} = {}^t(1\ 2\ \cdots\ p-1)$. For example, for $p=5$,

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix},$$

we see $A\boldsymbol{x} = \boldsymbol{x}$ and $A^p = I$. In general, W_U is the p-Sylow subgroup of Σ_p, which is not a pseudoreflection group. If $H^*(BT^{p-1};\mathbb{F}_p)^{\Sigma_p^*}$ is realizable, the space is p-torsion free. Consequently, its loop space is a p-torsion free p-compact group, [8], and hence the $\mathbb{F}_p$-cohomology would be a polynomial algebra. Thus the unstable algebra is not realizable.

Finally, we consider a little more about the representations of symmetric groups. Recall that the center of $SU(n)$ is isomorphic to $\mathbb{Z}_n$, and that if d divides n, the quotient $SU(n)/\mathbb{Z}_d$ is also a Lie group. For example, $SU(n)/\mathbb{Z}_n = PSU(n) = PU(n)$. The integral representations of Σ_n induced by the actions of the Weyl groups of $SU(n)/\mathbb{Z}_d$ on maximal tori are $\mathbb{Z}$-inequivalent, [6]. In fact, the $\mathbb{Z}$-representation of $W(SU(n)/\mathbb{Z}_d)$ on T^{n-1}, up to $\mathbb{Z}$-equivalence, is given by $\phi_d W(SU(n))\phi_d^{-1}$, where

$$\phi_d = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ \frac{1-d}{d} & \cdots & \frac{1-d}{d} & \frac{1}{d} \end{pmatrix}.$$

Thus if $p \nmid n$, the representations of Σ_n and Σ_n^* over $\mathbb{F}_p$ are equivalent. On the other hand, if p divides n, a complete answer to the question about $H^*(BT^{n-1};\mathbb{F}_p)^{\Sigma_n^*}$ is not available.

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