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MIXED CHORD-INTEGRALS OF STAR BODIES

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Abstract. The mixed chord-integrals are defined. The Fenchel-Aleksandrov inequality and a general isoperimetric inequality for the mixed chord-integrals are established. Furthermore, the dual general Bieberbach inequality is presented. As an application of the dual form, a Brunn-Minkowski type inequality for mixed intersection bodies is given.

1. Introduction and main results

In [6, 9] Lutwak posed the notion of the mixed width-integrals of convex bodies (compact, convex subsets with non-empty interiors) and obtained a great many properties in common with the dual quermassintegrals (see [7]). It exists closely relations (see [5, 16]) between the mixed width-integrals and mixed projection bodies (see [10, 12]). In [8] Lutwak established a general Bieberbach inequality. Motivated by the ideas of Lutwak, we shall introduce the definitions of the mixed chord-integrals of star bodies and establish a dual general Bieberbach inequality.

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) and $\mathcal{K}_o^n$ denote the subset of $\mathcal{K}^n$ that contains the origin in their interiors in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. The volume of the unit n-ball, U, will be denoted by ω_n. If $K \in \mathcal{K}^n$, then the support function of K, $h_K = h(K, \cdot) : \mathbb{R}^n \longrightarrow (0, \infty)$, is defined by

$$h(K, u) = \max\{u \cdot x : x \in K\}, \quad u \in S^{n-1} \tag{1.1}$$

where $u \cdot x$ denotes the standard inner product of u and x. For $K \in \mathcal{K}^n$ and $u \in S^{n-1}$, $b(K, u)$ is half the width of K in the direction u.

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Mixed width-integrals, $A(K_1,\dots,K_n)$, of $K_1,\dots,K_n\in\mathcal{K}^n$ was defined by Lutwak (see [9])

$$A(K_1,\dots,K_n)=\frac{1}{n}\int_{S^{n-1}}b(K_1,u)\cdots b(K_n,u)dS(u), \tag{1.2}$$

where dS is the $(n-1)$-dimensional volume element on S^{n-1}. More in general, for a real number $p\neq 0$, the mixed width-integrals of order p, $A_p(K_1,\dots,K_n)$, of $K_1,\dots,K_n\in\mathcal{K}^n$ was also defined by Lutwak (see [9])

$$A_p(K_1,\dots,K_n)=\omega_n\left[\frac{1}{n\omega_n}\int_{S^{n-1}}b(K_1,u)^p\cdots b(K_n,u)^p dS(u)\right]^{1/p}, \tag{1.3}$$

Furthermore, Lutwak in [9] showed an isoperimetric inequality involving the mixed width-integrals which generalized an inequality obtained by Chakerian (see [1, 2]).

Theorem 1*. *If $K_1,\dots,K_n\in\mathcal{K}^n$, then*

$$V(K_1)\cdots V(K_n)\leq A^n(K_1,\dots,K_n), \tag{1.4}$$

with equality if and only if $K_1,K_2,\dots,K_n$ are n-balls.

A more general version of inequality (1.4) is obtained by introducing indexed mixed width-integrals.

Theorem 2*. *If $K_1,\dots,K_n\in\mathcal{K}^n$, $p\neq 0$ and $-1\leq p\leq\infty$, then*

$$V(K_1)\cdots V(K_n)\leq A_p^n(K_1,\dots,K_n), \tag{1.5}$$

with equality if and only if $K_1,K_2,\dots,K_n$ are n-balls.

Let $\mathcal{S}_o^n$ denote the set of star bodies in $\mathbb{R}^n$ containing the origin in their interiors. In this paper, we give the definitions of the mixed chord-integrals, $B(L_1,\dots,L_n)$ and for a real number $p\neq 0$, the mixed chord-integrals of order p, $B_p(L_1,\dots,L_n)$, of $L_1,\dots,L_n\in\mathcal{S}_o^n$ and the p-chord, $\tilde{d}_p(L)$, of $L\in\mathcal{S}_o^n$, respectively. We mainly obtain the following results.

The analog of Theorem 1* (i.e., the general isoperimetric inequality involving the mixed width-integrals) for the mixed chord-integrals is obtained:

Theorem 1. *If $L_1,\dots,L_n\in\mathcal{S}_o^n$, then*

$$B^n(L_1,\dots,L_n)\leq V(L_1)\cdots V(L_n), \tag{1.6}$$

with equality if and only if $L_1,\dots,L_n$ are dilates and centered.

A more general version of inequality (1.6) is obtained:

Theorem 2. *If $L_1,\dots,L_n\in\mathcal{S}_o^n$ and $-\infty\leq p\leq 1$, then*

$$B_p^n(L_1,\dots,L_n)\leq V(L_1)\cdots V(L_n), \tag{1.7}$$

with equality if and only if $L_1,\dots,L_n$ are n-balls.

Theorem 3. *If $L_1, \dots, L_n \in \mathcal{S}_o^n$ and $1 < m \le n$, then*

$$B^m(L_1, \dots, L_n) \le \prod_{i=0}^{m-1} B(L_1, \dots, L_{n-m}, L_{n-i}, \dots, L_{n-i}), \tag{1.8}$$

with equality if and only if $L_{n-m+1}, L_{n-m+2}, \dots, L_n$ are all of similar chord.

A more general version of inequality (1.8) is obtained:

Theorem 4. *If $L_1, \dots, L_n \in \mathcal{S}_o^n$ and $1 < m \le n$, then for $p > 0$*

$$B_p^m(L_1, \dots, L_n) \le \prod_{i=0}^{m-1} B_p(L_1, \dots, L_{n-m}, L_{n-i}, \dots, L_{n-i}), \tag{1.9}$$

with equality if and only if $L_{n-m+1}, L_{n-m+2}, \dots, L_n$ are all of similar chord. For $p < 0$, inequality (1.9) is reversed.

Theorem 3 and Theorem 4 are just analogs of the Fenchel-Aleksandrov inequality for the dual mixed volumes (see [7]).

Moreover, we obtain the dual general Bieberbach inequality as follows.

Theorem 5. *If $L \in \mathcal{S}_o^n$ and $-\infty \le p < n$, then*

$$V(L) \ge \omega_n \tilde{d}_p(L)^n, \tag{1.10}$$

with equality if and only if L is an n-ball.

Theorem 5 is just a dual form of the following general Bieberbach inequality which was shown by Lutwak [8].

Theorem 5*. *If $K \in \mathcal{K}^n$ and $-n < p \le \infty$, then*

$$V(K) \le \omega_n \bar{b}_p(K)^n, \tag{1.11}$$

with equality if and only if K is an n-ball.

As an application of Theorem 5, we establish a Brunn-Minkowski type inequality for mixed intersection bodies which defined in [11, 15].

Theorem 6. *If $L_1, L_2 \in \mathcal{S}_o^n$ and $i \le n-1$, then*

$$\tilde{W}_i(I(L_1 \breve{+} L_2))^{1/(n-i)} \le \tilde{W}_i(IL_1)^{1/(n-i)} + \tilde{W}_i(IL_2)^{1/(n-i)},$$

with equality if and only if L_1 and L_2 are dilates. If $n-1 < i < n$ or $i > n$, then this inequality is reversed.

In particular, for $i = 0$, we obtain:

Corollary 1. *If $L_1, L_2 \in \mathcal{S}_o^n$, then*

$$V(I(L_1 \breve{+} L_2))^{1/n} \le V(IL_1)^{1/n} + V(IL_2)^{1/n},$$

with equality if and only if L_1 and L_2 are dilates.

2. Mixed chord-integrals

For a compact subset L of $\mathbb{R}^n$, which is star-shaped with respect to the origin, we shall use $\rho(L,\cdot)$ to denote its radial function; i.e., for $u \in S^{n-1}$,

$$\rho(L,u) = \max\{\lambda > 0 : \lambda u \in L\}. \tag{2.1}$$

If $\rho(L,\cdot)$ is continuous and positive, L will be called a star body. Two star bodies K and L are said to be dilates if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathcal{K}_o^n$, the polar body of K, K^*, is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1, x \in K\}. \tag{2.2}$$

Obviously, we have $(K^*)^* = K$. From the definition (2.2), we also know that: If $K \in \mathcal{K}_o^n$, then the support and radial function of K^*, the polar body of K, are respectively defined by

$$h_{K^*} = \frac{1}{\rho_K} \text{ and } \rho_{K^*} = \frac{1}{h_K}. \tag{2.3}$$

The polar coordinate formula for volume of body L in $\mathbb{R}^n$ is

$$V(L) = \frac{1}{n}\int_{S^{n-1}} \rho(L,u)^n dS(u). \tag{2.4}$$

If $L \in \mathcal{S}_o^n$ and $u \in S^{n-1}$, we let

$$d(L,u) = \frac{1}{2}\rho(L,u) + \frac{1}{2}\rho(L,-u), \tag{2.5}$$

i.e., $d(L,u)$ denotes half the chord of L in the direction u. Star bodies $L_1,\dots,L_n$ are said to have similar chord if there exist constants $\lambda_1,\dots,\lambda_n > 0$ such that $\lambda_1 d(L_1,u) = \cdots = \lambda_n d(L_n,u)$ for all $u \in S^{n-1}$; they are said to have constant chord jointly if the product $d(L_1,u)\cdots d(L_n,u)$ is constant for all $u \in S^{n-1}$. For reference see Gardner ([3]) and schneider ([14]).

Following Lutwak, we define the mixed chord integral of star bodies: For $L_1,\dots,L_n \in \mathcal{S}_o^n$, the mixed chord-integral, $B(L_1,\dots,L_n)$, of $L_1,\dots,L_n$ is defined by

$$B(L_1,\dots,L_n) = \frac{1}{n}\int_{S^{n-1}} d(L_1,u)\cdots d(L_n,u) dS(u). \tag{2.6}$$

By this definition, B is a map

$$B : \underbrace{\mathcal{S}_o^n \times \cdots \times \mathcal{S}_o^n}_{n} \longrightarrow \mathbb{R}.$$

We list some of its elementary properties.

(i) (Continuity) The mixed chord-integral $B(L_1,\dots,L_n)$ is a continuous function of $L_1,\dots,L_n \in \mathcal{S}_o^n$.

(ii) (Positivity) For $L_1,\dots,L_n \in \mathcal{S}_o^n$, $B(L_1,\dots,L_n) > 0$.

(iii) (Positively homogeneous) If $L_1,\dots,L_n \in \mathcal{S}_o^n$ and $\lambda_1,\dots,\lambda_n > 0$, then

$$B(\lambda_1 L_1,\dots,\lambda_n L_n) = \lambda_1\cdots\lambda_n B(L_1,\dots,L_n).$$

(iv) (Monotonicity for set inclusion) If $K_i, L_i \in \mathcal{S}_o^n$, $K_i \subset L_i$ and $1 \le i \le n$, then

$$B(K_1, \dots, K_n) \le B(L_1, \dots, L_n),$$

with equality if and only if $K_i = L_i$ for $1 \le i \le n$.

(v) (Change under linear transformations) If $L_1, \dots, L_n \in \mathcal{S}_o^n$ and $\phi \in GL(n)$, then

$$B(\phi L_1, \dots, \phi L_n) = |\det \phi| B(L_1, \dots, L_n).$$

Just as the width-integrals $B_i(K)$ (see [6]) of $K \in \mathcal{K}^n$, are defined to be the special mixed width-integrals

$$A(\underbrace{K, \dots, K}_{n-i}, \underbrace{U, \dots, U}_{i}),$$

the chord-integrals $D_i(L)$ of $L \in \mathcal{S}_o^n$, can be defined as the special mixed chord-integral

$$B(\underbrace{L, \dots, L}_{n-i}, \underbrace{U, \dots, U}_{i}).$$

Now we generalize the notion of the mixed chord-integral of star bodies: For $L_1, \dots, L_n \in \mathcal{S}_o^n$ and a real number $p \neq 0$, the mixed chord-integral of order p, $B_p(L_1, \dots, L_n)$, of $L_1, \dots, L_n$ is defined by

$$B_p(L_1, \dots, L_n) = \omega_n \left[\frac{1}{n\omega_n} \int_{S^{n-1}} d(L_1, u)^p \cdots d(L_n, u)^p dS(u) \right]^{1/p}. \tag{2.7}$$

Specially $p = 1$, then definition (2.7) is just definition (2.6). For p equal to $-\infty$, 0 or ∞ we respectively define the mixed chord-integral of order p by

$$B_p(L_1, \dots, L_n) = \lim_{s \to p} B_s(L_1, \dots, L_n).$$

As a direct consequence of Jensen's inequality [4] we have:

Proposition 1. *If $L_1, \dots, L_n \in \mathcal{S}_o^n$ and $-\infty \le p < q \le \infty$, then*

$$B_p(L_1, \dots, L_n) \le B_q(L_1, \dots, L_n),$$

with equality if and only if $L_1, \dots, L_n$ have constant chord jointly.

The well-known Blaschke-Santaló inequality (see [13]) can be stated: For $K \in \mathcal{K}_o^n$, then

$$V(K)V(K^*) \le \omega_n^2, \tag{2.8}$$

with equality if and only if K is an n-dimensional ellipsoid.

By combining the well-known Blaschke-Santaló inequality with Theorem 2, we obtain the Blaschke-Santaló type inequality for the mixed chord-integral of order p (the mixed chord-integral when $p = 1$).

Corollary 2. *If $K_1, \dots, K_n \in \mathcal{K}_o^n$, then*

$$B_p(K_1, \dots, K_n) B_p(K_1^*, \dots, K_n^*) \le \omega_n^2,$$

with equality if and only if $K_1, \dots, K_n$ are n-balls.

In particular, if $L_1 = \cdots = L_{n-i} = L$ and $L_{n-i+1} = \cdots = L_n = U$, then Theorem 1 becomes:

Corollary 3. *If $L \in \mathcal{S}_o^n$ and $0 \le i \le n$, then*

$$D_i^n(L) \le \omega_n^i V(L)^{n-i},$$

with equality if and only if L is an n-ball.

Combing Theorems 1^* and 2^* with Theorems 1 and 2, we obtain the following relations between $B(L_1, \ldots, L_n)$, $B_p(L_1, \ldots, L_n)$ and $A(L_1, \ldots, L_n)$, $A_p(L_1, \ldots, L_n)$, respectively.

Corollary 4. *If $K_1, \ldots, K_n \in \mathcal{K}^n$, then*

$$B(K_1, \ldots, K_n) \le A(K_1, \ldots, K_n),$$

with equality if and only if $K_1, K_2, \ldots, K_n$ are n-balls.

Corollary 5. *If $K_1, \ldots, K_n \in \mathcal{K}^n$ and $-1 \le p \le 1$, then*

$$B_p(K_1, \ldots, K_n) \le A_p(K_1, \ldots, K_n),$$

with equality if and only if $K_1, K_2, \ldots, K_n$ are n-balls.

Proof of Theorem 1. For $L_1, \ldots, L_n \in \mathcal{S}_o^n$. From definition (2.6), Hölder integral inequality [4], definition (2.5), Minkowski integral inequality [4] and formula (2.4), we have

$$\begin{aligned}
&B(L_1, \ldots, L_n)\\
&= \frac{1}{n}\int_{S^{n-1}} d(L_1, u)\cdots d(L_n, u)dS(u)\\
&\le n^{-1/n}\|d(L_1, u)\|_n \cdots n^{-1/n}\|d(L_n, u)\|_n\\
&= n^{-1/n}\|\frac{1}{2}\rho(L_1, u) + \frac{1}{2}\rho(L_1, -u)\|_n \cdots n^{-1/n}\|\frac{1}{2}\rho(L_n, u) + \frac{1}{2}\rho(L_n, -u)\|_n\\
&\le n^{-1/n}\|\rho(L_1, u)\|_n \cdots n^{-1/n}\|\rho(L_n, u)\|_n\\
&= V^{\frac{1}{n}}(L_1)\cdots V^{\frac{1}{n}}(L_n).
\end{aligned}$$

In view of the equality conditions of Hölder integral inequality and Minkowski integral inequality, equality of inequality (1.6) holds if and only if $L_1, \ldots, L_n$ are dilates and centered. Thus we obtain the conclusion. □

Proof of Theorem 2. For $L_1, \ldots, L_n \in \mathcal{S}_o^n$ and $-\infty \le p \le 1$. By combining Theorem 1 with Proposition 1 we obtain

$$\begin{aligned}
B_p^n(L_1, \ldots, L_n) &\le B_1^n(L_1, \ldots, L_n)\\
&\le V(L_1)\cdots V(L_n).
\end{aligned}$$

In view of the equality conditions of Theorem 1 and Proposition 1, equality holds if and only if $L_1, \ldots, L_n$ are n-balls. Thus we obtain the conclusion. □

In order to prove Theorem 3 and Theorem 4 in the introduction, we require the following simple extension of Hölder's inequality.

Lemma 1 (see [7]). *If $f_0, f_1, \dots, f_m$ are (strictly) positive continuous functions defined on S^{n-1} and $\lambda_1, \dots, \lambda_m$ are positive constants the sum of whose reciprocals is unity, then*

$$\int_{S^{n-1}} f_0(u) f_1(u) \cdots f_m(u) dS(u) \leq \prod_{i=1}^{m} \left[\int_{S^{n-1}} f_0(u) f_i^{\lambda_i}(u) dS(u) \right]^{1/\lambda_i},$$

with equality if and only if there exist positive constants $\alpha_1, \dots, \alpha_m$ such that $\alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u)$ for all $u \in S^{n-1}$.

Proof of Theorem 3. For $L_1, \dots, L_n \in \mathcal{S}_o^n$. Let

$$\lambda_i = m \ (1 \leq i \leq m),$$
$$f_0 = d(L_1, u) \cdots d(L_{n-m}, u) \ (f_0 = 1 \text{ if } m = n),$$
$$f_i = d(L_{n-i+1}, u) \ (1 \leq i \leq m).$$

Using Lemma 1, we have

$$\int_{S^{n-1}} d(L_1, u) \cdots d(L_n, u) dS(u)$$
$$\leq \prod_{i=1}^{m} \left[\int_{S^{n-1}} d(L_1, u) \cdots d(L_{n-m}, u) d(L_{n-i+1}, u)^m dS(u) \right]^{1/\lambda_i},$$

with equality if and only if $L_{n-m+1}, L_{n-m+2}, \dots, L_n$ are all of similar chord. Thus we obtain the conclusion. □

Proof of Theorem 4. For $L_1, \dots, L_n \in \mathcal{S}_o^n$. Let

$$\lambda_i = m \ (1 \leq i \leq m),$$
$$f_0 = d(L_1, u)^p \cdots d(L_{n-m}, u)^p \ (f_0 = 1 \text{ if } m = n),$$
$$f_i = d(L_{n-i+1}, u)^p \ (1 \leq i \leq m).$$

Using Lemma 1, we have

$$\int_{S^{n-1}} d(L_1, u)^p \cdots d(L_n, u)^p dS(u)$$
$$\leq \prod_{i=1}^{m} \left[\int_{S^{n-1}} d(L_1, u)^p \cdots d(L_{n-m}, u)^p d(L_{n-i+1}, u)^{pm} dS(u) \right]^{1/\lambda_i},$$

with equality if and only if $L_{n-m+1}, L_{n-m+2}, \dots, L_n$ are all of similar chord.

For $p > 0$, we get

$$\omega_n \left[\frac{1}{n\omega_n} \int_{S^{n-1}} d(L_1, u)^p \cdots d(L_n, u)^p dS(u) \right]^{1/p}$$
$$\leq \omega_n \prod_{i=1}^{m} \left[\frac{1}{n\omega_n} \int_{S^{n-1}} d(L_1, u)^p \cdots d(L_{n-m}, u)^p d(L_{n-i+1}, u)^p dS(u) \right]^{1/p\lambda_i},$$

with equality if and only if $L_{n-m+1}, L_{n-m+2}, \dots, L_n$ are all of similar chord. For $p < 0$, inequality above is reversed.

Thus we obtain the conclusion. □

3. Dual general Bieberbach inequality

In [8] Lutwak established a general Bieberbach inequality which has the Bieberbach, Urysohn and harmonic Urysohn inequalities as special cases. Following Lutwak, we give a dual general Bieberbach inequality.

If $L \in \mathcal{S}_o^n$ and $i \in \mathbb{R}$, then the i-th dual quermassintegrals is defined by Lutwak (see [7])

$$\tilde{W}_i(L) = \frac{1}{n} \int_{S^{n-1}} \rho(L,u)^{n-i} dS(u). \tag{3.1}$$

Specifically, $\tilde{W}_0(K) = V(K)$, and $\tilde{W}_n(K) = \omega_n$.

For $L \in \mathcal{S}_o^n$, the intersection body of L, IL is the centrally symmetric body whose radial function on S^{n-1} is given by (see [11]),

$$\rho(IL,u) = \frac{1}{n-1} \int_{S^{n-1} \cap u^\perp} \rho(L,v)^{n-1} d\lambda_{n-2}(v), \tag{3.2}$$

where λ_{n-2} denotes $(n-2)$-dimensional Lebesgue measure. For $u \in S^{n-1}$, $L \cap u^\perp$ denotes the intersection of L with the subspace $u^\perp$ that passes through the origin and is orthogonal to u.

For $L_1, L_2 \in \mathcal{S}_o^n$, the radial addition $L_1 \tilde{+} L_2$ is defined as the star body whose radial function is given by (see [11]),

$$\rho(L_1 \tilde{+} L_2, \cdot) = \rho(L_1, \cdot) + \rho(L_2, \cdot). \tag{3.3}$$

The radial Blaschke linear addition, $\lambda \cdot K \check{+} \mu \cdot L$, is defined by Lutwak (see [11]), whose radial function satisfies for $u \in S^{n-1}$ (see [11])

$$\rho(\lambda \cdot K \check{+} \mu \cdot L, u)^{n-1} = \lambda \rho(K,u)^{n-1} + \mu \rho(L,u)^{n-1}.$$

The following properties will be used later: If $L_1, L_2 \in \mathcal{S}_o^n$ and $\lambda, \mu > 0$, then

$$d(L_1 \tilde{+} L_2, \cdot) = d(L_1, \cdot) + d(L_2, \cdot), \tag{3.4}$$

$$I(\lambda \cdot K \check{+} \mu \cdot L) = \lambda IK \tilde{+} \mu IL. \tag{3.5}$$

For $K \in \mathcal{K}^n$ and a real number $p \neq 0$, the p-width, $\bar{b}_p(K)$ of K was defined by Lutwak in ([8])

$$\bar{b}_p(K) = \left[\frac{1}{n\omega_n} \int_{S^{n-1}} b^p(K,u) dS(u) \right]^{1/p}, \tag{3.6}$$

where the definition (3.6) differs slightly from that of Lutwak (see [8]) in that we multiply a constant factor. For p equal to $-\infty$, 0 or ∞ the p-width of K was defined by

$$\bar{b}_p(K) = \lim_{s \to p} \bar{b}_s(K).$$

In order to establish the dual general Bieberbach inequality, we shall also introduce the dual concept of the p-width of convex body: For $L \in \mathcal{S}_o^n$ and a real number $p \neq 0$, the p-chord, $\tilde{d}_p(L)$ of L was defined by

$$\tilde{d}_p(L) = \left[\frac{1}{n\omega_n}\int_{S^{n-1}} d^p(L,u)dS(u)\right]^{1/p}. \tag{3.7}$$

For p equal to $-\infty$, 0 or ∞ the p-chord of L was defined by

$$\tilde{d}_p(L) = \lim_{s\to p} \tilde{d}_s(L).$$

For a fixed p, the p-chord is a map

$$\tilde{d}_p : \mathcal{S}_o^n \longrightarrow \mathbb{R}.$$

Similar to the mixed chord-integral, it is positive, continuous, bounded, homogeneous of degree one and monotone under set inclusion.

From Theorem 5 and Theorem 5* we can obtain the relation between the dual general Bieberbach inequality and the general Bieberbach inequality as follows.

Corollary 6. *If $K \in \mathcal{K}^n$ and $-n < p < n$, then*

$$\tilde{d}_p(K) \le \bar{b}_p(K),$$

with equality if and only if K is an n-ball.

We shall prove the following relation between the p-chord of convex body and the p-chord of polar for convex body.

Theorem 7. *If $K \in \mathcal{K}_o^n$ and $-\infty \le p < n$, then*

$$\tilde{d}_p(K)\tilde{d}_p(K^*) \le 1,$$

with equality if and only if K is an n-ball.

Theorem 7 is just the dual of the following relation between the p-width of convex body and the p-width of polar for convex body which was shown by Lutwak (see [8]).

Theorem 7*. *If $K \in \mathcal{K}^n$ and $-n < p \le \infty$, then*

$$\bar{b}_p(K)\bar{b}_p(K^*) \ge 1,$$

with equality if and only if K is an n-ball.

Furthermore, we establish the following Brunn-Minkowski inequality for the p-chord of star bodies.

Theorem 8. *For $L_1, L_2 \in \mathcal{S}_o^n$, $p \ge 1$ and $\alpha \in [0,1]$, then*

$$\begin{aligned} \tilde{d}_p(L_1\tilde{+}L_2) &\le \tilde{d}_p(\alpha L_1\tilde{+}(1-\alpha)L_2) + \tilde{d}_p((1-\alpha)L_1\tilde{+}\alpha L_2) \\ &\le \tilde{d}_p(L_1) + \tilde{d}_p(L_2), \end{aligned} \tag{3.8}$$

in each inequalities, with equality if and only if L_1 and L_2 have similar chord. If $p < 1$ and $p \neq 0$, then inequality (3.8) *is reversed.*

Proof of Theorem 5. For $L \in \mathcal{S}_o^n$, taking $p = n$, from definition (3.7), definition (2.5), Minkowski integral inequality (see [4]) and formula (2.4), we get

$$\begin{aligned} \tilde{d}_n(L) &= \left[\frac{1}{n\omega_n}\int_{S^{n-1}} d^n(L,u)dS(u)\right]^{1/n} \\ &= \left[\frac{1}{n\omega_n}\int_{S^{n-1}} [\frac{1}{2}\rho(L,u) + \frac{1}{2}\rho(L,-u)]^n dS(u)\right]^{1/n} \\ &\leq \omega_n^{-1/n}\left[\frac{1}{n}\int_{S^{n-1}} \rho^n(L,u)dS(u)\right]^{1/n} \\ &= \omega_n^{-1/n} V(L)^{1/n}, \end{aligned} \tag{3.9}$$

with equality if and only if L is centered. From Jensen's inequality (see [4]), it follows that for $-\infty \leq p < n$

$$\tilde{d}_n(L) \geq \tilde{d}_p(L), \tag{3.10}$$

with equality if and only if L is of constant chord.

Combing these inequalities above, we have

$$V(L) \geq \omega_n \tilde{d}_p(L)^n,$$

with equality if and only if L is an n-ball. □

Proof of Theorem 7. For $K \in \mathcal{K}_o^n$, combing inequality (3.9) with Blaschke-Santaló inequality, we obtain

$$\omega_n \tilde{d}_n(K)^{-n} \geq V(K^*), \tag{3.11}$$

with equality if and only if L is an n-ball. By combing inequality (3.10) and inequality (3.11), we have for $-\infty \leq p < n$,

$$\omega_n \tilde{d}_p(K)^{-n} \geq V(K^*), \tag{3.12}$$

with equality if and only if L is an n-ball.

According to the inequality (3.11), inequality (3.12) and the dual general Bieberbach inequality, we obtain the desired result. □

Proof of Theorem 8. For $L_1, L_2 \in \mathcal{S}_o^n$, $p \geq 1$ and $\alpha \in [0,1]$. From definition (3.7) and formula (3.4), Minkowski integral inequality, definition (3.7) and (3.4) again, Minkowski integral inequality again, definition (3.7) again, it follows that

$$\begin{aligned} \tilde{d}_p(L_1 \tilde{+} L_2) &= (n\omega_n)^{-1/p}\|d(L_1 \tilde{+} L_2, u)\|_p \\ &= (n\omega_n)^{-1/p}\|d(L_1,u) + d(L_2,u)\|_p \end{aligned}$$

$$\begin{aligned}
&= (n\omega_n)^{-1/p}\|[\alpha d(L_1,u) + (1-\alpha)d(L_2,u)] \\
&\quad + [(1-\alpha)d(L_1,u) + \alpha d(L_2,u)]\|_p \\
&\le (n\omega_n)^{-1/p}\|\alpha d(L_1,u) + (1-\alpha)d(L_2,u)\|_p \\
&\quad + (n\omega_n)^{-1/p}\|(1-\alpha)d(L_1,u) + \alpha d(L_2,u)]\|_p \\
&= \tilde{d}_p(\alpha L_1 \tilde{+} (1-\alpha)L_2) + \tilde{d}_p((1-\alpha)L_1 \tilde{+} \alpha L_2) \\
&\le (n\omega_n)^{-1/p}\|\alpha d(L_1,u)\|_p + (n\omega_n)^{-1/p}\|(1-\alpha)d(L_2,u)\|_p \\
&\quad + (n\omega_n)^{-1/p}\|(1-\alpha)d(L_1,u)\|_p + (n\omega_n)^{-1/p}\|\alpha d(L_2,u)\|_p \\
&= \tilde{d}_p(L_1) + \tilde{d}_p(L_2),
\end{aligned}$$

in each inequalities, with equality if and only if L_1 and L_2 have similar chord. In view of the inverse Minkowski integral inequality, similar above the proof, the cases of $p < 1$ and $p \neq 0$ can also be proved. Here we omit the details, i.e., if $p < 1$ and $p \neq 0$, then this inequality is reversed. □

Proof of Theorem 6. For $L_1, L_2 \in \mathcal{S}_o^n$ and $i \le n-1$. Since $I(L_1 \breve{+} L_2)$, IL_1 and IL_2 are centered, then

$$d(I(L_1 \breve{+} L_2), u) = \rho(I(L_1 \breve{+} L_2), u),$$

$$d(IL_1, u) = \rho(IL_1, u),\ d(IL_2, u) = \rho(IL_2, u).$$

Combing Theorem 8 with definition (3.1) and equalities above, we have

$$\tilde{W}_i(I(L_1 \breve{+} L_2))^{1/(n-i)} \le \tilde{W}_i(IL_1)^{1/(n-i)} + \tilde{W}_i(IL_2)^{1/(n-i)},$$

with equality if and only if L_1 and L_2 are dilates. If $n-1 < i < n$ or $i > n$, then this inequality is reversed. □

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