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# UPPER TRIANGULAR OPERATORS WITH SVEP

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Abstract. A Banach space operator $A \in B(\mathcal{X})$ is polaroid if the isolated points of the spectrum of $A$ are poles of the resolvent of $A$; $A$ is hereditarily polaroid, $A \in (\mathcal{HP})$, if every part of $A$ is polaroid. Let $\mathcal{X}^n = \bigoplus_{i=1}^n \mathcal{X}_i$, where $\mathcal{X}_i$ are Banach spaces, and let $\mathcal{A}$ denote the class of upper triangular operators $A = (A_{ij})_{1\leq i,j\leq n}$, $A_{ij} \in B(\mathcal{X}_j, \mathcal{X}_i)$ and $A_{ij} = 0$ for $i > j$. We prove that operators $A \in \mathcal{A}$ such that $A_{ii}$ for all $1 \leq i \leq n$, and $A^*$ have the single–valued extension property have spectral properties remarkably close to those of Jordan operators of order $n$ and $n$-normal operators. Operators $A \in \mathcal{A}$ such that $A_{ii} \in (\mathcal{HP})$ for all $1 \leq i \leq n$ are polaroid and have SVEP; hence they satisfy Weyl's theorem. Furthermore, $A + R$ satisfies Browder's theorem for all upper triangular operators $R$, such that $\oplus_{i=1}^n R_{ii}$ is a Riesz operator, which commutes with $A$.

## 1. Introduction

Let $B(\mathcal{H})$ denote the algebra of operators on a separable infinite dimensional Hilbert space $\mathcal{H}$, and let $n \in \mathbb{N}$ be a fixed natural number. An operator $A \in B(\mathcal{H})$ is a $\mathcal{C}_n = \mathcal{C}_n(\mathcal{H})$ operator if there exists a decomposition $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$ and an $n \times n$ upper triangular operator matrix $A = (A_{ij})_{1\leq i,j\leq n}$, $A_{ij} \in B(\mathcal{H}_j, \mathcal{H}_i)$ and $A_{ij} = 0$ for all $i > j$, such that $A_{ii}$ is normal for all $1 \leq i \leq n$. $\mathcal{C}_n$ operators have recently been considered by Jung, Ko, and Pearcy in [10], where it is proved that the classes consisting of Jordan operators of order $n$ and $n$–normal operators are subclasses of $\mathcal{C}_n$. $\mathcal{C}_n$ operators share a number of spectral properties with normal operators. More precisely, one has [10, Theorem 2.3 and Lemma 2.4]:

**Theorem 1.1.** *If $A \in \mathcal{C}_n$, then $\sigma(A) = \sigma_a(A) = \cup_{i=1}^n \sigma(A_{ii}) = \sigma_e(A) \cup \sigma_p(A)$, $\sigma_e(A) = \cup_{i=1}^n \sigma_e(A_{ii}) = \sigma_{le}(A) \cap \sigma_{re}(A) = acc\sigma(A) \cup \sigma_p^\infty(A)$ and $\sigma_p(A)$ is at most countable. Furthermore, every hole in $\sigma_e(A)$ has associated Fredholm index zero and intersects $\sigma(A)$ in at most countably many points.*

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Here, and in the sequel, $\sigma_a(A)$ denotes the approximate point spectrum, $\sigma_p(A)$ the eigenvalues, $\sigma_p^\infty(A)$ the set of eigenvalues of infinite multiplicity, $\sigma_{le}(A)$ and $\sigma_{re}(A)$ the upper semi–Fredholm and the lower semi–Fredholm spectrum (respectively), $\sigma_e(A)$ the Fredholm spectrum, and acc$\sigma(A)$ the set of accumulation points of the spectrum of $A$. Recall that a hole in $\sigma_e(A)$ is a bounded component of $\mathbb{C} \setminus \sigma_e(A)$ [13].

A part of a Banach space operator $A \in B(\mathcal{X})$ is its restriction to a closed invariant subspace. The operator $A$ is polaroid, [8], if the isolated points $\lambda$ of $\sigma(A)$, $\lambda \in \text{iso}\sigma(A)$, are poles of the resolvent of $A$ (i.e., if $\lambda \in \text{iso}\sigma(A)$ implies $\text{asc}(A-\lambda) = \text{dsc}(A-\lambda) < \infty$, where the ascent $\text{asc}(A-\lambda)$ of $A-\lambda$ is the least non-negative integer $n$ such that $(A-\lambda)^{-n}(0) = (A-\lambda)^{(n+1)}(0)$ and the descent $\text{dsc}(A-\lambda)$ of $A-\lambda$ is the least non-negative integer $n$ such that $(A-\lambda)^n\mathcal{X} = (A-\lambda)^{n+1}\mathcal{X})$. We say that $A \in B(\mathcal{X})$ is hereditarily polaroid, $A \in (\mathcal{HP})$, if every part of $A$ is polaroid [6]. Class $(\mathcal{HP})$ is large: it contains a number of the more commonly considered classes of operators, amongst them subscalar operators, multipliers of commutative semi–simple Banach algebras and paranormal operators (See [6] for an extensive, though by no means exhaustive, list of classes of operators which are $(\mathcal{HP})$ operators). Just like normal operators, $(\mathcal{HP})$ operators have SVEP, the single–valued extension property [6, Theorem 2.8]; however, unlike normal operators, the conjugate operator $A^*$ of an operator $A \in (\mathcal{HP})$ does not (necessarily) have SVEP.

Let $\mathcal{X}^n = \bigoplus_{i=1}^n \mathcal{X}_i$, where each $\mathcal{X}_i$, $1 \le i \le n$, is an infinite dimensional Banach space. Let $\mathcal{A}$ denote the class of upper triangular operator matrices

$$A = (A_{ij})_{1\le i,j\le n} \in B(\mathcal{X}^n), \;\; A_{ij} \in B(\mathcal{X}_j, \mathcal{X}_i) \;\text{ and }\; A_{ij} = 0 \;\text{ for all }\; i > j,$$

and let $(HP)_n$ denote the class of operators $A \in \mathcal{A}$ such that $A_{ii} \in (\mathcal{HP})$ for all $1 \le i \le n$. We study operators $A \in (HP)_n$, and operators $A \in \mathcal{A}$ such that $A_{ii}$ has SVEP for all $1 \le i \le n$. It is proved that operators $A \in \mathcal{A}$ such that each $A_{ii}$ has SVEP have spectral properties remarkably similar to those of operators $A \in \mathcal{C}_n$. Indeed, if $A \in \mathcal{A}$ is such that $A^*$ along with each $A_{ii}$ has SVEP, then $A$ shares with operators in $\mathcal{C}_n$ the spectral properties of Theorem 1.1 (except for the countability properties). We prove that if the Banach spaces $\mathcal{X}_i$ are separable and the operators $A_{ii}$ are hereditarily normaloid $(\mathcal{HP})$ operators (see the following sections for any undefined terminology), then $\sigma_p(A)$ is at most countable. Operators $A \in (HP)_n$ are polaroid and have SVEP. It is proved that if $A \in \mathcal{A}$ and $A_0 = \bigoplus_{i=1}^n A_{ii}$ is a polynomially $(HP)_n$ operator which commutes with the diagonal algebraic operator $B = \bigoplus_{i=1}^n B_{ii} \in B(\mathcal{X}^n)$, then $f(A+B)$ satisfies Weyl's theorem and $f(A^*+B^*)$ satisfies $a$–Weyl's theorem for every function $f$ which is analytic on a neighbourhood of $\sigma(A+B)$. Perturbation of operators $A \in \mathcal{A}$ such that $A_{ii}$ has SVEP for all $1 \le i \le n$ by quasi–nilpotent operators is considered.

## 2. Upper triangular operators with SVEP

An operator $A \in B(\mathcal{X})$ is upper semi–Fredholm (lower semi–Fredholm) at

$\lambda \in \mathbb{C}$, $\lambda \in \Phi_+(A)$ (resp., $\lambda \in \Phi_-(A)$), if $(A-\lambda)\mathcal{X}$ is closed and $\alpha(A-\lambda) = \dim(A-\lambda)^{-1}(0) < \infty$ (resp., $(A-\lambda)\mathcal{X}$ is closed and $\beta(A-\lambda) = \dim(\mathcal{X} \setminus (A-\lambda)\mathcal{X}) < \infty$). The upper semi–Fredholm spectrum (resp., lower semi–Fredholm spectrum) of $A$ is the set $\sigma_{le}(A) = \{\lambda \in \mathbb{C} : \lambda \notin \Phi_+(A)\}$ (resp., $\sigma_{re}(A) = \{\lambda \in \mathbb{C} : \lambda \notin \Phi_-(A)\}$), the semi–Fredholm spectrum of $A$ is the set $\sigma_{sF}(A) = \sigma_{le}(A) \cap \sigma_{re}(A)$, and the Fredholm spectrum of $A$ is the set $\sigma_e(A) = \sigma_{le}(A) \cup \sigma_{re}(A) = \{\lambda \in \sigma(A) : \lambda \notin \Phi(A) = \Phi_+(A) \cap \Phi_-(A)\}$. The Weyl spectrum (resp., Browder spectrum) of $A$ is the set $\sigma_w(A) = \{\lambda \in \mathbb{C} : A-\lambda$ is not Fredholm or $\mathrm{ind}(A-\lambda) \neq 0\}$ (resp., $\sigma_b(A) = \{\lambda \in \mathbb{C} : A-\lambda$ is not Fredholm or $\mathrm{asc}(A-\lambda) \neq \mathrm{dsc}(A-\lambda)\}$. Here $\mathrm{ind}(A-\lambda) = \alpha(A-\lambda) - \beta(A-\lambda)$ is the Fredholm index of $A-\lambda$. Let $\pi_{00}(A) = \{\lambda \in \mathrm{iso}\sigma(A) : 0 < \alpha(A-\lambda) < \infty\}$, and let $p_0(A) = \{\lambda \in \mathrm{iso}\sigma(A) : \lambda$ is a finite rank pole of the resolvent of $A\}$. In common with current terminology, we say that $A$ satisfies Weyl's theorem, shortened to $A$ satisfies $Wt$, if $\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A)$; $A$ satisfies Browder's theorem, or $Bt$, if $\sigma_w(A) = \sigma_b(A)$ ($\Longleftrightarrow \sigma(A) \setminus \sigma_w(A) = p_0(A)$).

An operator $A \in B(\mathcal{X})$ has the single-valued extension property at a point $\lambda_0 \in \mathbb{C}$, SVEP at $\lambda_0$, if for every open disc $\mathcal{D}$ centered at $\lambda_0$ the only analytic function $f : \mathcal{D} \longrightarrow \mathcal{X}$ satisfying $(A-\lambda)f(\lambda) = 0$ is the function $f \equiv 0$. The single valued extension property plays an important role in local spectral theory and Fredholm theory (see [11] and [1]). Evidently, every $A$ has SVEP at points in the resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and the boundary $\partial\sigma(A)$ of $\sigma(A)$. It is easily verified that SVEP is inherited by restrictions. We say that $A$ has SVEP if it has SVEP at every $\lambda \in \sigma(A)$.

**Lemma 2.1.** *If $\lambda \in \Phi_{\pm}(A)$, then the following implications hold*:

$$A \ \ (\textit{resp.}, A^*) \ \ \textit{has SVEP at} \ \ \lambda \Longrightarrow \textit{asc}(A-\lambda) < \infty \ \ (\textit{resp.}, \textit{dsc}(A-\lambda) < \infty);$$

$$\textit{asc}(A-\lambda) < \infty \Longrightarrow \textit{ind}(A-\lambda) \leq 0 \ \ (\textit{resp.}, \textit{dsc}(A-\lambda) < \infty \Longrightarrow \textit{ind}(A-\lambda) \geq 0).$$

*Proof.* See [1, Theorems 3.16, 3.17 and Corollary 3.19]. □

The following theorem is known (see, for example, [2, Lemma 2.18] and [3, Theorem 2.2]).

**Theorem 2.2.** *Let $A \in B(\mathcal{X})$.*

(i) *A necessary and sufficient condition for $A$ to satisfy Browder's theorem, $\sigma(A) \setminus \sigma_w(A) = p_0(A)$ (equivalently, $\sigma_b(A) = \sigma_w(A)$), is that $A$ has SVEP at points $\lambda \notin \sigma_w(A)$.*

(ii) *A necessary and sufficient condition for $A$ to satisfy Weyl's theorem is that $A$ satisfies Browder's theorem and $A$ is polaroid at points $\lambda \in \pi_{00}(A)$.*

It is well known that $A$ satisfies Browder's theorem if and only if $A^*$ satisfies Browder's theorem: this, however, fails for Weyl's theorem.

The quasinilpotent part $H_0(A)$ and the analytic core $K(A)$ of an operator $A \in B(\mathcal{X})$ are the sets

$$H_0(A) = \{x \in \mathcal{X} : \lim_{n \longrightarrow \infty} ||A^n x||^{\frac{1}{n}} = 0\}$$

and

$$K(A) = \{x \in \mathcal{X} : \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which } x = x_0, Ax_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n \in \mathbb{N}\}$$

[1, 11]. $H_0(A)$ and $K(A)$ are generally non-closed hyperinvariant subspaces of $A$ such that $A^{-m}(0) \subseteq H_0(A)$ for all $m \in \mathbb{N}$ and $AK(A) = K(A)$ [12]. If $\lambda \in \text{iso}\sigma(A)$, then

$$\mathcal{X} = H_0(A - \lambda) \oplus K(A - \lambda)$$

[1, Theorem 3.74]; furthermore, if also $\lambda \in \Phi_{\pm}(A)$, then, [1, Theorem 3.77], $H_0(A - \lambda) = (A - \lambda)^{-p}(0)$ for some $p \in \mathbb{N}$ and

$$\mathcal{X} = (A - \lambda)^{-p}(0) \oplus (A - \lambda)^p \mathcal{X}$$

(so that $\lambda$ is a finite rank pole of the resolvent of $A$).

Operators $A \in (\mathcal{HP})$ have SVEP [6, Theorem 2.8], and SVEP survives quasi–affine transforms. Indeed, if $SX = XT$, where $S \in B(\mathcal{X})$, $X \in B(\mathcal{Y}, \mathcal{X})$ is injective and $T \in B(\mathcal{Y})$, then $(T - \lambda)f(\lambda) = 0 \Longrightarrow (S - \lambda)Xf(\lambda) = 0$: hence if $S$ has SVEP at a point $\lambda_0$ and $\lambda$ is a point in an open disc centered at $\lambda_0$, then $Xf(\lambda) \equiv 0 \Longrightarrow f(\lambda) \equiv 0$. The polaroid property survives similarities, but it does not survive quasi–affine transforms, as the following example shows. Let $T \in \ell^2(\mathbb{N})$ be the forward unilateral shift, let $S \in \ell^2(\mathbb{N})$ be the weighted forward unilateral shift with the weight sequence $\{\frac{1}{n+1}\}$, and let $X \in \ell^2(\mathbb{N})$ be the multiplication operator defined by $Xx = \{\frac{x_n}{n!}\}_{n \in \mathbb{N}}$ for all $x = \{x_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$. Then $T \in (\mathcal{HP})$, $X$ is a quasi–affinity and $SX = XT$, but $S$ is quasi–nilpotent (hence not polaroid).

Operators $A \in \mathcal{A}$ such that $A_{ii}$ has SVEP for all $1 \leq i \leq n$ have SVEP [4, Lemma 3.1]. Furthermore:

**Theorem 2.3.** *If $A \in \mathcal{A}$ is such that $A_{ii}$ has SVEP for all $1 \leq i \leq n$, then:*

(i) $\sigma(A) = \sigma_a(A^*) = \bigcup_{i=1}^n \sigma_a(A_{ii}^*) = \bigcup_{i=1}^n \sigma(A_{ii}) = \sigma_w(A) \cup \pi_{00}(A) = \sigma_b(A) \cup \pi_{00}(A)$;

(ii) $\sigma_e(A) = \sigma_{re}(A) = \bigcup_{i=1}^n \sigma_{re}(A_{ii}) = \bigcup_{i=1}^n \sigma_e(A_{ii})$;

(iii) $\sigma_{sF}(A) = \sigma_{le}(A) = acc\sigma_a(A) \cup \{\lambda \in iso\sigma_a(A) : \dim H_0(A - \lambda) = \infty$ *or* $(A - \lambda)\mathcal{X}^n$ *is not closed*$\}$;

(iv) $\sigma_w(A) = \bigcup_{i=1}^n \sigma_w(A_{ii})$, *and* $\sigma_w(A) = \sigma_e(A)$ *if and only if* $A^*$ *has SVEP at points* $\lambda \in \sigma_w(A) \setminus \sigma_e(A)$;

(v) *if $C$ is a hole in $\sigma_e(A)$, then, for all $\lambda \in C$, either ind$(A - \lambda) = 0$ and $C \cap \rho(A) \neq \emptyset$ or ind$(A - \lambda) < 0$ and the eigenvalues do not have a limit point in $C$ (and every point of $C$ is a deficiency value).*

*Furthermore, if also $A^*$ has SVEP, then:*

(vi) $\sigma_e(A) = \sigma_{le}(A) \cap \sigma_{re}(A) = \sigma_{sF}(A) = \sigma_w(A)$ *and* $\sigma(A) = \sigma_a(A) = \sigma_e(A) \cup \pi_{00}(A)$;

(vii) *for every hole $C$ in $\sigma_e(A)$ and $\lambda \in C$, ind$(A - \lambda) = 0$ and $C \cap \rho(A) \neq \emptyset$.*

*Proof.* A proof of parts (i) to (iv) of the theorem, except for the equality $\sigma(A) = \sigma_w(A) \cup \pi_{00}(A) = \sigma_b(A) \cup \pi_{00}(A)$ of part (i), the equality $\sigma_{sF}(A) = \sigma_{le}(A)$ of part (iii), and the necessary and sufficient condition for the equality of the spectra $\sigma_w(A)$ and $\sigma_e(A)$ of part (iv), appears in [4, Proposition 3.5]. We divide the proof into parts (**A**) and (**B**), with (**A**) completing the proof of parts (i) to (v), and (**B**) proving parts (vi) and (vii).

(**A**). Start by observing that $A$ has SVEP implies $\sigma_w(A) = \sigma_b(A)$. Evidently, $\sigma_b(A) \cup \pi_{00}(A) \subseteq \sigma(A)$. (Recall that $\pi_{00}(A) = \{\lambda \in \text{iso}\sigma(A) : 0 < \alpha(A-\lambda) < \infty\}$.) For the reverse inclusion, if $\lambda \in \sigma(A) \setminus \sigma_b(A)$, then $\lambda \in \Phi(A) = \Phi_+(A) \cap \Phi_-(A)$ and $\text{asc}(A-\lambda) = \text{dsc}(A-\lambda) < \infty$. Hence $\lambda$ is a finite rank pole of the resolvent of $A$. Thus $\lambda \in \pi_{00}(A)$ and $\sigma(A) \subseteq \sigma_b(A) \cup \pi_{00}(A)$. To complete the proof of (iii), let $\lambda \notin \sigma_{re}(A)$. Then $\lambda \in \Phi_-(A)$. Since $A$ has SVEP, $\text{ind}(A-\lambda) = \alpha(A-\lambda) - \beta(A-\lambda) \leq 0$. Thus $\lambda \notin \sigma_{re}(A) \Longrightarrow (A-\lambda)\mathcal{X}^n$ is closed and $\alpha(A-\lambda) \leq \beta(A-\lambda) < \infty$. Hence $\lambda \notin \sigma_{le}(A)$, which implies that $\sigma_{le}(A) \subseteq \sigma_{re}(A)$. Thus $\sigma_{sF}(A) = \sigma_{le}(A) \cap \sigma_{re}(A) = \sigma_{le}(A)$.

To complete the proof of (iv), we start by observing that $A$ has SVEP implies $A$ satisfies Browder's theorem, which in turn implies that $A^*$ satisfies Browder's theorem, i.e., $A^*$ has SVEP at $\lambda \notin \sigma_w(A^*) = \sigma_w(A)$. Thus if $\sigma_w(A) = \sigma_e(A)$, then (vacuously) $A^*$ has SVEP at points $\lambda \in \sigma_w(A) \setminus \sigma_e(A)$. Conversely, if $A^*$ has SVEP at points $\lambda \in \sigma_w(A) \setminus \sigma_e(A)$, then $A^*$ has SVEP at all $\lambda \in \Phi(A^*) = \Phi(A)$. This, since $A$ has SVEP, implies that if $\lambda \in \Phi(A)$, then $\text{ind}(A-\lambda) = 0$. Hence $\sigma_w(A) \subseteq \sigma_e(A)$. Since $\sigma_e(A) \subseteq \sigma_w(A)$ for every $A$, $\sigma_w(A) = \sigma_e(A)$.

To prove (v), we observe that if $C$ is a hole in $\sigma_e(A)$ and $\lambda \in C$, then $\lambda \in \Phi(A)$ and (because of SVEP) $\text{ind}(A-\lambda) \leq 0$. If $\text{ind}(A-\lambda) = 0$, then (both $A$ and $A^*$ have SVEP at $\lambda$ and) $\text{asc}(A-\lambda) = \text{dsc}(A-\lambda) < \infty$; if, instead, $\text{ind}(A-\lambda) < 0$, then ($A$ has SVEP at $\lambda$ implies) $\text{asc}(A-\lambda) < \infty$. Now apply [9, Theorem 51.1 $(a_1)$, $(b_1)$].

(**B**). Observe that if $A^*$ has SVEP, then $\sigma(A) = \sigma_a(A)$ [11, Proposition 1.3.2]. Thus, if both $A$ and $A^*$ have SVEP, then (see (iv)) $\sigma(A) = \sigma_a(A) = \sigma_e(A) \cup \pi_{00}(A)$. Again, if $A^*$ has SVEP, then $\lambda \notin \sigma_{le}(A)$ implies that $\lambda \in \Phi_+(A)$ and $\text{ind}(A-\lambda) \geq 0 \Longrightarrow \lambda \in \Phi(A)$ (and $\text{ind}(A-\lambda) \geq 0$). Thus $\lambda \notin \sigma_{re}(A)$, which in view of the fact that $\sigma_{le}(A) \subseteq \sigma_{re}(A)$, see the proof above, implies that $\sigma_{le}(A) = \sigma_{re}(A)$. This proves (vi). The proof of (vii) follows from [9, Theorem 51.1]. □

It is apparent from the theorem above that a number of the properties of operators $A \in \mathcal{C}_n$ proved in Theorem 1.1 are typical of operators $A \in \mathcal{A}$ such that $A_{ii}$, $1 \leq i \leq n$, and $A^*$ have SVEP. The following theorem supplements Theorem 2.3 to prove that if $\mathcal{X}$ is separable and $A \in \mathcal{A}$, then $\sigma_p(A)$ is countable for certain classes of operators $A_{ii}$.

Recall that an operator $T \in B(\mathcal{X})$ is normaloid if its norm equals its spectral radius. We say that $T$ is hereditarily normaloid if every part of $T$ is normaloid.

We shall call an operator simply polaroid if the isolated points of its spectrum are simple poles, i.e., poles of order one, of its resolvent.

**Theorem 2.4.** *Let $\mathcal{X}$ be a separable Banach space, and let $T \in B(\mathcal{X})$ be a hereditarily normaloid ($\mathcal{HP}$) operator. Then $\sigma_p(T)$ is countable.*

*Proof.* Let $\alpha$, $\beta \in \sigma_p(T)$, where $\beta \neq 0$, $\alpha \neq \beta$ and $|\alpha| \leq |\beta|$. Let $M$ denote the subspace generated by $(T-\alpha)^{-1}(0)$ and $(T-\beta)^{-1}(0)$. Then $T_1 = T|_M$ is a normaloid ($\mathcal{HP}$) operator with $\sigma(T_1) = \{\alpha, \beta\}$, and $||T_1|| = |\beta|$. Furthermore, $\mathrm{asc}(T_1 - \beta) \leq 1$ [9, Proposition 54.2], which implies that $\beta$ is a simple pole of the resolvent of $T_1$. Let $P_\beta$ denote the spectral projection corresponding to $\beta$; then $||P_\beta|| = 1$ [9, Proposition 54.4]. Since

$$M = P_\beta M \oplus (1-P_\beta)M = (T_1-\beta)^{-1}(0) \oplus (T_1-\beta)M = (T-\beta)^{-1}(0) \oplus (T-\alpha)^{-1}(0),$$

it follows that

$$||x|| = ||P_\beta x|| = ||P_\beta(x+y)|| \leq ||x+y||$$

for every $x \in (T-\beta)^{-1}(0)$ and $y \in (T_1-\beta)M = (T-\alpha)^{-1}(0)$. Suppose now that $\sigma_p(T)$ is not countable. Then there exists an uncountable set of unit vectors $\{x_i\}$ and $\{y_i\}$, determined by the unit eigen–vectors corresponding to distinct eigen–values of $T$ such that

$$1 \leq ||x_i + y_i||.$$

Since $\mathcal{X}$ is separable, this is impossible. Hence $\sigma_p(T)$ is countable. □

Theorem 2.4 implies that if $\mathcal{X}$ is separable, each $A_{ii}$, $1 \leq i \leq n$, is a hereditarily normaloid ($\mathcal{HP}$) operator and $A$ is the upper triangular operator $A = (A_{ij})_{1 \leq i,j \leq n}$, then $\sigma_p(A)$ is at most countable. The hypotheses of Theorem 2.4 are satisfied by a number of classes of Banach space operators, amongst them $\mathcal{THN}$ and $\mathcal{CHN}$ operators of [2]. Suffice it to say here that paranormal operators, [9, page 229], are operators of the type considered in the theorem.

**Perturbations.** An operator $R \in B(\mathcal{X})$ is a Riesz operator if $\lambda \in \Phi(R)$ for every non–zero $\lambda \in \mathbb{C}$: equivalently, $R$ is a Riesz operator if the essential numerical radius $r_e(R) = \lim_{n \longrightarrow \infty} ||\pi(R)^n||^{\frac{1}{n}} = 0$, where $\pi : B(\mathcal{X}) \longrightarrow B(\mathcal{X})/\mathcal{K}(\mathcal{X})$ is the Calkin map and $\mathcal{K}(\mathcal{X}) \subset B(\mathcal{X})$ is the ideal of compact operators. Evidently, quasi–nilpotent operators are Riesz operators. The following theorem, the main result of this section, considers perturbations of operators $A \in \mathcal{A}$ such that $A_{ii}$ has SVEP for all $1 \leq i \leq n$ by upper triangular operators $R$ such that $[A, R] = 0$ and $R_0 = \oplus_{i=1}^n R_{ii}$ is a Riesz operator. But before that we introduce some terminology and complementary results specific to this and the following results.

For an operator $T \in B(\mathcal{X})$, let $T \in \Phi(\mathcal{X})$ (resp., $T \in \Phi_+(\mathcal{X})$, $T \in \Phi_-(\mathcal{X})$) denote that $T$ is Fredholm (resp., upper semi-Fredholm, lower semi-Fredholm).

The Weyl essential approximate point spectrum $\sigma_{wa}(T)$ and the Weyl essential surjectivity spectrum $\sigma_{ws}(T)$ of $T \in B(\mathcal{X})$ are the sets

$$\sigma_{wa}(T) = \bigcap_{K \in \mathcal{K}(\mathcal{X})} \sigma_a(T+K) = \{\lambda \in \sigma_a(T) : T-\lambda \notin \Phi_+(\mathcal{X}) \text{ or } \operatorname{ind}(T-\lambda) \not\leq 0\}$$

and

$$\sigma_{ws}(T) = \bigcap_{K \in \mathcal{K}(\mathcal{X})} \sigma_s(T+K) = \{\lambda \in \sigma_s(T) : T-\lambda \notin \Phi_-(\mathcal{X}) \text{ or } \operatorname{ind}(T-\lambda) \not\geq 0\}.$$

Evidently, $\sigma_{ws}(T) = \sigma_{wa}(T^*)$, $\sigma_{wa}(T) = \sigma_{ws}(T^*)$ and $\sigma_w(T) = \sigma_{wa}(T) \cup \sigma_{ws}(T)$. The Browder essential approximate point spectrum $\sigma_{ba}(T)$ and the Browder essential surjectivity spectrum $\sigma_{bs}(T)$ of $T \in B(\mathcal{X})$ are the sets

$$\sigma_{ba}(T) = \bigcap_{K \in \mathcal{K}(\mathcal{X}),\, KT=TK} \sigma_a(T+K) = \{\lambda \in \sigma_a(T) : T-\lambda \notin \Phi_+(\mathcal{X}) \text{ or } \operatorname{asc}(T-\lambda) = \infty\}$$

and

$$\sigma_{bs}(T) = \bigcap_{K \in \mathcal{K}(\mathcal{X}),\, KT=TK} \sigma_s(T+K) = \{\lambda \in \sigma_s(T) : T-\lambda \notin \Phi_-(\mathcal{X}) \text{ or } \operatorname{dsc}(T-\lambda) = \infty\}.$$

Apparently, $\sigma_{bs}(T) = \sigma_{ba}(T^*)$, $\sigma_{ba}(T) = \sigma_{bs}(T^*)$, $\sigma_b(T) = \sigma_{ba}(T) \cup \sigma_{bs}(T)$, $\sigma_{wa}(T) \subseteq \sigma_{ba}(T)$ and $\sigma_{ws}(T) \subseteq \sigma_{bs}(T)$. We say that $T$ satisfies $a$-Browder's theorem, or $a-Bt$ (resp., $s$-Browder's theorem, or $s-Bt$) if $\sigma_{wa}(T) = \sigma_{ba}(T)$ (resp., $\sigma_{ws}(T) = \sigma_{bs}(T)$).

**Lemma 2.5.** *$T$ satisfies $a-Bt$ (rep., $s-Bt$) if and only $T$ has SVEP at points $\lambda \notin \sigma_{wa}(T)$ (resp., $\lambda \notin \sigma_{ws}(T)$).*

*Proof.* The lemma is proved in [2, Lemma 2.18] for the case in which $T$ satisfies $a-Bt$; the case in which $T$ satisfies $s-Bt$ is similarly proved. □

**Lemma 2.6.** *If $T$ has SVEP, then $T$ and $T^*$ satisfy $a-Bt$ and $s-Bt$.*

*Proof.* It is clear from Lemma 2.5 that $T$ satisfies $a-Bt$ and $s-Bt$. To prove that $T^*$ satisfies $a-Bt$ and $s-Bt$, we have only to prove that $\sigma_{ba}(T^*) \subseteq \sigma_{wa}(T^*)$ and $\sigma_{bs}(T^*) \subseteq \sigma_{ws}(T^*)$. Observe that if $\lambda \notin \sigma_{wa}(T^*)$ (resp., $\lambda \notin \sigma_{ws}(T^*)$), then $T^* - \lambda I^* \in \Phi_+(\mathcal{X}^*)$ and $\operatorname{ind}(T^* - \lambda I^*) \leq 0$ (resp., $T^* - \lambda I^* \in \Phi_-(\mathcal{X}^*)$ and $\operatorname{ind}(T^* - \lambda I^*) \geq 0$). Thus, if $T$ has SVEP and $\lambda \notin \sigma_{wa}(T^*)$ (resp., $\lambda \notin \sigma_{ws}(T^*)$), then, by Lemma 2.1, $T^* - \lambda I^* \in \Phi(\mathcal{X}^*)$ and $\operatorname{asc}(T^* - \lambda I^*) = \operatorname{dsc}(T^* - \lambda I^*) < \infty$ (resp., $T^* - \lambda I^* \in \Phi(\mathcal{X}^*)$ and $\operatorname{dsc}(T^* - \lambda I^*) < \infty$). Hence $\lambda \notin \sigma_{ba}(T^*)$ (resp., $\lambda \notin \sigma_{bs}(T^*)$). □

The following lemma follows from [14, Theorem 5] and the argument of the proof of [5, Theorem 1.5(iii)].

**Lemma 2.7.** *If $R \in B(\mathcal{X})$ is a Riesz operator such that $[T, R] = 0$, then $\sigma_{wx}(T+R) = \sigma_{wx}(T)$ and $\sigma_{bx}(T+R) = \sigma_{bx}(T)$, where $\sigma_{.x}$ stands for $\sigma_{.a}$ or $\sigma_{.s}$.*

Taken together, Lemmas 2.6 and 2.7 imply that if $T$ has SVEP and the Riesz operator $R$ commutes with $T$, then $T+R$ and $T^*+R^*$ satisfy $a-Bt$ and $s-Bt$. Combining with Lemma 2.5, this implies the following.

**Corollary 2.8.** *If $T$ has SVEP and the Riesz operator $R$ commutes with $T$, then $T+R$ has SVEP at points $\lambda \notin \sigma_{wx}(T+R)$ and $T^*+R^*$ has SVEP at points $\lambda \notin \sigma_{wx}(T^*+R^*)$, where $\sigma_{.x}$ stands for $\sigma_{.a}$ or $\sigma_{.s}$.*

**Theorem 2.9.** *Let $A \in \mathcal{A}$ be such that $A_{ii}$ has SVEP for all $1 \leq i \leq n$. Let $R = (R_{ij})_{1\leq i,j\leq n} \in B(\mathcal{X}^n)$, $R_{ij} = 0$ for all $i > j$, be an upper triangular operator such that $[T, R] = 0$.*

(i) *If $R_0 = \oplus_{i=1}^n R_{ii}$ is a Riesz operator, then $\sigma(A+R) \setminus \sigma_w(A+R) = p_0(A+R)$.*
(ii) *If $R_0 = \oplus_{i=1}^n R_{ii}$ is a quasi–nilpotent operator, then $A+R$ satisfies conclusions* (i) *to* (v) *of Theorem 2.3.*
(iii) *If $R_0 = \oplus_{i=1}^n R_{ii}$ is an injective quasi–nilpotent operator, then $(A+R$ satisfies conclusions* (i) *to* (v) *of Theorem 2.3 and) $\sigma(A+R) = \sigma_w(A+R) = \sigma_b(A+R)$.*

*Proof.* The hypothesis $R$ commutes with $A$ implies that $R_{ii}$ commutes with $A_{ii}$ for all $1 \leq i \leq n$.

(i) Since the restriction of a Riesz operator to an invariant subspace is again a Riesz operator, the hypotheses imply that $R_{ii}$ is a Riesz operator such that $[A_{ii}, R_{ii}] = 0$, $\sigma_w(A_{ii}+R_{ii}) = \sigma_w(A_{ii})$, $\sigma_b(A_{ii}+R_{ii}) = \sigma_b(A_{ii})$ for all $1 \leq i \leq n$. Furthermore, $A$ has SVEP (implies $A$ satisfies $Bt$, i.e., $\sigma_w(A) = \sigma_b(A)$), $\sigma_w(A) = \bigcup_{i=1}^n \sigma_w(A_{ii})$ and $\sigma_b(A) = \bigcup_{i=1}^n \sigma_b(A_{ii})$. Hence $\sigma_w(A) = \bigcup_{i=1}^n \sigma_w(A_{ii}+R_{ii}) = \bigcup_{i=1}^n \sigma_b(A_{ii}+R_{ii}) = \sigma_b(A)$. Evidently, $\sigma_w(A+R) \subseteq \bigcup_{i=1}^n \sigma_w(A_{ii}+R_{ii})$ and $\sigma_b(A+R) \subseteq \bigcup_{i=1}^n \sigma_b(A_{ii}+R_{ii})$. We prove the reverse inclusions: this would then imply that $A+R$ satisfies $Bt$.

Define the operator $T_s \in B(\oplus_{i=s}^n \mathcal{X}_i)$, $1 \leq s \leq n$, by

$$T_s = \begin{pmatrix} A_{ss}+R_{ss} & A_{s(s+1)}+R_{s(s+1)} & \cdots & A_{sn}+R_{sn} \\ 0 & A_{(s+1)(s+1)}+R_{(s+1)(s+1)} & \cdots & A_{(s+1)n}+R_{(s+1)n} \\ \vdots & & & \\ 0 & 0 & \cdots & A_{nn}+R_{nn} \end{pmatrix}.$$

Considering the operator

$$A+R = \begin{pmatrix} A_{11}+R_{11} & * \\ 0 & T_2 \end{pmatrix},$$

it follows that if $\lambda \notin \sigma_w(A+R)$, then $A_{11}+R_{11}-\lambda \in \Phi_+(\mathcal{X}_1)$, $T_2-\lambda \in \Phi_-(\oplus_{i=2}^n \mathcal{X}_i)$ and $\mathrm{ind}(A_{11}+R_{11}-\lambda)+\mathrm{ind}(T_2-\lambda) = 0$. Recall from Corollary 3.2 that $A_{11}+R_{11}$ has SVEP at points $\lambda \notin \sigma_{wa}(A_{11}+R_{11})$ and $A_{11}^*+R_{11}^*$ has SVEP at points $\lambda \notin \sigma_{ws}(A_{11}^*+R_{11}^*) = \sigma_{wa}(A_{11}+R_{11})$. Hence, see Lemma 2.1, if $A_{11}+R_{11}-\lambda \in \Phi_+(\mathcal{X}_1)$ and if either $\mathrm{ind}(A_{11}+R_{11}-\lambda) \geq 0$ or $\mathrm{ind}(A_{11}+R_{11}-\lambda) \leq 0$, then $\mathrm{ind}(A_{11}+R_{11}-\lambda) = 0$ and $A_{11}+R_{11}-\lambda \in \Phi(\mathcal{X}_1)$ (i.e.,

$\lambda \notin \sigma_w(A_{11}+R_{11})$). Consequently, $T_2-\lambda \in \Phi_-(\oplus_{i=2}^n \mathcal{X}_i)$ and $\operatorname{ind}(T_2-\lambda)=0$, i.e., $T_2-\lambda \in \Phi(\oplus_{i=2}^n \mathcal{X}_i)$. Repeating this argument, successively considering $T_2$, $T_3$ etc., it follows that if $\lambda \notin \sigma_w(A+R)$, then $\lambda \notin \sigma_w(A_{ii}+R_{ii})$ for all $1 \leq i \leq n$. Hence $\bigcup_{i=1}^n \sigma_w(A_{ii}+R_{ii}) \subseteq \sigma_w(A+R)$. A similar argument proves that $\bigcup_{i=1}^n \sigma_b(A_{ii}+R_{ii}) \subseteq \sigma_b(A+R)$.

(ii) The argument in this case is straightforward. Since $R_{ii}$ is a quasinilpotent which commutes with $A_{ii}$ for all $1 \leq i \leq n$, $A_{ii}$ and $A_{ii}+R_{ii}$ are quasinilpotent equivalent [11, p. 253]. Since $A_{ii}$ has SVEP for all $1 \leq i \leq n$, $A_{ii}+R_{ii}$ has SVEP [11, Proposition 3.4.1]. Hence $A+R$ has SVEP (so that Theorem 2.3 applies).

(iii) Evidently, $A_{ii}+R_{ii}$ has SVEP for all $1 \leq i \leq n$, Theorem 2.3 applies and we conclude that $\sigma_w(A+R)=\sigma_b(A+R)$ and $\sigma(A+R)\setminus\sigma_w(A+R)=p_0(A+R) \subseteq \pi_{00}(A+R)$. Since, this is easily verified, $\sigma_p(A+R) \subseteq \bigcup_{i=1}^n \sigma_p(A_{ii}+R_{ii})$ and $\pi_{00}(A_{ii}+R_{ii})=\emptyset \Longrightarrow \pi_{00}(A+R)=\emptyset$, to complete the proof it would suffice to prove that the set $\pi_{00}(A_{ii}+R_{ii})$ is empty. To this end, we start by observing from the commutativity of $A_{ii}$ and $R_{ii}$ that if $(0 \neq)x \in (A_{ii}+R_{ii}-\lambda)^{-1}(0)$ for some $\lambda \in \sigma(A_{ii}+R_{ii})$, then $R_{ii}^m x \in (A_{ii}+R_{ii}-\lambda)^{-1}(0)$ for all $m \in \mathbb{N}$. Let $g(t)=\sum_{i=1}^r c_i t^i = c_r \prod_{i=1}^r t-\lambda_i$ be a polynomial such that $g(R_{ii})=0$. Then, since the injectivity of $R$ implies that of $R_{ii}$ for all $1 \leq i \leq n$, $c_r=0$, and hence, by a finite induction argument, that $c_i=0$ for all $0 \leq i \leq r$. Consequently, $\{R_{ii}^n x\}$ is a linearly independent set of vectors in $(A_{ii}+R_{ii}-\lambda)^{-1}(0)$ and the eigenvalues of $A_{ii}+R_{ii}$ (and so also of $A_{ii}=A_{ii}+R_{ii}-R_{ii}$) have infinite multiplicity. Hence $\pi_{00}(A_{ii}+R_{ii})=\emptyset$. □

## 3. Operators $A \in (HP)_n$: Weyl's theorem

For an upper triangular operator $A=(A_{ij})_{1 \leq i,j \leq n} \in B(\mathcal{X}^n)$, $A$ polaroid (even, hereditarily polaroid) does not imply $A_{ii}$ is hereditarily polaroid (even, polaroid) for all $1 \leq i \leq n$. For example, let $A=A_{11} \oplus A_{22}$, where $A_{11}=U \in B(\ell^2)$ is the forward unilateral shift and $A_{22} \in B(\ell^2)$ is a non–nilpotent quasi–nilpotent operator. Then $\sigma(A)$ is the closed unit disc, $A$ is vacuously polaroid, and $A_{22}$ is not polaroid. Again, if $A=\left(\begin{smallmatrix} U & 1-UU^* \\ 0 & U^* \end{smallmatrix}\right)$, where $U \in B(\ell^2)$ is the forward unilateral shift, then $A$ is unitary (hence $(\mathcal{HP})$), $U \in (\mathcal{HP})$, $U^*$ is polaroid, but $U^* \notin (\mathcal{HP})$. The following theorem considers the converse problem.

**Theorem 3.1.** *Operators $A \in (HP)_n$ are polaroid.*

*Proof.* Since $(\mathcal{HP})$ operators have SVEP [6], $\sigma(A)=\cup_{i=1}^n \sigma(A_{ii})$. Hence

$$\operatorname{iso}\sigma(A) \subseteq \bigcup_{i=1}^n \operatorname{iso}\sigma(A_{ii}).$$

Let $\lambda \in \operatorname{iso}\sigma(A)$. Then the polaroid hypothesis on $A_{ii}$ implies the existence of non–negative integers $p_i$ such that $H_0(A_{ii}-\lambda)=(A_{ii}-\lambda)^{-p_i}(0)$. (Evidently, $p_j=0$ whenever $\lambda \notin \sigma(A_{jj})$.) We consider the operator $A_2=\left(\begin{smallmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{smallmatrix}\right)$: the proof for the general case follows from a finitely repeated argument applied to $A_j = \begin{pmatrix} A_{j-1} & * \\ 0 & A_{jj} \end{pmatrix}$, $3 \le j \le n$. Let $p = \max\{p_1, p_2\}$. Then $H_0(A_{ii} - \lambda) = (A_{ii} - \lambda)^{-p}(0)$; $1 \le i \le 2$. Observe that if ($\lambda \in \mathrm{iso}\sigma(A_2)$ and) $x = x_1 \oplus x_2 \in H_0(A_2 - \lambda)$, then

$$||(A_2 - \lambda)^m x||^{\frac{1}{m}} = ||\{(A_{11} - \lambda)^m x_1 + \sum_{j=0}^{m-1} (A_{11} - \lambda)^{m-j-1} A_{12}(A_{22} - \lambda)^j x_2\} \oplus (A_{22} - \lambda)^m x_2||^{\frac{1}{m}} \longrightarrow 0 \text{ as } m \longrightarrow \infty$$

implies that

$$\lim_{m \longrightarrow \infty} ||(A_{22} - \lambda)^m x_2||^{\frac{1}{m}} = 0 \quad (\Longrightarrow x_2 \in (A_{22} - \lambda)^{-p}(0))$$

and

$$\lim_{m \longrightarrow \infty} ||(A_{11} - \lambda)^m x_1 + \sum_{j=0}^{m-1} (A_{11} - \lambda)^{m-j-1} A_{12}(A_{22} - \lambda)^j x_2||^{\frac{1}{m}} = 0.$$

Choose $m = (t+1)p$, $t$ some positive integer, and set

$$(A_{11} - \lambda)^p x_1 + \sum_{j=0}^{p-1} (A_{11} - \lambda)^{p-j-1} A_{12}(A_{22} - \lambda)^j x_2 = y.$$

Then, since $(A_{22} - \lambda)^r x_2 = 0$ for integers $r \ge p$,

$$\begin{aligned} &(A_{11} - \lambda)^m x_1 + \sum_{j=0}^{m-1} (A_{11} - \lambda)^{m-j-1} A_{12}(A_{22} - \lambda)^j x_2 \\ &= (A_{11} - \lambda)^{(t+1)p} x_1 + \sum_{j=0}^{(t+1)p-1} (A_{11} - \lambda)^{(t+1)p-j-1} A_{12}(A_{22} - \lambda)^j x_2 \\ &= (A_{11} - \lambda)^{tp}\{(A_{11} - \lambda)^p x_1 + \sum_{j=0}^{p-1} (A_{11} - \lambda)^{p-j-1} A_{12}(A_{22} - \lambda)^j x_2\} \\ &= (A_{11} - \lambda)^{tp} y. \end{aligned}$$

Since

$$||(A_{11} - \lambda)^{tp} y||^{\frac{1}{tp}} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty,$$

$y \in H_0(A_{11} - \lambda) = (A_{11} - \lambda)^{-p}(0)$. Hence

$$H_0(A_2 - \lambda) \subseteq \oplus_{i=1}^2 (A_{ii} - \lambda)^{-p}(0) = (A_2 - \lambda)^{-p}(0).$$

Since $(T - \lambda)^{-p}(0) \subseteq H_0(T - \lambda)$ for every operator $T$ (and non–negative integer $p$), $H_0(A_2 - \lambda) = (A_2 - \lambda)^{-p}(0)$.

To complete the proof, we now recall from above that if $\lambda \in \mathrm{iso}\sigma(A_2)$, then

$$\mathcal{X}^2 = H_0(A_2 - \lambda) \oplus K(A_2 - \lambda) = (A_2 - \lambda)^{-p}(0) \oplus (A_2 - \lambda)^p \mathcal{X}^2,$$

i.e., $\lambda$ is a pole of order $\le p$ of the resolvent of $A_2$. $\square$

Recall, [1, p. 177], that a Banach space operator $A$ satisfies $a$–Weyl's theorem if $\sigma_a(A)\backslash\sigma_{aw}(A) = \pi_{00}^a(A)$, where $\pi_{00}^a(A) = \{\lambda \in \text{iso}\sigma_a(A) : 0 < \alpha(A-\lambda) < \infty\}$. Let $H(\sigma(A))$ denote the class of functions $f$ which are analytic on a neighbourhood of $\sigma(A)$. For a polaroid operator with SVEP, $f(A)$ satisfies Weyl's theorem and $f(A^*)$ satisfies $a$–Weyl's theorem for every $f \in H(\sigma(A))$ [7, Lemma 3.1]. Hence:

**Corollary 3.2.** *For operators $A \in (HP)_n$, $f(A)$ satisfies Weyl's theorem and $f(A^*)$ satisfies $a$–Weyl's theorem for every $f \in H(\sigma(A))$.*

More is true, as we now proceed to prove. But before that some terminology. An operator $A$ is said to be algebraic if there exists a (non-trivial) polynomial $q(\cdot)$ such that $q(A) = 0$. Banach space operators $F$ such that $F^n$ is finite dimensional for some positive integer $n$ are algebraic. We say that an operator $A$ is polynomially $(\mathcal{HP})$ if there exists a polynomial $g(\cdot)$ such that $g(A) \in (\mathcal{HP})$.

**Theorem 3.3.** *Let $A \in \mathcal{A}$ be such that the operator $A_0 = \bigoplus_{i=1}^n A_{ii}$ is a polynomially $(HP)_n$ operator. If $B = \bigoplus_{i=1}^n B_{ii} \in B(\mathcal{X}^n)$ is an algebraic operator which commutes $A_0$, then $f(A+B)$ satisfies Weyl's theorem and $f(A^*+B^*)$ satisfies $a$–Weyl's theorem for every $f \in H(\sigma(A+B))$.*

*Proof.* The hypothesis that $A_0$ is polynomially $(HP)_n$ implies that $A_{ii}$ is polynomially $(\mathcal{HP})$ for all $1 \leq i \leq n$. Hence $A_{ii}$ is polaroid and has SVEP for all $1 \leq i \leq n$ [7, Lemma 3.3]. Furthermore, since $B$ is algebraic and commutes with $A_0$, $B_{ii}$ is algebraic and commutes with $A_{ii}$ for all $1 \leq i \leq n$. Consequently, $A_{ii} + B_{ii}$ is polaroid and has SVEP for all $1 \leq i \leq n$. [7, Lemma 3.4]; hence $A + B$ has SVEP, a consequence of the fact that the diagonal elements $A_{ii}+B_{ii}$ of $A+B$ have SVEP, and $A+B$ is polaroid (by Theorem 3.1). This, by [7, Theorem 3.8], implies that $f(A+B)$ satisfies Weyl's theorem and $f(A^*+B^*)$ satisfies $a$–Weyl's theorem for every $f \in H(\sigma(A+B))$. □

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