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SOME APPLICATIONS OF q-DIFFERENTIAL OPERATOR

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Abstract. In this paper, we use q-differential operator to recover the finite Heine $_2\Phi_1$ transformations given in [3]. Applying that, we also obtain some terminating q-series transformation formulas.

1. Introduction

Recently, G. E. Andrews [3] derived several finite Heine $_2\Phi_1$ transformations from the terminating Sears $_3\Phi_2$ transformation. Then he used them to give two finite Rogers-Ramanujan type identities. In this paper, by using the properties of q-differential operators, we also obtain the finite Heine $_2\Phi_1$ transformation and the following finite q-series transformations

$$\sum_{j=0}^{M} \begin{bmatrix} M \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}+2j} \frac{1}{1-a_1 q^j} = \frac{(q;q)_M}{(a_1;q)_{M+1}} \sum_{j=0}^{M} (a_1;q)_j q^j, \tag{1}$$

$$\sum_{k=0}^{M} \frac{(-1)^k q^{k^2}(-q;q^2)_k}{(q^4;q^4)_k (q^2;q^2)_{M-k}} = \frac{(q;q^2)_M}{(q^4;q^4)_M}, \tag{2}$$

$$\sum_{k=0}^{M} \frac{q^{k^2-sk}(-q^s;q^2)_k}{(q;q)_{2k}(q^2;q^2)_{M-k}} = \frac{(-q^{1-s};q^2)_M}{(q;q)_{2M}}, \ s=0,1, \tag{3}$$

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$$\sum_{k=0}^{M} \frac{q^{3k^2-sk}}{(q^{2-(1-s)};q^2)_k(q^{4-2s};q^4)_k(q^2;q^2)_{M-k}} \tag{4}$$
$$= \frac{1}{(-q^{2-s};q^2)_M}\sum_{k=0}^{M}\frac{q^{k^2+(1-s)k}}{(q;q)_{2k}(q^2;q^2)_{M-k}},\ s=0,1,$$

where

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n=1,2,\ldots$$

and

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

As $M\to\infty$, the second identity reduces to the identity appearing in [14, p. 152, Eq. (4)] (or [13, p. 99, Eq. (A.4)]). If $s=0$, $M\to\infty$, the third tends to the identity appearing in Slater's paper [14, p. 156, Eq. (47)] (or [4, p. 252, Eq. (11.2.1)], [13, p. 104, Eq. (A.47)]).

Throughout the paper, we take $0<|q|<1$. And we also use the following notations

$$(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$
$$\begin{bmatrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{bmatrix}_n = \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n},$$

$${}_r\Phi_s\left(\begin{matrix} a_1, & \ldots, & a_r \\ b_1, & \ldots, & b_s \end{matrix};q,x\right)$$
$$= \sum_{n=0}^{\infty}\begin{bmatrix} a_1, & a_2, & \ldots, & a_r \\ q, & b_1, & \ldots, & b_s \end{bmatrix}_n \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} x^n,$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

2. Some lemmas

Recall that the q-differential operator D_q and q-shifted operator η (cf. [6, 7, 10-12]), acting on the variable x, are defined by:

$$D_q\{f(x)\} = \frac{f(x)-f(xq)}{x} \quad \text{and} \quad \eta\{f(x)\} = f(xq).$$

We can prove, by means of induction, the explicit formulae (cf. [10, 11])

$$D_q{}^n\left\{\frac{(x\omega;q)_\infty}{(xs;q)_\infty}\right\} = s^n\frac{(\omega/s;q)_n(x\omega q^n;q)_\infty}{(xs;q)_\infty}, \tag{5}$$

$$D_q^n\{f(x)\} = x^{-n}\sum_{k=0}^{n} q^k \frac{(q^{-n};q)_k}{(q;q)_k} f(q^k x) \tag{6}$$

and the q-Leibniz rule for the product of two functions

$$D_q^n\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\{f(x)\} D_q^{n-k}\left\{g(xq^k)\right\}. \tag{7}$$

In [6, 7], we have constructed the following q-exponential operator

$$ {}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix}; q, cD_q\right) = \sum_{n=0}^{\infty} \frac{(b;q)_n (cD_q)^n}{(q;q)_n}, \tag{8}$$

and gave some applications of it. In this paper, we will use the case of $b = q^{-M}$,

$$ {}_1\Phi_0\left(\begin{matrix} q^{-M} \\ - \end{matrix}; q, cD_q\right) = \sum_{n=0}^{M} \frac{(q^{-M};q)_n (cD_q)^n}{(q;q)_n} \tag{9}$$

and the following more general finite q-exponential operator

$$ {}_2\Phi_1\left(\begin{matrix} q^{-M}, & a_1 \\ & b_1 \end{matrix}; q, cD_q\right) = \sum_{n=0}^{M} \begin{bmatrix} q^{-M}, & a_1 \\ q, & b_1 \end{bmatrix}_n (cD_q)^n, \tag{10}$$

where M is a non-negative integer.

Letting

$$F(x) = \begin{bmatrix} xc_1, & xc_2, & \ldots, & xc_r \\ xd_1, & xd_2, & \ldots, & xd_r \end{bmatrix}_\infty,$$

from (6), we have:

Lemma 2.1. *For complex numbers x, a_i, b_i, $i = 1, 2, \ldots, r$,*

$$D_q^n\{F(x)\} = x^{-n}F(x)\sum_{k=0}^{n} \begin{bmatrix} q^{-n}, & xd_1, & xd_2, & \ldots, & xd_r \\ q, & xc_1, & xc_2, & \ldots, & xc_r \end{bmatrix}_k q^k. \tag{11}$$

From (7) and (9), we obtain the next lemma.

Lemma 2.2. *We have*

$$\begin{aligned} &{}_1\Phi_0\left(\begin{matrix} q^{-M} \\ - \end{matrix}; q, cD_q\right)\left\{\frac{(c_1x;q)_\infty}{(d_1x, d_2x)_\infty}\right\} \\ &= (cd_2q^{-M};q)_M \frac{(c_1x;q)_\infty}{(d_1x, d_2x)_\infty}\, {}_3\Phi_2\left(\begin{matrix} q^{-M}, & c_1/d_1, & xd_2 \\ & cd_2q^{-M}, & xc_1 \end{matrix}; q, cd_1\right). \end{aligned} \tag{12}$$

The identity above is a special case of an identity in [6, p. 21, Eq. (7)].

Lemma 2.3. *If $c = q/d_2$, then*

(13)

$$
\begin{aligned}
&{}_2\Phi_1\left(\begin{matrix} q^{-M}, & a_1 \\ & b_1 \end{matrix}; q, cD_q\right)\left\{\frac{(c_1x;q)_\infty}{(d_1x, d_2x)_\infty}\right\} \\
&= a_1^M \frac{(b_1/a_1;q)_M}{(b_1;q)_M}\frac{(c_1x;q)_\infty}{(d_1x,d_2x)_\infty}\,{}_4\Phi_2\left(\begin{matrix} q^{-M}, & a_1, & d_2x, & c_1/d_1 \\ & & c_1x, & q^{1-M}a_1/b_1 \end{matrix}; q, qd_1/b_1d_2\right).
\end{aligned}
$$

Proof. From (7), we have

$$
\begin{aligned}
&{}_2\Phi_1\left(\begin{matrix} q^{-M}, & a_1 \\ & b_1 \end{matrix}; q, cD_q\right)\left\{\frac{(c_1x;q)_\infty}{(d_1x, d_2x)_\infty}\right\} \\
&= \sum_{k=0}^{M}\begin{bmatrix} q^{-M}, & a_1 \\ q, & b_1 \end{bmatrix}_k c^k D_q^k\left\{\frac{(c_1x;q)_\infty}{(d_1x;q)_\infty}\right\} \\
&\quad\times\sum_{j=0}^{M-k}\begin{bmatrix} q^{-(M-k)}, & a_1q^k \\ q, & b_1q^k \end{bmatrix}_j\left(\frac{c}{q^k}\right)^j D_q^j\left\{\frac{1}{(d_2xq^k;q)_\infty}\right\} \\
&= a_1^M \frac{(b_1/a_1;q)_M}{(b_1;q)_M}\frac{(c_1x;q)_\infty}{(d_1x,d_2x)_\infty}\,{}_4\Phi_2\left(\begin{matrix} q^{-M}, a_1, d_2x, & c_1/d_1 \\ c_1x, & q^{1-M}a_1/b_1 \end{matrix}; q, qd_1/b_1d_2\right).
\end{aligned}
$$

This completes the proof. □

3. Main results and special cases

Theorem 3.1 (cf. [8], p. 11)**.** *The q-Chu-Vandermonde summation*

$$
{}_2\Phi_1\left(\begin{matrix} q^{-n}, & a \\ & d \end{matrix}; q, q\right) = a^n\frac{(d/a;q)_n}{(d;q)_n}. \tag{14}
$$

Proof. Setting $F(x) = (xc_1;q)_\infty/(xd_1;q)_\infty$ in (11), and then putting $xc_1 = d, xd_1 = a$, we complete the proof. □

Theorem 3.2 (cf. [8, p. 16, Eq. (1.9.11)])**.** *Suppose $n > m_1 + \cdots + m_r$. Then we have*

$$
{}_{r+1}\Phi_r\left(\begin{matrix} q^{-n}, & xc_1q^{m_1}, & \ldots, & xc_rq^{m_r} \\ & xc_1, & \ldots, & xc_r \end{matrix}; q, q\right) = 0. \tag{15}
$$

Proof. Setting $d_i = c_iq^{m_i}, m_i = 0, 1, \ldots, \infty, i = 1, 2, \ldots, r$, in (11), we complete the proof. □

Theorem 3.3. *We have*

(16)

$$
\begin{aligned}
&\sum_{m=0}^{n}\sum_{j=0}^{M}\frac{(q^{-n},a;q)_m}{(q,d;q)_m}\frac{(q^{-M},a_1;q)_j}{(q,b_1;q)_j}q^{m+j+mj} \\
&= \frac{a_1^M(b_1/a_1;q)_M}{(b_1;q)_M}\frac{a^n(d/a;q)_n}{(d;q)_n}\,{}_4\Phi_2\left(\begin{matrix} q^{-M}, a_1, & a, & q^{-n} \\ & aq^{1-n}/d, & q^{1-M}a_1/b_1 \end{matrix}; q, q^2/db_1\right).
\end{aligned}
$$

Proof. We rewrite (14) as follows

$$\sum_{m=0}^{n}\frac{(q^{-n};q)_m}{(q,d;q)_m}q^m\frac{1}{(aq^m;q)_\infty}=\frac{(-1)^nq^{\binom{n}{2}}d^n}{(d;q)_n}\frac{(aq^{1-n}/d;q)_\infty}{(aq/d,a;q)_\infty}. \tag{17}$$

Applying the operator ${}_2\Phi_1\left(\begin{smallmatrix}q^{-M}, & a_1\\ & b_1\end{smallmatrix};q,qD_q\right)$ to both sides of (17) with respect to the variable a, from (13), we complete the proof. □

Corollary 3.1. *We have*

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{j=0}^{M}\frac{(q^{-n};q)_m(q^{-M};q)_j}{(q;q)_m(q;q)_j}\frac{q^{m+j+mj}}{(1-aq^m)(1-a_1q^j)}\\&=\frac{(q;q)_M}{(a_1;q)_{M+1}}\frac{(q;q)_n}{(a;q)_{n+1}}\sum_{k=0}^{\min\{M,n\}}\frac{(a,a_1;q)_k}{(q;q)_k}(-1)^kq^{-\binom{k}{2}}a_1^{M-k}a^{n-k}.\end{aligned} \tag{18}$$

Proof. If set $b_1=qa_1$ and $d=qa$ in (16), we complete the proof. □

Theorem 3.4. *We have*

(19)

$$\begin{aligned}&\sum_{m=0}^{n}\sum_{j=0}^{M}\frac{(q^{-n},a;q)_m}{(q,d;q)_m}\frac{(q^{-M},a_1;q)_jd^j}{(q,b_1;q)_j}q^{m+mj}\\&=\frac{a_1^M(b_1/a_1;q)_M}{(b_1;q)_M}\frac{a^n(d/a;q)_n}{(d;q)_n}{}_4\Phi_2\left(\begin{matrix}q^{-M}, & a_1, & aq/d, & q^{1-n}/d\\ & & aq^{1-n}/d, & q^{1-M}a_1/b_1\end{matrix};q,d/b_1\right).\end{aligned}$$

Proof. We rewrite (17) as follows

$$\sum_{m=0}^{n}\frac{(q^{-n};q)_m}{(q,d;q)_m}q^m\frac{1}{(aq^m;q)_\infty}=\frac{(-1)^nq^{\binom{n}{2}}d^n}{(d;q)_n}\frac{(aq^{1-n}/d;q)_\infty}{(a,aq/d;q)_\infty}. \tag{20}$$

Applying the operator ${}_2\Phi_1\left(\begin{smallmatrix}q^{-M}, & a_1\\ & b_1\end{smallmatrix};q,dD_q\right)$ to both sides of (20) with respect to the variable a, using (13), we complete the proof. □

Corollary 3.2 (cf. [8, p. 23])**.** *Jackson's transformation formula*

(21)

$$\sum_{j=0}^{M}\frac{(q^{-M},a_1;q)_j}{(q,b_1;q)_j}d^j=\frac{a_1^M(b_1/a_1;q)_M}{(b_1;q)_M}{}_3\Phi_1\left(\begin{matrix}q^{-M}, & a_1, & q/d\\ & & q^{1-M}a_1/b_1\end{matrix};q,d/b_1\right).$$

Proof. If set $a\to 1$ in (19), we complete the proof. □

Corollary 3.3. *We have*

(22)

$$\sum_{j=0}^{M}\begin{bmatrix}M\\ j\end{bmatrix}(-1)^jq^{\binom{j+1}{2}}\frac{(a_1;q)_j}{(b_1;q)_j}\left(\frac{db_1}{a_1}\right)^j=\frac{(b_1/a_1;q)_M}{(b_1;q)_M}\sum_{j=0}^{M}\frac{(q^{-M},a_1,dq;q)_j}{(q,a_1q^{1-M}/b_1;q)_j}q^j.$$

Proof. If set $q \to 1/q$ in (21), then replacing a_1 by $1/a_1$, b_1 by $1/b_1$, we complete the proof. ☐

Setting $d = 1, b_1 = qa_1$, (22) tends to:

Corollary 3.4. *We have*

$$\sum_{j=0}^{M} \begin{bmatrix} M \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}+2j} \frac{1}{1-a_1q^j} = \frac{(q;q)_M}{(a_1;q)_{M+1}} \sum_{j=0}^{M} (a_1;q)_j q^j. \tag{23}$$

Theorem 3.5. *We have*

$$\begin{aligned} &{}_3\Phi_2 \left(\begin{matrix} q^{-M}, & c_1/d_2, & xd_1 \\ & cd_1q^{-M}, & xc_1 \end{matrix} ; q, cd_2 \right) \\ &= \frac{(q/cd_2;q)_M}{(q/cd_1;q)_M} \left(\frac{d_2}{d_1}\right)^M {}_3\Phi_2 \left(\begin{matrix} q^{-M}, & c_1/d_1, & xd_2 \\ & cd_2q^{-M}, & xc_1 \end{matrix} ; q, cd_1 \right). \end{aligned} \tag{24}$$

Proof. For

$${}_1\Phi_0 \left(\begin{matrix} q^{-M} \\ - \end{matrix} ; q, cD_q \right) \left\{ \frac{(c_1x;q)_\infty}{(d_1x,d_2x)_\infty} \right\} = {}_1\Phi_0 \left(\begin{matrix} q^{-M} \\ - \end{matrix} ; q, cD_q \right) \left\{ \frac{(c_1x;q)_\infty}{(d_2x,d_1x)_\infty} \right\},$$

and applying (12), we complete the proof. ☐

In the identity (24), taking $q \to 1/q$, then replacing (x, c, c_1, d_i) by $(1/x$, c/q, $1/c_1$, $1/d_i)$ respectively, where $i = 1, 2$, we obtain the following identity:

Theorem 3.6. *We have*

$${}_3\Phi_2 \left(\begin{matrix} q^{-M}, & c_1/d_2, & xd_1 \\ & d_1q^{1-M}/c, & xc_1 \end{matrix} ; q, q \right) = \frac{(c/d_2;q)_M}{(c/d_1;q)_M} {}_3\Phi_2 \left(\begin{matrix} q^{-M}, & c_1/d_1, & xd_2 \\ & d_2q^{1-M}/c, & xc_1 \end{matrix} ; q, q \right). \tag{25}$$

Remark. An equivalent identity can be found in Andrews' paper [3, Corollary 4].

Theorem 3.7. *We have*

$${}_3\Phi_2 \left(\begin{matrix} q^{-M}, & a, & b \\ & c, & d \end{matrix} ; q, cdq^M/ab \right) = \frac{(cd/ab;q)_M}{(d;q)_M} {}_3\Phi_2 \left(\begin{matrix} q^{-M}, & c/a, & c/b \\ & c, & cd/ab \end{matrix} ; q, dq^M \right). \tag{26}$$

Proof. In (24), letting $c \to cq^M$, then replacing xc_1 by c, cd_1 by d, c_1/d_2 by a, last step setting $cd_1/ad_2 = b$, we complete the proof. ☐

Remark. (26) follows from setting $a = q^{-M}$ in the Sears ${}_3\Phi_2$ transformation [8, p. 62, Eq. (3.2.7)].

Corollary 3.5 (cf. [8, p. 10, Eq. (1.4.6)])**.** *Heine's ${}_2\Phi_1$ transformation formula*

$${}_2\Phi_1 \left(\begin{matrix} a, & b \\ & c \end{matrix} ; q, z \right) = \frac{(abz/c;q)_\infty}{(z;q)_\infty} {}_2\Phi_1 \left(\begin{matrix} c/a, & c/b \\ & c \end{matrix} ; q, abz/c \right). \tag{27}$$

Proof. In (25), letting $M \to \infty$, then replacing c/d_1 by z, xc_1 by c, c_1/d_1 by a, last step setting $d_1c/ad_2 = b$, we complete the proof. □

Corollary 3.6. *We have*

$${}_2\Phi_2\left(\begin{matrix} a, & b \\ c, & d \end{matrix}; q, cd/ab\right) = \frac{(cd/ab;q)_\infty}{(d;q)_\infty}\, {}_2\Phi_2\left(\begin{matrix} c/a, & c/b \\ c, & cd/ab \end{matrix}; q, d\right).$$

Proof. In (25), letting $M \to \infty$, we complete the proof. □

4. Some other special cases

Theorem 4.1. *We have*

$$\sum_{k=0}^{M} \frac{(c_1/d_2;q)_k q^k}{(q, xc_1;q)_k} = \sum_{k=0}^{M} \frac{(-1)^k q^{\binom{k+1}{2}} (xd_2;q)_k}{(q, xc_1;q)_k (q;q)_{M-k}} \left(\frac{c_1}{d_2}\right)^k. \tag{28}$$

Proof. In (25), putting $c = d_1q$, then letting $d_1 \to 0$, we complete the proof. □

Theorem 4.2. *We have*

$$\sum_{k=0}^{M} \frac{(c_1/d_2;q)_k (-xd_2)^k q^{\binom{k}{2}}}{(q, xc_1;q)_k} = \sum_{k=0}^{M} \frac{(-1)^k q^{\binom{k+1}{2}} (xd_2;q)_{M-k}}{(q;q)_k (q, xc_1;q)_{M-k}}. \tag{29}$$

Proof. In (28), taking $q \to 1/q$, then replacing (x, c_1, d_2) by $(1/x, 1/c_1, 1/d_2)$ respectively, we complete the proof. □

Corollary 4.1. *We have*

$$\sum_{k=0}^{M} \frac{q^k}{(xd_2;q)_{k+1}} = \sum_{k=0}^{M} \frac{(-1)^k q^{\binom{k}{2}} q^{2k}}{(1 - xd_2q^k)(q;q)_k (q;q)_{M-k}}. \tag{30}$$

Proof. In (28), putting $c_1 = d_2q$, we complete the proof. □

Corollary 4.2. *We have*

$$\sum_{k=0}^{M} \frac{q^k}{(q, xc_1;q)_k} = \sum_{k=0}^{M} \frac{q^{k^2}(xc_1)^k}{(q, xc_1;q)_k (q;q)_{M-k}}. \tag{31}$$

Proof. In (28), letting $d_2 \to \infty$, we complete the proof. □

Corollary 4.3. *We have*

$$\sum_{k=0}^{M} \frac{q^k}{(q;q)_k^2} = \sum_{k=0}^{M} \frac{q^{k^2+k}}{(q;q)_k^2 (q;q)_{M-k}}. \tag{32}$$

Proof. In (31), putting $xc_1 = q$, we complete the proof. □

Corollary 4.4. *We have*

$$\sum_{k=0}^{M} \frac{q^{k^2-k}(xc_1)^k}{(q, xc_1;q)_k} = \sum_{k=0}^{M} \frac{(-1)^k q^{\binom{k+1}{2}}}{(q;q)_k (q, xc_1;q)_{M-k}}. \tag{33}$$

Proof. In (29), setting $d_2 = 0$, we complete the proof. □

Corollary 4.5. *We have*

$$\sum_{k=0}^{M} \frac{q^{\binom{k+1}{2}}(-1;q)_k}{(q;q)_k^2} = \sum_{k=0}^{M} \frac{(-1)^k q^{\binom{k+1}{2}}(-q;q)_{M-k}}{(q;q)_k(q;q)_{M-k}^2}. \tag{34}$$

Proof. In (29), taking $xc_1 = q, c_1 = -d_2$, we complete the proof. □

Theorem 4.3. *We have*

$$\begin{aligned} &\sum_{k=0}^{M} \frac{(-1)^k q^{k^2-k}(a,b;q^2)_k}{(q^2,c,d;q^2)_k(q^2;q^2)_{M-k}} \left(\frac{cd}{ab}\right)^k \\ &= \frac{(cd/ab;q^2)_M}{(d;q^2)_M} \sum_{k=0}^{M} \frac{d^k(-1)^k q^{k^2-k}(c/a,c/b;q^2)_k}{(q^2,c,cd/ab;q^2)_k(q^2;q^2)_{M-k}}. \end{aligned} \tag{35}$$

Proof. In (26), letting $q \to q^2$, we complete the proof. □

Corollary 4.6. *We have*

$$\sum_{k=0}^{M} \frac{(-1)^k q^{k^2}(-q;q^2)_k}{(q^4;q^4)_k(q^2;q^2)_{M-k}} = \frac{(q;q^2)_M}{(q^4;q^4)_M}. \tag{36}$$

Proof. In (35), letting $c = b, d = -q^2, a = -q$, we complete the proof. □

Corollary 4.7. *We have*

$$\sum_{k=0}^{M} \frac{q^{k^2-sk}(-q^s;q^2)_k}{(q;q)_{2k}(q^2;q^2)_{M-k}} = \frac{(-q^{1-s};q^2)_M}{(q;q)_{2M}}, \tag{37}$$

where $s = 0, 1$.

Proof. In (35), letting $c = b, d = q, a = -q, -1$, we complete the proof. □

Corollary 4.8. *We have*

$$\sum_{k=0}^{M} \frac{q^{k^2+(2-s)k}(-q^s;q^2)_k}{(q;q)_{2k+1}(q^2;q^2)_{M-k}} = \frac{(-q^{3-s};q^2)_M}{(q;q)_{2M+1}}, \tag{38}$$

where $s = 0, 1, 2$.

Proof. In (35), letting $c = b, d = q^3, a = -q^2, -q, -1$, we complete the proof. □

Corollary 4.9. *We have*

$$\begin{aligned} &\sum_{k=0}^{M} \frac{q^{3k^2-sk}}{(q^{2-(1-s)};q^2)_k(q^{4-2s};q^4)_k(q^2;q^2)_{M-k}} \\ &= \frac{1}{(-q^{2-s};q^2)_M} \sum_{k=0}^{M} \frac{q^{k^2+(1-s)k}}{(q;q)_{2k}(q^2;q^2)_{M-k}}, \end{aligned} \tag{39}$$

where $s = 0, 1$.

Proof. In (35), letting $c = q, d = -q^2, -q, a, b \to \infty$, we complete the proof. ☐

Corollary 4.10. *We have*

(40)
$$\sum_{k=0}^{M} \frac{q^{3k^2+sk}}{(q;q)_{2k+1}(-q^s;q^2)_k(q^2;q^2)_{M-k}} = \frac{1}{(-q^s;q^2)_M} \sum_{k=0}^{M} \frac{q^{k^2+(s-1)k}}{(q;q)_{2k+1}(q^2;q^2)_{M-k}},$$

where $s = 0, 1, 2, 3$.

Proof. In (35), letting $c = q^3, d = -q^3, -q^2, -q, -1, a, b \to \infty$, we complete the proof. ☐

Corollary 4.11. *We have*

$$\sum_{k=0}^{M} \frac{q^{2k^2}(q;q^2)_k}{(-q;q^2)_k(q^4;q^4)_k(q^2;q^2)_{M-k}} = \frac{1}{(-q;q^2)_M} \sum_{k=0}^{M} \frac{q^{k^2}(-q;q^2)_k}{(q^4;q^4)_k(q^2;q^2)_{M-k}}, \tag{41}$$

where $s = 0, 1, 2, 3$.

Proof. In (35), letting $c = q^2, d = -q, a = q, b \to \infty$, we complete the proof. ☐

Letting $M \to \infty$, if $xc_1 = q$, (33) tends to (cf. [1, p. 33, Eq. (1.1)] or [3, p. 1, Eq. (1.2)])

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k^2} = \frac{1}{(q;q)_\infty}.$$

Equation (34) reduces to

$$\sum_{k=0}^{\infty} \frac{q^{\binom{k+1}{2}}(-1;q)_k}{(q;q)_k^2} = \frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{42}$$

Equation (39) turns to

$$\sum_{k=0}^{\infty} \frac{q^{3k^2-sk}}{(q^{2-(1-s)};q^2)_k(q^{4-2s};q^4)_k} = \frac{1}{(-q^{2-s};q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+(1-s)k}}{(q;q)_{2k}}, \tag{43}$$

where $s = 0, 1$.

Equation (40) tends to

$$\sum_{k=0}^{\infty} \frac{q^{3k^2+sk}}{(q;q)_{2k+1}(-q^s;q^2)_k} = \frac{1}{(-q^s;q^2)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+(s-1)k}}{(q;q)_{2k+1}}, \tag{44}$$

where $s = 0, 1, 2, 3$.

Applying these relations above, then using the identities

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_{2k}} = \frac{(q^2, q^8, q^{10}; q^{10})_\infty (q^6, q^{14}; q^{20})_\infty}{(q;q)_\infty}, \tag{45}$$

$$\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_{2k+1}}=\frac{(q^3,q^7,q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty}, \tag{46}$$

$$\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q;q)_{2k}}=\frac{(q,q^9,q^{10};q^{10})_\infty(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty}, \tag{47}$$

$$\sum_{k=0}^{\infty}\frac{q^{k^2+2k}}{(q;q)_{2k+1}}=\frac{(q^4,q^6,q^{10};q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q;q)_\infty}, \tag{48}$$

shown in Slater's paper [14, p. 162, Eq. (98), (94), (99), (96), respectively] (or cf. [4, p. 252, Eq. (11.2.1)–Eq. (11.2.4)]), we have

$$\sum_{k=0}^{\infty}\frac{q^{3k^2}}{(q;q^2)_k(q^4;q^4)_k}=\frac{(q,q^9,q^{10};q^{10})_\infty(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty(-q^2;q^2)_\infty}, \tag{49}$$

$$\sum_{k=0}^{\infty}\frac{q^{3k^2-k}}{(q^2;q^2)_k(q^2;q^4)_k}=\frac{(q^2,q^8,q^{10};q^{10})_\infty(q^6,q^{14};q^{20})_\infty}{(q;q)_\infty(-q;q^2)_\infty}, \tag{50}$$

$$\sum_{k=0}^{\infty}\frac{q^{3k^2+2k}}{(q;q^2)_{k+1}(q^4;q^4)_k}=\frac{(q^3,q^7,q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty(-q^2;q^2)_\infty}, \tag{51}$$

$$\sum_{k=0}^{\infty}\frac{q^{3k^2+3k}}{(q;q)_{2k+1}(-q;q^2)_{k+1}}=\frac{(q^4,q^6,q^{10};q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q;q)_\infty(-q;q^2)_\infty}. \tag{52}$$

Equations (49), (50), (51) and (52) are equivalent to the identities [14, p. 154, Eq. (19)], [14, p. 156, Eq. (46)], [4, p. 252, Eq. (11.2.7)] and [14, p. 156, Eq. (44)] respectively. In [2, 3, 4, 9, 15], the authors used q-series transformations to obtain many Rogers-Ramanujan type identities. Here, we will present a new identity by using this method. From the identity in Slater's list [14, p. 154, Eq. 25], combined with (41), we get the new identity.

Corollary 4.12. *We have*

$$\sum_{k=0}^{\infty}\frac{q^{2k^2}(q;q^2)_k}{(-q;q^2)_k(q^4;q^4)_k}=\frac{(q^3,q^3,q^6;q^6)_\infty}{(q^2;q^2)_\infty}.$$

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